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CENTRAL EXTENSIONS AND QUANTUM PHYSICS *

M. Calixto[1,3†] and V. Aldaya[2,3‡]

1. Department of Physics, University of Wales Swansea, Singleton Park, Swansea, SA2 8PP, U.K.
2. Instituto de Astrofísica de Andalucía, Apartado Postal 3004, 18080 Granada, Spain.
3. Instituto Carlos I de Física Teórica y Computacional, Facultad de Ciencias, Universidad de Granada, Campus de Fuentenueva, Granada 18002, Spain.

Abstract

The unitary implementation of a symmetry group G of a classical system in the corresponding quantum theory entails unavoidable deformations $\tilde{G}$ of G, namely, central extensions by the typical phase invariance group $U(1)$. The appearance of central charges in the corresponding Lie-algebra quantum commutators, as a consequence of non-trivial responses of the phase of the wave function under symmetry transformations, lead to a quantum generation of extra degrees of freedom with regard to the classical counterpart. In particular, symmetries of the Hall effect, Yang-Mills and conformally invariant classical field theories are affected when passing to the quantum realm.

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1 Introduction

The importance of *phase invariance* in Quantum Mechanics and its physical implications has been widely discussed in the study of, for example, geometric phases (see [1] for a collection of papers). Fundamental mathematical structures and objects like *fibre bundles* and *holonomies*, and important physical concepts like *Berry's phase*, are directly attached

*Work partially supported by the DGICYT.
†E-mail: pymc@swansea.ac.uk / calixto@ugr.es
‡E-mail: valdaya@iaa.es

to this relevant invariance of quantum phenomena. Also, the quantization of some physical magnitudes and the stability of some topological quantum numbers can be directly attributed to the cyclic (compact) character of the quantum mechanical phase $\zeta = e^{i\alpha}$.

However, despite the relevance of phase invariance, the standard (canonical) approach to Quantum Theory does not stress enough its central role in the quantization procedure. Concepts like *anomaly* or *no-go theorem*, which usually make reference to failures associated with the quantization of a classical system, have been serious obstacles for the traditional quantization methods, even though *they* sometimes prove to be essential ingredients of the corresponding quantum system. Anomalies, for example, which manifest themselves through the appearance of central terms at the right-hand side of some commutators, which their classical counterpart (Poisson bracket) does not possess, prove to have an important physical meaning in connection with the *spontaneous breakdown* of some underlying symmetry. This central term contribution traces back to a non-trivial transformation of the phase of the wave function under the consecutive action of two different symmetry operations which, nevertheless, leave the corresponding classical equations of motion unchanged (see below in Sec. 5 in connection with the breakdown in the unitary implementation of the conformal symmetry for massless fields).

All of these failures in quantization should disappear if one gives up the idea of "quantizing classical systems". Also, any reasonable attempt to reconcile Gravity with Quantum Mechanics should pass through an *inherently* quantum approach to the formulation of the quantum theory of the gravitational field. Suitable candidates for an *intrinsic* formulation of Quantum Mechanics (without any reference to an underlying classical system) should incorporate tools and concepts of Algebra, Geometry and Topology to take into account both local and global aspects of Quantum Theory. In fact, physical systems possessing a gauge symmetry share and exhibit a *principal bundle* internal structure where the phase invariance plays a "central role" in, for example, determining the structure of constraints (first- and/or second-class). Symmetry is another essential component of Quantum Mechanics; after all, any consistent (non-perturbative) quantization is mostly a unitary irreducible representation of a suitable (Lie, Poisson) algebra. All these previous requirements advocate a group structure as a firm candidate to define a quantum system. Central extensions $\tilde{G}$ of a group G by $U(1)$ (the group of phase invariance) have a natural fibre bundle structure $\tilde{G} \to \tilde{G}/U(1)$, where $U(1)$ is the *structure subgroup* of $\tilde{G}$ [2]. Other fibrations $\tilde{G} \to \tilde{G}/\tilde{T}$ of $\tilde{G}$ by $\tilde{T} \sim T \times U(1)$ correspond to a constrained quantum system [3] with "gauge" group T and, as already mentioned, the structure of $\tilde{T}$ (as a central extension of T by $U(1)$) determines the nature of constraints: first- and/or second-class (see later). The group law $\tilde{g}'' = \tilde{g}' * \tilde{g}$ for $\tilde{G}$ can be written in general form as:

$$\tilde{g}'' = (g''; \zeta'') = (g' * g; \zeta'\zeta e^{\frac{i}{\hbar}\xi(g',g)}), \;\; g, g', g'' \in G; \;\; \zeta'', \zeta', \zeta \in U(1), \tag{1}$$

where $\xi : G \times G \to \Re$ is a two-cocycle, which verifies the general property:

$$\xi(g_2, g_1) + \xi(g_2 * g_1, g_3) = \xi(g_2, g_1 * g_3) + \xi(g_1, g_3) \;\; , \forall g_i \in G \, , \tag{2}$$

and the constant $\hbar$ (namely, the Planck constant) is intended to kill any possible dimension of ξ (namely, action dimension). In the general theory of central extensions [4], two-cocycles are said to be equivalent if they differ by a coboundary, i.e. a two-cocycle which can be written in the form $\xi(g', g) = \eta(g' * g) - \eta(g') - \eta(g)$, where $\eta(g)$ is called the generating function of the coboundary (from now on we shall omit the prefix "two" when referring to two-cocycles). However, although cocycles differing by a coboundary lead to equivalent central extensions as such, there are some coboundaries which provide a non-trivial connection

$$\Theta = \frac{\partial}{\partial g^j}\xi(g', g)\bigg|_{g'=g^{-1}} dg^j - i\hbar\zeta^{-1}d\zeta\,, \tag{3}$$

on the fibre bundle $\tilde{G}$ and Lie-algebra structure constants different from those of the direct product $G \times U(1)$. These are generated by a function η with a non-trivial gradient at the identity $d\eta(g)|_{g=e} = \frac{\partial\eta(g)}{\partial g^j}\Big|_{g=e} dg^j \neq 0$, and can be divided into pseudo-cohomology equivalence subclasses: two *pseudo-cocycles* are equivalent if they differ by a coboundary generated by a function with trivial gradient at the identity [5, 6, 7]. Pseudo-cohomology plays an important role in the theory of finite-dimensional semi-simple groups, as they have trivial cohomology. For them, pseudo-cohomology classes are associated with coadjoint orbits [7]. We shall have the opportunity of showing how, in fact, the introduction of coboundaries in some physical systems alters the corresponding quantum theory.

