FINITE-DIFFERENCE EQUATIONS IN CIRCULAR-WAVEGUIDE COUPLING PROBLEMS

Indexing terms: Circular waveguides, Difference equations, Perturbation techniques, Waveguide couplers

A numerical method is developed to verify perturbation results in imperfect-circular-waveguide computations, and some results are presented for the most critical cases. Conclusions encourage both the perturbation solution and the proposed numerical solution by finite-difference equations.

The degradation of the transmission characteristics, due to random conversion and reconversion phenomena, in a circular-waveguide link can be readily computed if a perturbation solution is used.[1,2] However, the degree of approximation of the solution obtained cannot be satisfactorily controlled, and we are facedwith the problem of the accuracy of the results in determining the design parameters of the link. A different approach is contained in Reference 3; very interesting results are given for coupled covariance differential equations. Unfortunately, these results are exact only if the imperfection power spectrum is white. If the computations are to be extended to nonwhite spectra, we need a control.[4] In this letter, the following basic idea is presented: it is possible to follow a different path in deriving the equations of Reference 3. In this way, an 'intermediate' solution, in terms of finite-difference equations, is obtained for every power spectrum. This solution is approximate, and the approximation can be improved by reducing the step of the solution. Thus, as it will be shown, control of the approximation is possible.

Let us start from coupled-line equations of the form:

$$\left.\begin{aligned}\frac{dG_0(z)}{dz} &= jc(z)\exp(\Delta\Gamma z)\,G_1(z)\\ \frac{dG_1(z)}{dz} &= jc(z)\exp(\Delta\Gamma z)\,G_0(z)\end{aligned}\right\} \quad \ldots\ldots \quad (1)$$

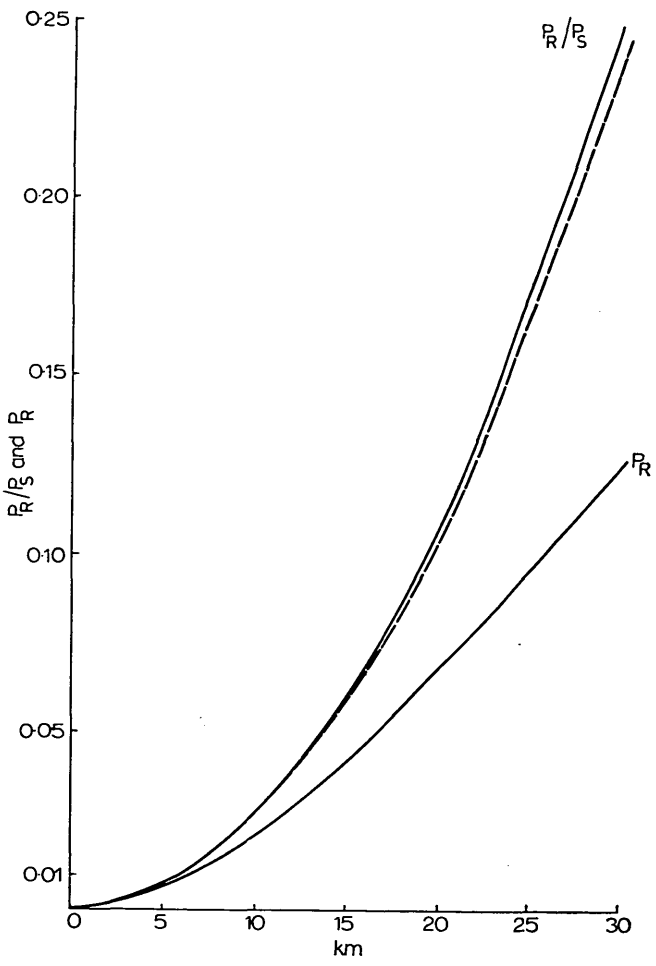

Fig. 1 *P_R and P_R/P_s*
——— finite-difference equations
— — — perturbation solution for $\Delta\alpha = 0$

where

$$\Delta\Gamma = \Gamma_0 - \Gamma_1 = \alpha_0 - \alpha_1 + j(\beta_0 - \beta_1) = \Delta\alpha + j\Delta\beta$$

where Γ_0 and Γ_1 are complex propagation coefficients of the TE_{01} mode and the unwanted mode, respectively, and $c(z)$ is the coupling coefficient, assumed real. In eqn. 1, G_0 and G_1 are the normalised complex-wave amplitudes related to the complex-wave amplitudes I_0 and I by

$$\left.\begin{aligned}G_0(z) &= I_0(z)\exp(\Gamma_0 z)\\ \text{and}\quad G_1(z) &= I_1(z)\exp(\Gamma_1 z)\end{aligned}\right\} \quad \ldots\ldots \quad (2)$$