Wave functions ψ are defined in this group-framework as complex functions on $\tilde{G}$, $\psi : \tilde{G} \to C$, which verify the $\tilde{T}$*-equivariance condition*

$$L_{\tilde{g}_t}\psi(\tilde{g}) \equiv \psi(\tilde{g}_t * \tilde{g}) = D^{(\epsilon)}_{\tilde{T}}(\tilde{g}_t)\psi(\tilde{g})\,, \quad \forall \tilde{g}_t \in \tilde{T}, \ \forall \tilde{g} \in \tilde{G}\,, \tag{4}$$

where $D^{(\epsilon)}_{\tilde{T}}$ symbolizes a specific representation D of $\tilde{T}$ with ϵ-index (in particular, the $\epsilon = \vartheta$-angle [8] of non-Abelian gauge theories; see below) and generalizes the typical phase invariance of quantum mechanics, $D^{(\epsilon)}_{\tilde{T}}(\zeta) = \zeta, \ \forall \zeta \in U(1) \subset \tilde{T}$, when extra *constraints* are considered in the theory. The left-action

$$L_{\tilde{g}'}\psi(\tilde{g}) \equiv \psi(\tilde{g}' * \tilde{g}), \ \tilde{g}', \tilde{g} \in \tilde{G} \tag{5}$$

defines a reducible (in general) representation of $\tilde{G}$ on the linear space of $\tilde{T}$-equivariant wave functions (4). The reduction is achieved by means of those right restrictions on wave functions

$$R_{\tilde{g}_p}\psi(\tilde{g}) \equiv \psi(\tilde{g} * \tilde{g}_p) = \psi(\tilde{g}), \ \forall \tilde{g}_p \in G_p, \forall \tilde{g} \in \tilde{G} \tag{6}$$

(which commute with the left action) compatible with the $U(1)$-equivariant condition $D^{(\epsilon)}_{\tilde{T}}(\zeta) = \zeta$; the right restrictions (6) generalize the notion of *polarization conditions* of Geometric Quantization (see [2] and references therein), the Schrödinger equation being a particular case of polarization condition in the Group Approach to Quantization (GAQ) framework [2]. The subgroup G_p is called the *polarization subgroup* (see [9, 7] for anomalous cases) and contains non-symplectic (without dynamical content) coordinates like time, rotations, etc, and half of the symplectic ones (either 'positions' or 'momenta').

The *characteristic subalgebra* $\mathcal{G}_c$ of left-invariant vector fields (l.i.v.f.) related to non-symplectic coordinates admits an algebraic characterization as follows:

$$\mathcal{G}_c = \mathrm{Ker}\Theta \cap \mathrm{Ker}d\Theta = \{\text{l.i.v.f. } \tilde{X} \,/\, \Theta(\tilde{X}) = 0,\, d\Theta(\tilde{X}) = 0\}\,. \tag{7}$$

The representation (5) is also unitary with respect to the scalar product

$$\langle \psi | \psi' \rangle = \int_{\tilde{G}} \mu(\tilde{g}) \psi^*(\tilde{g}) \psi'(\tilde{g}) \tag{8}$$

where $\mu(\tilde{g}) = \theta_1^L \wedge \overset{\dim(\tilde{G})}{\cdots} \wedge \theta_n^L$ is the natural left-invariant measure of $\tilde{G}$ (exterior product of left-invariant one-forms).

The classical limit of the theory corresponds to the replacement $U(1) \leftrightarrow \Re$; that is, to a central extension $\bar{G}$ of G by the additive group $\Re$ with group law

$$\bar{g}'' = (g''; r'') = (g' * g; r' + r + \xi(g', g)), \;\; r, r' \in \Re\,, \tag{9}$$

the classical analogue of ψ being Hamilton's principal function when the $U(1)$-equivariance condition (4) is replaced by an $\Re$-equivariance condition

$$L_r \psi(\bar{g}) = \psi(\bar{g}) + r\,, \;\; \forall r \in \Re,\; \forall \bar{g} \in \bar{G}\,, \tag{10}$$

(see [2] for more details). Note that the constant $\hbar$ is dispensable in the classical limit $\bar{G}$ of $\tilde{G}$.

In what follows, we are going to make use of some relevant examples where symmetry determines the quantum physical system, trying to capture much of the GAQ flavour. Let us begin working out a simple, although general, example of a quantizing group $\tilde{G}$ which eventually applies to a diversity of physical systems, for example, the quantum Hall effect and quantum Yang-Mills theories.

2 Generalized Conformal Symmetry and Extended Objects from the Free Particle

The simplest quantum commutation relations are given by the Heisemberg-Weyl commutators:

$$[\hat{q}_j, \hat{p}_k] = i\hbar \delta_{jk} \hat{I} \tag{11}$$

which means that, after successive translations q' and p' of the position q and the momentum p, the wave function gains a non-trivial phase factor $\zeta = e^{\frac{i}{2\hbar}\sum_{j=1}^{d}(q'_j p_j - p'_j q_j)}$ which can be read from the Heisemberg-Weyl group law:

$$\begin{array}{rcl} \vec{V}'' &=& \vec{V}' + \vec{V} \\ \zeta'' &=& \zeta' \zeta e^{\frac{i}{2\hbar}\vec{V}'^t W \vec{V}} \end{array}\,, \tag{12}$$

where we have denoted $\vec{V} = \begin{pmatrix} \vec{q} \\ \vec{p} \end{pmatrix}$ and $W = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, a $2d \times 2d$ symplectic matrix. This is the simplest example where all the operators have a canonically-conjugated counterpart. The addition of quadratic polynomials

$$sp(2d,\Re) = \{\hat{Q}_{jk} \sim \hat{q}_j\hat{q}_k,\, \hat{P}_{jk} \sim \hat{p}_j\hat{p}_k,\, \hat{R}_{jk} \sim \hat{q}_j\hat{p}_k\} \tag{13}$$

(more precisely, the generators of the symplectic group $Sp(2d,\Re) = \{S \in M_{2d\times 2d}/\; S^tWS = W\}$) yields the Weyl-symplectic group $WSp(2d,\Re)$, whose group law is

$$\begin{aligned} S'' &= S'S\,, \\ \vec{V}'' &= \vec{V}' + S'\vec{V}\,, \\ \zeta'' &= \zeta'\zeta e^{\frac{i}{2\hbar}\vec{V}'^t WS'\vec{V}}\,. \end{aligned} \tag{14}$$

The unitary irreducible representations of the group $WSp(2d,\Re)$ describe a quantum free-like dynamics (Galilean particle, harmonic oscillator, etc) when the structure group is just $\tilde{T} = U(1)$. Other choices of $\tilde{T}$ lead to constrained (non-linear, in general) quantum dynamics on a submanifold Γ of the phase space $\Re^{2d}$; for example, the Affine structure subgroup $\tilde{T} = \tilde{A}(1)$, which is generated by $\{\hat{\vec{q}}^2 - r^2\hat{I},\, \frac{1}{2}(\hat{\vec{q}}\hat{\vec{p}} + \hat{\vec{p}}\hat{\vec{q}}),\, \hat{I}\}$, leads to a particle moving on the S^{d-1} sphere of radius r (see [10]). Nevertheless, in comparison with the Heisenberg-Weyl group, the Weyl-symplectic group, as such, does not incorporate new degrees of freedom into the theory unless extra central extensions are considered. For example, the appearance of internal degrees of freedom manifests itself through a non-trivial transformation $\zeta \stackrel{S'}{\rightarrow} \zeta e^{\frac{i}{\hbar}\xi_{\rm cob}(S',S)}$ of the phase under the action of the symplectic group $Sp(2n,\Re) \subset WSp(2n,\Re)$ (note that this possibility is not realized in the group law (14)), where $\xi_{\rm cob}(S',S) = \eta(S'S) - \eta(S) - \eta(S')$ is a (pseudo) cocycle generated by the linear function $\eta(S) = \alpha_{jk}q_{jk} + \beta_{jk}p_{jk} + \rho_{jk}r_{jk}$ in the coordinates q_{jk}, p_{jk}, r_{jk} of the symplectic group (attached to the Lie-algebra generators (13)) and $\alpha_{jk}, \beta_{jk}, \rho_{jk}$ are real parameters. When the cocycle (in fact, coboundary) $\xi_{\rm cob}(S',S)$ is added to the cocycle $\xi(\vec{V}'S'^t, \vec{V}) = \frac{1}{2}\vec{V}'^t WS'\vec{V}$ in the group law (14), i.e. when the phase transform as

$$\zeta'' = \zeta'\zeta e^{\frac{i}{\hbar}\xi(\vec{V}'S'^t,\vec{V})} e^{\frac{i}{\hbar}\xi_{\rm cob}(S',S)}\,, \tag{15}$$

it induces the appearance of central charges in the Lie-algebra commutators of $sp(2d,\Re)$. For example, a non-trivial response of the phase under rotations $SU(2) \subset Sp(6,\Re)$, given by the generating function $\eta(S) = \rho_{12}(r_{12} - r_{21})$, deforms the Lie-algebra of angular momentum operators $\hat{L}_i = \epsilon_{ijk}\hat{R}_{jk}$ and leads to the appearance of central terms

$$\left[\hat{L}_1, \hat{L}_2\right] = i\hbar\hat{L}_3 + \rho_{12}\hbar\hat{I} \tag{16}$$

proportional to the *spin* parameter $s = \rho_{12}/\hbar$. Thus, the couple $(\hat{L}_1, \hat{L}_2)$ becomes a conjugated pair of operators, on an equal footing with $(\hat{q}_j, \hat{p}_j)$; in fact, the operators $\hat{L}_j$ are no longer written in terms of the basic operators $\hat{q}_j, \hat{p}_k$, since they, themselves, are now basic (spin operators) and represent a new degree of freedom of the quantum theory given by a unitary irreducible representation of the "deformed" Weyl-symplectic group (the "one" which incorporates $\xi_{\rm cob}$). A similar argument applies for non-trivial responses of the phase under the action of the $SL(2,\Re) \subset Sp(2d,\Re)$ subgroup, given by the generating function $\eta(S) = \rho \sum_{j=1}^{d} r_{jj}$, which deforms the Lie-algebra of *squeezing* operators $\{\hat{Q} = \frac{1}{2}\sum_{j=1}^{d}\hat{Q}_{jj}, \hat{P} = \frac{1}{2}\sum_{j=1}^{d}\hat{P}_{jj}, \hat{R} = \sum_{j=1}^{d}\hat{R}_{jj}\}$ by introducing a central term

$$\left[\hat{Q}, \hat{P}\right] = i\hbar\hat{R} + \rho\hbar\hat{I} \tag{17}$$

proportional to the *symplin* (symplectic spin) parameter $\bar{s} = \rho/\hbar$ (see [10]). The symplin degree of freedom shares with spin its internal character, although it possesses an infinite number of internal states which correspond to the carrier space of the irreducible representations of the non-compact group $SL(2,\Re)$ with Bargmann index $k = \bar{s}$. Unlike spin, whose physical significance is well understood in terms of fermionic and bosonic objects, symplin does not seem to fit any known characteristic of the elementary particle. Only the (quantum) *critical* value of $\bar{s}_0 = d/2$ (for which an *anomalous* reduction of the representation is possible [10]) seems to have a direct physical meaning related to the *zero point energy* $E_0 = \bar{s}_0\hbar\omega$ of a d-dimensional harmonic oscillator (observable, for example, in the Casimir effect [11]). This particular value of $\bar{s}_0 = d/2$ tells us that the quantization map "$\hat{\ }$" assigns the operator $\hat{R} - i\hbar\bar{s}_0\hat{I} = \frac{1}{2}\sum_{j=1}^{d}(\hat{q}_j\hat{p}_j + \hat{p}_j\hat{q}_j)$ (actually, the *symmetrized* operator) to the classical function $r = \sum_{j=1}^{d} q_j p_j$. This anomalous reduction for the critical value of $\bar{s}_0 = d/2$, which allows the operator $\hat{R}$ to be written in terms of the basic operators $\hat{q}$ and $\hat{p}$, is of the same nature as the anomalous reduction which allows the Virasoro operators $\hat{L}_k$ to be written in terms of the string modes $\hat{\alpha}_n^\nu$, more explicitly

$$\hat{L}_k = \frac{1}{2} g_{\mu\nu} : \sum_n \hat{\alpha}_{k-n}^\mu \hat{\alpha}_n^\nu : \tag{18}$$

(the Sugawara construction [12]), for the critical values $c_0 = c_0' = d$ (dimension of the space-time) of the central extension parameters of the Virasoro algebra

$$\left[\hat{L}_n, \hat{L}_m\right] = \hbar(n-m)\hat{L}_{n+m} + \frac{\hbar^2}{12}(cn^3 - c'n)\delta_{n,-m}\hat{I}\,. \tag{19}$$

However, the possibility of a non-critical value of $\bar{s}$ prompts us to wonder whether or not quantizing outside the value $\bar{s}_0$ makes any sense. The fact is that, the symplin degree of freedom can be understood as forming part of a larger set of degrees of freedom originating in the free particle and conforming to an extended object which proves to generalize other physical systems bearing conformal symmetry. More precisely, this extended object arises when one is trying to quantize more classical observables than those allowed by the well-known *no-go* theorems of Groenwald and van Hove [13, 14, 15]. Indeed, the standard canonical quantization fails to go beyond any Poisson algebra containing polynomials in q and p of degree greater than two. Ambiguities in quantization arise mainly due to ordering problems; for example, the quantization mapping "$\hat{\ }$" is not unique for the classical function q^2p^2. From the point of view of group quantization, the quantum morphism

"$\hat{\ }$" can be distorted because of the appearance of inescapable terms (central ones, in particular) in the quantum Lie-algebra commutators, whose classical counterpart (Poisson algebra brackets) do not possess. In fact, let us choose the next set of classical functions of the harmonic oscillator variables $a = \frac{1}{\sqrt{2}}(q+ip)$, $a^* = \frac{1}{\sqrt{2}}(q-ip)$ (we are using $m = 1 = \omega$, for simplicity):

$$L^{\alpha}_{|n|} = \frac{1}{2}a^{2|n|}(aa^*)^{\alpha-|n|+1}, \;\; L^{\beta}_{-|m|} = \frac{1}{2}a^{*2|m|}(aa^*)^{\beta-|m|+1}, \tag{20}$$

where $n, m, \alpha, \beta \in Z/2$. A straightforward calculation from the basic Poisson bracket $\{a, a^*\} = i$ provides the following formal Poisson algebra:

$$\{L^{\alpha}_{n}, L^{\beta}_{m}\} = -i[(1+\beta)n - (1+\alpha)m]L^{\alpha+\beta}_{n+m}\,, \tag{21}$$

which formally generalizes the Poisson algebra w_∞ of functions on a two-dimensional phase space [16] (to 'half-integer' indices). The (conformal-spin-2) generators $L_n \equiv L^0_n$ close the Virasoro algebra (19) without central extension, and the (conformal-spin-1) generators $\alpha_m \equiv L^{-1}_m$ close the non-extended Abelian Kac-Moody algebra of "string-modes"; in general L^α_n has conformal-spin $N = \alpha + 2$ and conformal-dimension n (the eigenvalue of L^0_0).

If the analyticity of the classical functions L^α_n is taken into account, then one should worry about a restriction of the range of the indices n, α. The subalgebra of polynomial functions $w_\wedge \equiv \{L^\alpha_n, /\ \alpha - |n| + 1 \geq 0,\ n, \alpha \in Z\}$ closes a subalgebra of (21) and corresponds to the 'classical limit' $\hbar \rightarrow 0$ of the so-called *wedge* subalgebra $W_\wedge$ [17] (isomorphic to $sl(\infty, \Re)$ algebra), which is a particular case ($\tau(0)$) of $SL(2, \Re)$ tensor-operator algebras denoted collectively by $\tau(\mu)$ [the quotient of the enveloping algebra $\mathcal{U}(sl(2, \Re))$ by the ideal generated by $Q - \mu$, where Q is the Casimir of $sl(2, \Re)$ and μ is related to the symplin $\bar{s}$ by $\mu = \frac{\bar{s}}{2}(\frac{\bar{s}}{2} - 1)$]. The tensor algebras $\tau(\mu)$ can be considered as "quantum" deformations (mainly due to ordering ambiguities)

$$\begin{aligned} \left[\hat{L}^{\alpha}_{n}, \hat{L}^{\beta}_{m}\right] &= \sum_{j=0} \hbar^{2j+1} f^{\alpha\beta}_{2j}(n, m; \mu)\hat{L}^{\alpha+\beta-2j}_{n+m}\,, \\ f^{\alpha\beta}_{0}(n, m; \mu) &= [(1+\beta)n - (1+\alpha)m]\,, \end{aligned} \tag{22}$$

of $w_\wedge$, and their essence can be captured in a classical construction by extending the Poisson bracket $\{\cdot, \cdot\}$ to the Moyal bracket $\{\cdot, \cdot\}_M$ [18] of functions on the co-adjoint orbits of $SL(2, \Re)$ (the wedge algebra $W_\wedge$ corresponds to the case where the co-adjoint orbit is a cone $\Re^+ \times S^1$). For the critical value of the symplin $\bar{s}_0 = \frac{1}{2}$, the algebra $\tau(\mu = -\frac{3}{16})$ is the "symplecton" algebra of Biedenharn and Louck (see [17] and references therein), and the limit $\mu \rightarrow \infty$ corresponds to the area-preserving diffeomorphisms $SDiff(H)$ of a two-dimensional hyperboloid H.

Quantum deformations (22) of $w_\wedge$ do not introduce true central extensions (the situation changes when half-integer values of the indices are allowed). The inclusion of central terms in (22) requires the formal extension $W_\infty(\mu)$ of $\tau(\mu)$ beyond the wedge $|n| \leq \alpha + 1$

[17], that is, the consideration of non-polynomial functions (20). If we use polar coordinates $a = re^{i\theta}$, then the functions (20) acquire the form $L_n^\alpha = r^{2(1+\alpha)}e^{2in\theta}$, which are single-valued for all values of $\alpha, n \in Z/2$ with the condition $\alpha \geq -1$ (strictly positive conformal-spin). Central extensions of (22) —outside the wedge— of the form

$$\Sigma(\hat{L}_n^\alpha, \hat{L}_m^\beta) = \hbar^{2\alpha+2} c_\alpha(n;\mu)\delta^{\alpha,\beta}\delta_{n+m,0}\hat{I} \tag{23}$$

are known for the particular cases: $W_\infty \equiv W_\infty(0)$ and $W_{1+\infty} \equiv W_\infty(-\frac{1}{4})$. It is also precisely for these specific values of $\mu = 0, -\frac{1}{4}$ that the sequence of terms on the right-hand side of (22) turns out to be zero whenever $\alpha + \beta - 2j \leq 0$ and $\alpha + \beta - 2j \leq -1$, respectively, and therefore W_∞ (resp. $W_{1+\infty}$) can be consistently truncated to a closed algebra containing only those generators with conformal-spins ≥ 2 (resp. ≥ 1). The aforementioned central term provides extensions for all (positive) conformal-spin currents, in particular the standard central extension (19) for the Virasoro sector, and the central extension $[\hat{\alpha}_n, \hat{\alpha}_m] = nc\delta_{n+m,0}\hat{I}$ for the Abelian Kac-Moody subalgebra of conformal-spin-1 'string modes', when $\mu = -\frac{1}{4}$ (see [17]).

The algebras w_∞ and $w_{1+\infty}$ generalize the underlying Virasoro gauged symmetry of the light-cone two-dimensional induced gravity discovered by Polyakov [19], and induced actions for these w-gravity theories have been proposed in [20]. For them, the quantization procedure deforms w to W symmetry due to the presence of anomalies. Also, hidden $SL(\infty, \Re)$ and $GL(\infty, \Re)$ Kac-Moody symmetries exist for W_∞ and $W_{1+\infty}$ gravity (see [21] and [22] for a review), respectively, generalizing the hidden $SL(2, \Re)$ Kac-Moody symmetry of Polyakov's induced gravity [23].

Thus, $W_\infty(\mu)$ theories for the 'critical' values $\mu = 0$ and $\mu = -\frac{1}{4}$ are fairly well understood (also induced W_∞ gravity has being obtained as a WZNW model [24]). In contrast, gauge theories of $W_\infty(\mu)$-algebras, for general μ, are said to be less interesting because they include, in general, *negative* conformal-spins $N = \alpha + 2 \leq 0$. However, the consideration of negative conformal-spins provide new central extensions which, as far as we know, have not been considered in the literature. In fact, in addition to the Virasoro-sector central term

$$\Sigma_0(\hat{L}_n^\alpha, \hat{L}_m^\beta) = \hbar^2 \frac{c}{12}(n^3 - n)\delta^{\alpha,0}\delta^{\beta,0}\delta_{n+m,0}\hat{I} \,, \tag{24}$$

which is the only possible central extension (23) of (21) as such for positive conformal-spins $N > 0$ (see [17]), extra central extensions providing new couples of conjugated generators

$$\left[\hat{L}_n^\alpha, \hat{L}_{-n}^{-1-\alpha}\right] = \hbar n \hat{L}_0^{-1} + O(\hbar^2) = \frac{1}{2}\hbar n \hat{I} + O(\hbar^2) \tag{25}$$

are possible when negative conformal-spins are considered. Indeed, the generator $\hat{L}_0^{-1}$ (the quantum counterpart of the the classical constant function $L_0^{-1} = \frac{1}{2}$) commutes with all the other generators and plays the role of the central generator $\hat{I}$. Classical functions L_n^α with negative conformal-spins $\alpha < -1$ are singular at the origin $aa^* = 0$. Now, from a quantum point of view, each operator $\hat{L}_n^\alpha$ with $\alpha < -1$ has a conjugated counterpart $\hat{L}_{-n}^{-1-\alpha}$ (related to a non-singular classical function) and no longer needs to be written in terms of other effectively basic operators such as $\hat{L}_{\frac{1}{2}}^{-\frac{1}{2}} = \frac{1}{2}\hat{a}$ and $\hat{L}_{-\frac{1}{2}}^{-\frac{1}{2}} = \frac{1}{2}\hat{a}^\dagger$, thus avoiding problems of lack of analyticity. Even more, the operator $\hat{L}_0^0$ associated with the classical function $L_0^0 = \frac{1}{2}aa^*$ is never zero because of the existence of a "zero point energy" (critical value of the symplin degree of freedom) associated with the inherent ordering ambiguity; also, square roots of aa^* are a minor harm from the quantum point of view.

In other words, the removal of the origin from the original symplectic manifold (leading to a punctured complex plane) creates new cohomology on the Poisson algebra of functions, i.e. new couples of conjugated generators (in fact, an infinite amount) emerge when the topology of the phase-space turns less trivial ($\Re^2 \to \Re^+ \times S^1$) and, as a result, new quantum *extended* objects possessing conformal symmetry appear. The field content of these extended objects and the possible deformations (renormalizations) caused by the quantization procedure are being investigated from a GAQ framework [25]. An explicit expression for the corresponding invariant action functional S can be given from the integral $S = \int \Theta$ of the one-form Θ in (3), which generalizes the Poincaré-Cartan form Θ_{PC} of Classical Mechanics for the quantum phase term $-i\hbar\zeta^{-1}d\zeta$, along the trajectories of the l.i.v.f. in the characteristic subalgebra (7). This way of constructing field models from two-cocycles (see Ref. [27]) differs from the analysis in related works based on the coadjoint-orbit approach [13, 14, 28, 29].

Also, the study of tensor-operator algebras of more general simple groups, their precise relation with the Poisson algebra of functions on the corresponding co-adjoint orbits and the physical meaning of its extensions 'beyond the wedge' is in progress [26]. For the case of $SU(p)$, the generators L_n^α are now labelled by a $(p-1)$-dimensional —the rank of $SU(p)$— vector $\alpha = (\alpha_1, \ldots, \alpha_{p-1})$, which is taken to lie on an integral lattice, and an upper-triangular $p \times p$ matrix n, with $p(p-1)/2$ integral entries —the dimension of the biggest coadjoint orbit— (see [26] for more details).

Let us see how $W_{1+\infty}$ symmetry also arises in a two-dimensional electron gas in a perpendicular magnetic field, and its relevance in the classification of all the universality classes of *incompressible quantum fluids* and the identification of the quantum numbers of the excitations in the Quantum Hall Effect (QHE).

3 Quantum Hall Universality Classes and $W_{1+\infty}$ Symmetry

QHE can be considered the paradigm of the two-dimensional quantum physical systems. The extreme precision of the rational values of the Hall conductivity $\sigma_{x_1x_2}$ (in e^2/h units) suggests that the dynamics of planar electrons in a perpendicular magnetic field H is constrained by symmetry and topology. In fact, *magnetic translations* $x_j = -\epsilon_{jk}q_k + \frac{p_j}{m\omega_c}$, $j = 1, 2$ ($\omega_c = \frac{eH}{mc}$ denotes the cyclotron frequency), which are standard space translations combined with gauge transformations, are the most important underlying symmetry of this system. The commutator of two infinitesimal magnetic translations is $[\hat{x}_j, \hat{x}_k] = i\frac{m\omega_c}{\hbar}\epsilon_{jk}\hat{I}$, which reproduces the Heisemberg-Weyl commutation relations given in (11). This symmetry is responsible for the infinite degeneracy of the Landau levels, degeneracy that becomes *finite* when boundary conditions (for example, periodicity conditions) are imposed. This constrained symmetry can be described by means of the Heisemberg-Weyl group $\tilde{G}$ (12) in $d = 1$, that is:

$$x''_j = x'_j + x_j, \quad \zeta'' = \zeta'\zeta e^{i\frac{m\omega_c}{2\hbar}(x'_1 x_2 - x_1 x'_2)}, \tag{26}$$

where finite translations $x_j \to x_j + k_j l_j$ by an integer amount of l_j (spatial period) play the role of the structure subgroup $\tilde{T} \sim Z \times Z \times U(1) \subset \tilde{G}$ (see [30] for more details). One can easily check from the last group law that, for integer values of $\phi = \frac{m\omega_c l_1 l_2}{2\pi\hbar} = n$, the structure subgroup $\tilde{T}$ is Abelian (i.e. $\tilde{T}$ is a trivial central extension $\tilde{T} = Z \times Z \times U(1)$), so that all the constraints are first-class, and the unitary irreducible representations of $\tilde{G}$ are n-dimensional (in contrast with the infinite-dimensional character of the $\tilde{T} = U(1)$ case). The condition $\phi = n$ represents a quantization of the magnetic flux through the torus surface (the quotient $\Re \times \Re / Z \times Z$) in the same manner as in the Dirac monopole case. Nevertheless, this quantization condition is not strictly necessary and fractional values of the flux $\phi = \frac{n}{r}$ are also allowed (the irrational case is more involved and requires techniques from Non-Commutative Geometry [31]). The structure subgroup $\tilde{T}$ is non-Abelian for this case (it is a non-trivial central extension), so that constraints are second-class, and the unitary irreducible representations of $\tilde{G}$ are $n \times r$-dimensional ("spinorial"-like representations) [30].

There is a reciprocal relation between the flux ϕ and the *filling factor* $\nu = h\sigma_{x_1 x_2}/e^2$, which is a very stable (topological) number that appears with *odd* denominators for fermionic carrying systems (perhaps modular invariance could explain this 'odd denominator rule' [32]). Very accurate trial wave functions were proposed by Laughlin (see [33] for a review) to describe the behavior of the ground state for different values of ν. These are macroscopic quantum states with uniform density and a gap for density fluctuations, which lead to the key idea of *two-dimensional incompressible quantum fluids*, and can be classically thought of as *droplets of liquid* without density waves. Thus, at the classical level, all possible configurations of droplets of incompressible fluid (*edge excitations*) can be generated by area-preserving diffeomorphisms from a reference droplet, that is, by applying generators of $w_{1+\infty}$ on the ground-state distribution function Ω. This dynamical symmetry, which has been identified by [34], traces back to the extra canonical transformations of the four-dimensional phase space that leave invariant the Hamiltonian; that is, functions like (20) of magnetic translations $a = \frac{1}{2}(x_2 + ix_1)$.

The quantization of these edge excitations can be achieved by studying the irreducible, unitary highest-weight representations of $W_{1+\infty}$ (a complete classification has been given in [35]), which correspond to $(1+1)$-dimensional effective conformal field theories. In this formalism, the incompressible quantum fluid ground state $|\Omega\rangle$ appears as a highest-weight state satisfying $\hat{L}^\alpha_n|\Omega\rangle = 0$, $\forall n \geq 0$, $\alpha \geq -1$. Particle-hole edge excitations above the ground state are obtained by applying generators with negative mode index, $\hat{L}^\alpha_{-|n|}$, to $|\Omega\rangle$. These highest-weight conditions are automatically fulfilled from the GAQ point of view. Indeed, according to general settings [3, 36], the vacuum of a quantum theory defined through a quantizing group $\tilde{G}$ must be annihilated by the right-version of the polarization subalgebra dual to $\mathcal{G}_p$ in (6). For the case of the quantizing algebra $\tilde{\mathcal{G}} = W_{1+\infty}$, the characteristic subalgebra (7) is $\mathcal{G}_c = \{\hat{X}^\alpha_0,\ \alpha \geq -1\}$ ($\hat{X}$ means the left-invariant vector field counterpart of $\hat{L}$), which can be enlarged to a polarization subalgebra $\mathcal{G}_p = \{\hat{X}^\alpha_n,\ \alpha \geq -1,\ n < 0\}$. The polarization conditions (6) $\hat{X}\psi = 0,\ \hat{X} \in \mathcal{G}_p$ reduce the representation (5) of $\tilde{G}$ on $U(1)$-equivariant wave functions (4).

The point is that one can presumably characterize *any* quantum incompressible fluid as a $W_{1+\infty}$ theory, so that the hierarchy problem (that is, the classification of stable ground states and their excitations corresponding to all observed *plateaus* or filling fractions) reduces to a complete classification of $W_{1+\infty}$ theories (see [34] for more details).

The general idea that central extensions of the symmetry group of a physical system provide new (quantum) degrees of freedom is also applicable to the appearance of mass (in a *non*-standard way) in Yang-Mills quantum theories, as we are going to see now.

4 Group Quantization of Yang-Mills Theories: a Cohomological Origin of Mass

Some of the essential issues we have discussed in Sec. 2 about the irreducible representations of $WSp(2d,\Re)$ bearing internal degrees of freedom, will be useful in showing how mass can enter Yang-Mills theories through central (pseudo) extensions of the corresponding gauge group. This mechanism does not involve extra (Higgs) scalar particles and could provide new clues for the better understanding of the nature of the Symmetry Breaking Mechanism. We are going to outline the essential points and refer the interested reader to Refs. [38, 39] for further information.

Let us denote by $A^\mu(x) = r^a_b A^\mu_a(x) T^b$, $\mu = 0, ..., 3; a, b = 1, ..., n$ the Lie-algebra valued vector potential attached to a non-Abelian gauge group which, for simplicity, we suppose to be unitary, say $T = \mathrm{Map}(\Re^4, SU(N)) = \{U(x) = \exp \varphi_a(x) T^a\}$, where T_a are the generators of $SU(N)$, which satisfy the commutation relations $[T_a, T_b] = C^c_{ab} T_c$, and the coupling constant matrix r^a_b reduces to a multiple of the identity $r^a_b = r\delta^a_b$. We shall also make partial use of the gauge freedom to set the temporal component $A^0 = 0$, so that the Lie-algebra valued electric field is simply $E^j(x) \equiv r^a_b E^j_a(x) T^b = -\dot{A}^j(x)$. In this case, there is still a residual gauge invariance $T = \mathrm{Map}(\Re^3, SU(N))$ (see [40]).

The proposed (infinite dimensional) quantizing group for quantum Yang-Mills theories will be a central extension $\tilde{G}$ of $G = (G_A \times G_E) \times_s T$ (semi-direct product of the cotangent group of the Abelian group of Lie-algebra valued vector potentials and the non-Abelian gauge group T) by $U(1)$. More precisely, the group law for $\tilde{G}$, $\tilde{g}'' = \tilde{g} * \tilde{g}$, with $\tilde{g} = (A^j_a(x), E^j_a(y), U(z); \zeta)$, can be explicitly written as (in natural units $\hbar = 1 = c$):

$$\begin{aligned} U''(x) &= U'(x)U(x)\,, \\ \vec{A}''(x) &= \vec{A}'(x) + U'(x)\vec{A}(x)U'(x)^{-1}\,, \\ \vec{E}''(x) &= \vec{E}'(x) + U'(x)\vec{E}(x)U'(x)^{-1}\,, \end{aligned}$$

$$\zeta'' = \zeta'\zeta \exp\left\{-\frac{i}{r^2}\sum_{j=1}^{2}\xi_j(\vec{A}',\vec{E}',U'|\vec{A},\vec{E},U)\right\}; \quad (27)$$
$$\xi_1(g'|g) \equiv \int d^3x \,\mathrm{tr}\left[\left(\begin{array}{cc}\vec{A}' & \vec{E}'\end{array}\right) W \left(\begin{array}{c} U'\vec{A}U'^{-1} \\ U'\vec{E}U'^{-1}\end{array}\right)\right],$$
$$\xi_2(g'|g) \equiv \int d^3x \,\mathrm{tr}\left[\left(\begin{array}{cc}\nabla U'U'^{-1} & \vec{E}'\end{array}\right) W \left(\begin{array}{c} U'\nabla U U^{-1}U'^{-1} \\ U'\vec{E}U'^{-1}\end{array}\right)\right],$$

where $W = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0\end{array}\right)$ is a symplectic matrix and we have split up the cocycle ξ into two distinguishable and typical cocycles $\xi_j,\ j = 1, 2$. The first cocycle ξ_1 is meant to provide *dynamics* for the vector potential, so that the couple (A, E) corresponds to a canonically-conjugate pair of coordinates. The second cocycle ξ_2, the *mixed* cocycle, provides a non-trivial (non-diagonal) action of the structure subgroup $\tilde{T}$ on vector potentials and determines the number of degrees of freedom of the constrained theory; in fact, it represents the "quantum" counterpart of the "classical" unhomogeneous term $U(x)\nabla U(x)^{-1}$ we miss at the right-hand side of the gauge transformation of $\vec{A}$ (second line of (27)), that is, the vector potential $\vec{A}$ has to transform homogeneously under the action of the gauge group T in order to define a proper group law, whereas the inhomogeneous term $U(x)\nabla U(x)^{-1}$ modifies the *phase* ζ of the wave function according to ξ_2 (see [39, 38] for a covariant form of this "quantum" transformation).

To make more explicit the intrinsic significance of these two quantities $\xi_j\,,\ j = 1, 2$, let us calculate the non-trivial Lie-algebra commutators of the right-invariant vector fields (that is, the generators of the left-action $L_{\tilde{g}'}(\tilde{g}) = \tilde{g}' * \tilde{g}$ of $\tilde{G}$ on itself) from the group law (27). They are explicitly:

$$\begin{aligned}
\left[\hat{A}^j_a(x), \hat{E}^k_b(y)\right] &= i\delta_{ab}\delta^{jk}\delta(x-y)\hat{I}\,, \\
\left[\hat{A}^j_a(x), \hat{\varphi}_b(y)\right] &= -iC^c_{ab}\delta(x-y)\hat{A}^j_c(x) - \frac{i}{r}\delta_{ab}\partial^j_x\delta(x-y)\hat{I}\,, \\
\left[\hat{E}^j_a(x), \hat{\varphi}_b(y)\right] &= -iC^c_{ab}\delta(x-y)\hat{E}^j_c(x) \\
\left[\hat{\varphi}_a(x), \hat{\varphi}_b(y)\right] &= -iC^c_{ab}\delta(x-y)\hat{\varphi}_c(x)\,,
\end{aligned} \quad (28)$$

which agree with those of Ref. [40].

The unitary irreducible representations of $\tilde{G}$ with structure subgroup $\tilde{T} = T \times U(1)$ (a direct product for this case) represent a quantum theory of $n = N^2 - 1 = \dim(SU(N))$ interacting massless vector bosons. Indeed, we start with $f = 3n$ field degrees of freedom, corresponding to the basic operators $\{\hat{A}^j_a(x), \hat{E}^j_a(x)\}$ (the ones that have a conjugated counterpart); the constraints (4) provide $c = n$ independent restrictions $\hat{\varphi}_a(x)\psi = 0,\ a = 1, \ldots, n$ (the quantum implementation of the non-Abelian Gauss law), since they are first-class constraints and we choose the trivial representation $D^{(\epsilon)}_{\tilde{T}}(\tilde{g}_t) = 1,\ \forall \tilde{g}_t = (0, 0, U(x); 1) \in T$, restrictions which lead to $f_c = f - c = 2n$ field degrees of freedom corresponding to an interacting theory of n massless vector bosons.

However, more general representations $D^{(\epsilon)}_{\tilde{T}}(U) = e^{i\epsilon_U}$ can be considered when we impose additional boundary conditions like $U(x) \stackrel{x\to\infty}{\longrightarrow} \pm I$, that is, when we compactify the space $\Re^3 \to S^3$ so that the gauge group T falls into disjoint homotopy classes $\{U_l\,,\, \epsilon_{U_l} = l\vartheta\}$ labeled by integers $l \in Z = \pi_3(SU(N))$ (the third homotopy group). The index ϑ (the *ϑ-angle* [8]) parametrizes *non-equivalent quantizations*, as the Bloch momentum ϵ does for particles in periodic potentials, where the wave function acquires a phase $\psi(q + 2\pi) = e^{i\epsilon}\psi(q)$ after a translation of, let us say, 2π. The phenomenon of non-equivalent quantizations can also be reproduced by keeping the constraint condition $D^{(\epsilon)}_{\tilde{T}}(U) = 1$ unchanged at the price of introducing a new (pseudo) cocycle ξ_ϑ which is added to the previous cocycle $\xi = \xi_1 + \xi_2$ in (27). The generating function η_ϑ of ξ_ϑ is

$$\eta_\vartheta(g) = \vartheta \int d^3x\, \mathcal{C}^0(x)\,, \quad \mathcal{C}^\mu = -\frac{1}{16\pi^2}\epsilon^{\mu\alpha\beta\gamma}\mathrm{tr}(\mathcal{F}_{\alpha\beta}\mathcal{A}_\gamma - \frac{2}{3}\mathcal{A}_\alpha\mathcal{A}_\beta\mathcal{A}_\gamma)\,, \tag{29}$$

where $\mathcal{A} \equiv A + \nabla U U^{-1}$ and $\mathcal{C}^0$ is the temporal component of the *Chern-Simons secondary characteristic class* $\mathcal{C}^\mu$, which is the vector whose divergence equals the Pontryagin density $\mathcal{P} = \partial_\mu\mathcal{C}^\mu = -\frac{1}{16\pi^2}\mathrm{tr}({}^*\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu})$ (see [40], for instance). Like some total derivatives (namely, the Pontryagin density), which do not modify the classical equations of motion when added to the Lagrangian but have a non-trivial effect in the quantum theory, pseudo-cocycles like ξ_ϑ give rise to non-equivalent quantizations when the topology of the space is affected by the imposition of certain boundary conditions ("compactification of the space"), even though they are trivial cocycles of the "unconstrained" theory. The phenomenon of non-equivalent quantizations can be also sometimes understood as a *Aharonov-Bohm-like effect* (an effect experienced by the quantum particle but not by the classical particle) and $d\eta(g) = \frac{\partial\eta(g)}{\partial g^j}dg^j$ can be seen as an *induced gauge connection* (see [30] for the example of a super-conducting ring threaded by a magnetic flux) which modifies momenta according to the minimal coupling.

We can also go further and consider more general representations $D^{(\epsilon)}_{\tilde{T}}$ of $\tilde{T}$ (in particular, non-Abelian representations) by adding extra pseudo-cocycles to ξ. This is the case of

$$\xi_\lambda(g'|g) \equiv -2\int d^3x\,\mathrm{tr}[\lambda\,(\log(U'U) - \log U' - \log U)]\,, \tag{30}$$

which is generated by $\eta_\lambda(g) = -2\int d^3x\,\mathrm{tr}[\lambda\log U]$, where $\lambda = \lambda^a T_a$ is a matrix carrying some parameters λ^a which actually characterize the representation of $\tilde{G}$. In fact, this pseudo-cocycle alters the gauge group commutators and leads to the appearance of new central terms at the right-hand side of the last equation in (28), more explicitly:

$$[\hat{\varphi}_a(x), \hat{\varphi}_b(y)] = -iC^c_{ab}\delta(x-y)\hat{\varphi}_c(x) - iC^c_{ab}\frac{\lambda_c}{r^2}\delta(x-y)\hat{I}\,. \tag{31}$$

For this case, new couples of generators become conjugated because of the appearance of these new central terms proportional to the parameter λ_c at the right-hand side of the commutators of the gauge generators $\hat{\varphi}_a(x)$. That is, new basic operators $(\hat{\varphi}_a(x), \hat{\varphi}_b(x))$, such that $C^c_{ab}\lambda_c \neq 0$, join the previous $(\hat{A}^j_a(x), \hat{E}^j_a(x))$, since they do not need to be written in terms of them. Constraints are second-class for this case and only the set

$\mathcal{T}_c = \{\hat{\varphi}_a(x), \, / \, C^c_{ab}\lambda_c \neq 0 \;\forall b\}$ (that is, the subalgebra of non-dynamical operators of $\tilde{T}$), together with half of the dynamical operators of $\tilde{T}$ (that is, a polarization subgroup T_p of $\tilde{T}$), can be consistently imposed as constraints. These facts lead to an increasing of field degrees of freedom for the constrained theory, with regard to the former (massless) case. In fact, the irreducible representations of the algebra (28) with the new commutator (31) define an interacting theory of $n_c = \dim(T_c)$ massless vector bosons, corresponding to the unbroken gauge subgroup $T_c \subset \tilde{T}$ with Lie algebra $\mathcal{T}_c$, and $n - n_c$ massive-like vector bosons with mass cubed $m_c^3 = \lambda_c$ (see [38, 39] for more details).

Pseudo-cocycle parameters such as λ_c are usually hidden in a redefinition of the generators involved in the pseudo-extension $\hat{\varphi}_c(x) + \frac{\lambda_c}{r^2} \equiv \hat{\varphi}'_c(x)$, as it happens for example with the parameter c' in the Virasoro algebra (19), which is a redefinition of $\hat{L}_0$. However, whereas the vacuum expectation value $\langle 0_\lambda|\hat{\varphi}_c(x)|0_\lambda\rangle$ is zero,[§] the vacuum expectation value $\langle 0_\lambda|\hat{\varphi}'_c(x)|0_\lambda\rangle = \lambda_c/r^2$ of the redefined operators $\hat{\varphi}'_c(x)$ is non-null and proportional to the cubed mass in the 'direction' c of the unbroken gauge symmetry T_c, which depends on the particular choice of the mass matrix λ. Thus, the effect of the pseudo-extension manifests also in a different choice of a vacuum in which some gauge operators have a non-zero expectation value. This fact reminds us of the Higgs mechanism in non-Abelian gauge theories, where the Higgs fields point to the direction of the non-null vacuum expectation values. However, the spirit of the Higgs mechanism, as an approach to supply mass, and the one discussed here are radically different, even though they have some common characteristics. In fact, we are not making use of extra scalar fields in the theory to provide mass to the vector bosons, but it is the gauge group itself that acquires dynamics for the massive case and transfers degrees of freedom to the vector potentials to form massive vector bosons. Thus, the appearance of mass seems to have a *cohomological origin*, beyond any introduction of extra scalar particles (Higgs bosons). The physical implications of this alternative approach deserve further study, although some important steps have been already done (see [37, 38, 39]). Also, it would be worth exploring the richness of the case $SU(\infty)$ (infinite number of colours), the Lie-algebra of which is related to the (infinite-dimensional) Lie-algebra of area preserving diffeomorphisms of the torus $SDiff(T^2)$, for which *true* central extensions

$$[T_{\vec{m}}, T_{\vec{n}}] = (\vec{m} \times \vec{n}) T_{\vec{m}+\vec{n}} + \vec{\lambda} \cdot \vec{m}\delta_{\vec{m}+\vec{n},0}\hat{I}, \qquad \vec{m}, \vec{n} \in Z \times Z \tag{32}$$

exist (see e.g. [41]; see also [42] for a change basis relating the generators $T_{\vec{m}}$ and L^α_n in Sec. 2,3). Like in String Theory, the appearance of central terms in the constraint subalgebra does not spoil gauge invariance but forces us to impose a polarization subgroup T_p of $\tilde{T}$ only (namely, the 'positive modes' $\hat{L}_{n\geq 0}$ of the Virasoro algebra) as restrictions (4) on physical wave functions; for this case, constraints are said to be of second-class.

[§]it can be easily proven taking into account that the vacuum is annihilated by the right version of the polarization subalgebra dual to $\mathcal{G}_p$ [36]; also, $\hat{\varphi}_c = \tilde{X}^R_{\varphi_c}$ is always in $\mathcal{T}_p$; that is, it is zero on constrained wave functionals $\Psi_{\rm phys.}$, including the physical vacuum.

5 Vacuum Radiation and Symmetry Breaking in Conformally Invariant Quantum Field Theory

The quantum theory demands an extra requirement of a symmetry in comparison with the classical theory, namely the requirement of *unitarity*, and this proves to be the reason for some breakdown in the quantum implementation of some classical symmetries of certain physical systems. For example, the conformal group $SO(4,2)$ has always been recognized as a symmetry of the Maxwell equations for *classical* electro-dynamics [43], and more recently considered as an invariance of general, non-Abelian, massless gauge field theories at the classical level; however, the *quantum* theory raises, in general, serious problems in the implementation of conformal symmetry and much work has been devoted to the study of the physical reasons for that (see e.g. Ref. [44]). Basically, the main trouble associated with this quantum symmetry (at the second quantization level) lies in the difficulty of finding a vacuum for massless fields, which is *stable* under special conformal transformations acting on the Minkowski space in the form:

$$x^\mu \to x'^\mu = \frac{x^\mu + c^\mu x^2}{\sigma(x,c)}, \quad \sigma(x,c) = 1 + 2cx + c^2x^2. \tag{33}$$

These transformations, which can be interpreted as transitions to systems of relativistic, uniformly accelerated observers (see e.g. Ref. [45]), cause *vacuum radiation*, a phenomenon analogous to the Fulling-Unruh effect [46, 47] in a non-inertial reference frame. To be more precise, if $a(k), a^+(k)$ are the Fourier components of a scalar massless field $\phi(x)$, satisfying the equation

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi(x) = 0\,, \tag{34}$$

then the Fourier components $a'(k), a'^+(k)$ of the transformed field $\phi'(x') = \sigma^{-l}(x,c)\phi(x)$ by (33) (l being the conformal dimension) are expressed in terms of both $a(k), a^+(k)$ through a Bogolyubov transformation

$$a'(\lambda) = \int dk\,\Big[A_\lambda(k)a(k) + B_\lambda(k)a^+(k)\Big]\,. \tag{35}$$

In the second quantized theory, the vacuum states defined by the conditions $\hat{a}(k)|0\rangle = 0$ and $\hat{a}'(\lambda)|0'\rangle = 0$, are not identical if the coefficients $B_\lambda(k)$ in (35) are not zero. In this case the new vacuum has a non-trivial content of untransformed particle states.

This situation is always present when quantizing field theories in curved space as well as in flat space, whenever some kind of global mutilation of the space is involved. This is the case of the natural quantization in Rindler coordinates [46], which leads to a quantization inequivalent to the normal Minkowski quantization, or that of a quantum field in a box, where a dilatation produces a rearrangement of the vacuum [46]. Nevertheless, it must be stressed that the situation for SCT is more peculiar. The rearrangement of the vacuum in a massless QFT due to SCT, even though they are a *symmetry* of the classical system, behaves as if the conformal group were *spontaneously broken*, and this fact can be interpreted as some sort dynamical *anomaly*.

Thinking of the underlying reasons for this anomaly, we are tempted to make the singular action of the transformations (33) in Minkowski space responsible for it, as has been in fact pointed out in [48]. However, a deeper analysis of the interconnection between symmetry and quantization reveals a more profound obstruction to the possibility of *unitarily* implementing STC in a generalized Minkowski space (homogeneous space of the conformal group), free from singularities. This obstruction is traced back to the impossibility of representing the entire $SO(4,2)$ group *unitarily* and irreducibly on a space of functions depending arbitrarily on $\vec{x}$ (see e.g. Ref. [44]), so that a Cauchy surface determines the evolution in time. Natural representations, however, can be constructed by means of wave functions having support on the whole space-time and evolving in some kind of *proper time* (related to a dilatation coordinate δ). It has been proved (see [36]) that *unitary* irreducible representations of the conformal group require the generator $\hat{P}_0$ of time translations to have dynamical character (i.e., it has a conjugated pair), as happens with the spatial component $\hat{P}_1$, due to the appearance of a central term $\hat{I}$ in the quantum commutators

$$[\hat{P}_\mu, \hat{K}_\mu] = -\eta_{\mu\mu}(2\hat{D} + 4N\hat{I}) \tag{36}$$

($\hat{K}_\mu$ and $\hat{D}$ denote the generators of SCT and scale transformations, respectively) proportional to the extension parameter N, which characterizes the unitary irreducible representations of the pseudo-extended conformal group $\tilde{G} = SO(4,2)\tilde{\otimes}U(1)$, i.e., a trivial central extension of $SO(4,2)$ by $U(1)$ with generating function $\eta_N(g) = N\delta$ linear in the dilatation coordinate δ. That is, time is a quantum observable subject to uncertainty relations in conformally invariant quantum mechanics on Minkowski space and this fact extends covariance rules to the quantum domain. As a consequence of this, conformal wave functions $\psi^{(N)}(\vec{x}, t)$ have support on the whole space-time. If we forced the functions $\psi^{(N)}$ to evolve in time according to the Klein-Gordon-like equation

$$\hat{Q}\psi^{(N)} = \hat{P}_\mu \hat{P}^\mu \psi^{(N)} = 0\,, \tag{37}$$

("null square mass condition", i.e., by selecting those functions annihilated by the Casimir operator $\hat{Q}$ of the Poincaré subgroup of $SO(4,2)$) we would find that the appearance of *quantum* terms proportional to N at the right-hand side of the quantum commutators

$$\left[\hat{K}_\mu, \hat{Q}\right] = f_\mu(x,t)\hat{Q} + 8N\hat{P}_\mu \tag{38}$$

(where $f_\mu(x,t)$ are some functions on the generalized Minkowski space), terms which do not appear at the classical level ($N = 0$), prevent the whole conformal group to be an exact symmetry of the massless quantum field. This way, the quantum time evolution itself destroys the conformal symmetry, leading to some sort of *dynamical symmetry breaking* which preserves the Weyl subgroup (Poincaré + dilatations). The SCT do not leave Eq. (37) invariant, as can be deduced from (38), and this fact manifests itself, at the second quantization level, through a *radiation* of the vacuum of a massless quantum field ("Weyl vacuum") under the action of SCT. That is, from the point of view of a uniformly accelerated observer, the Weyl vacuum (which proves to be a coherent conformal state made of *zero modes*) radiates like a black body. The spectrum of the outgoing particles can be calculated exactly [36] and proves to be a generalization of the Planckian one, this being recovered in the limit $N \to 0$. The temperature T of this thermal bath is linear in the acceleration parameter a, more precisely $T = \frac{\hbar}{2\pi ck} a$ where k denotes Boltzmann's constant and c is the speed of light. This simple, but profound, relation between temperature and acceleration was first considered by Unruh [47].

6 Comments

Several approaches to quantum theory (namely, canonical quantization, path integrals, geometric quantization, etc) exist and all of them are still rather rooted in the classical understanding of physical phenomena. Indeed, the reference to a classical action as a starting point seems to be the usual prerequisite to define a given quantum system. New perspectives on the approach to quantum physics can provide novel insights that could not be reached from classically-oriented formulations. Also, the present state of 'uncertainty' for a satisfactory quantum theory of the gravitational field demands a sound revision of Quantum Mechanics.

In this paper, we have tried to capture much of the GAQ flavour by making use of some relevant examples where symmetry determines the quantum physical system. Central extensions (and more general quantum deformations) of the underlying symmetry algebra provide the essential ingredient for a group-oriented approach to quantum theory. Among its main virtues are the inherent formulation of quantum theories without reference to a previous classical system, especially gauge theories, which exhibit a particular fibre-bundle structure. Nevertheless, the full richness and scope of this alternative point of view deserves further study.

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