G_0 and G_1 are functions of $\Delta\alpha$, $\Delta\beta$ and z. As $\Delta\beta$ can be considered proportional to the frequency, we choose it as the explicit variable. Let us define

$$\left.\begin{aligned}R_0 &= \langle G_0^*(\Delta\beta)\,G_0(\Delta\beta+\sigma)\rangle = \langle G_0^*\,G_{0\sigma}\rangle\\ R_1 &= \langle G_1^*(\Delta\beta)\,G_1(\Delta\beta+\sigma)\rangle = \langle G_1^*\,G_{1\sigma}\rangle\end{aligned}\right\} \quad (3)$$

where the operator $\langle\ \rangle$ indicates the ensemble average, σ is a variation from $\Delta\beta$ and, as a subscript, indicates that the computation is performed for $\Delta\beta+\sigma$. From eqn. 1, it is possible to compute the first derivative of eqns. 3 at the co-ordinate z, as a function of G_0 and G_1 in z. These expressions are significantly simplified by assuming a 1st-order perturbation solution for computing G_0 and G_1 in z; hence, expressions are found for the first derivative of eqn. 3, as functions of coupling, waveguide parameters, autocorrelation function of the deformation process, assumed stationary, and of R_0 and R_1 for $z = 0$. Assuming $c(z)$ is proportional to the deformation,

$$c(z) = C_0\,d(z) \quad \ldots\ldots \quad (4)$$

$$C(u) = \langle d(z)\,d(z+u)\rangle \quad \ldots\ldots \quad (5)$$

$$I(z,\Delta\Gamma) = \exp(\Delta\Gamma z)\int_0^z C(u)\exp(-\Delta\Gamma u)\,du \quad (6)$$

$$\begin{aligned}\frac{dR_0}{dz} &= -C_0^2\exp(\Delta\alpha z)\,R_0(0)[\exp\{j(\Delta\beta+\sigma)z\}\\ &\quad\times I(z,-\Delta\Gamma_\sigma)+\exp\{-j\Delta\beta z\}I(z,-\Delta\Gamma^*)]\\ &\quad+C_0^2\exp(\Delta\alpha z)\,R_1(0)[\exp\{j(\Delta\beta+\sigma)z\}\,I(z,\Delta\Gamma^*)\\ &\quad+\exp(-j\Delta\beta z)I(z,\Delta\Gamma_\sigma)] \quad \ldots\ldots \quad (7)\end{aligned}$$

and

$$\begin{aligned}\frac{dR_1}{dz} &= C_0^2\exp(-\Delta\alpha z)\,R_0(0)[\exp\{-j(\Delta\beta+\sigma)z\}\\ &\quad\times I(z,-\Delta\Gamma^*)+\exp(j\Delta\beta z)\,I(z,-\Delta\Gamma_\sigma)]\\ &\quad-C_0^2\exp(-\Delta\alpha z)\,R_1(0)[\exp\{-j(\Delta\beta+\sigma)z\}\,I(z,\Delta\Gamma_\sigma)\\ &\quad+\exp(j\Delta\beta z)I(z,\Delta\Gamma^*)] \quad \ldots\ldots \quad (8)\end{aligned}$$

It is important to note that the derivation of eqns. 7 and 8 involves an independent averaging of $d(z)$ and G_0 and G_1 that is always correct starting from $z = 0$. However, to obtain an iterative formulation, it is necessary to use eqns. 7 and 8 from a z co-ordinate and obtain dR_0/dz and dR_1/dz at $z+\Delta z$. For white spectra, all values of $d(z)$ for different co-ordinates are independent, and so $G_0(\bar{z})$ and $G_1(\bar{z})$ are independent of $d(z)$ in the $\bar{z}<z<\bar{z}+\Delta z$ interval, without limitation of Δz. For nonwhite spectra, a Δz step much greater than the correlation distance of the imperfection must be used in eqns. 7 and 8 if one wants to suppose that the correlation is practically extinguished. A limit of eqns. 7 and 8 for $z \to 0$ is allowed for white spectra, but, for other spectra, a finite-difference formulation derived from eqns. 7 and 8 is allowed. If

$$C(u) = A_0\,\delta(u) \quad \ldots\ldots \quad (9)$$

with A_0 a real constant and $\delta(u)$ the delta function, and, if

$$\tilde{R}_0(z) = R_0(z) \quad \ldots\ldots \quad (10)$$

and

$$\tilde{R}_1(z) = R_1(z)\exp(2\Delta\alpha z)\exp(j\sigma z) \quad \ldots\ldots \quad (11)$$

are defined, taking the limit for $\Delta z \to 0$ of eqns. 7 and 8,

written from z to $z+\Delta z$, the following equations are obtained, corresponding to those already given by Rowe and Young in Reference 3:

$$\left.\begin{aligned}\frac{d\tilde{R}_0}{dz} &= -C_0^2 A_0 \tilde{R}_0(z)+C_0^2 A_0 \tilde{R}_1(z)\\ \frac{d\tilde{R}_1}{dz} &= C_0^2 A_0 \tilde{R}_0(z)+(2\Delta\alpha+j\sigma-C_0^2 A_0)\tilde{R}_1(z)\end{aligned}\right\} \quad (12)$$

If $\sigma = 0$, a power formulation is obtained. For this, the finite-difference equations were derived to allow calculation of the ratio between the reconverted power P_R and the signal power P_s, and also to allow comparison of the results with those obtained by other methods.

Let us define

$$\left.\begin{aligned}P_0 &= \langle I_0 I_0\rangle\\ P_1 &= \langle I_1 I_1^*\rangle\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (13)$$

$$\boldsymbol{P} = \begin{bmatrix}P_0\\ P_1\end{bmatrix} \quad \ldots\ldots\ldots\ldots \quad (14)$$

and choose L as the step of solution. Then

$$\boldsymbol{P}(n+1) = \exp(-2\alpha_0 L)\,\boldsymbol{T}\boldsymbol{P}(n) \quad \ldots\ldots \quad (15)$$

The elements of the $\boldsymbol{T}$ matrix follow from integration of eqns. 7 and 8 for $\sigma = 0$, and the initial conditions are $P_0(0) = 1$ and $P_1(0) = 0$. The general solution of eqn. 15 is analitically available, using z transform or eigenvalue techniques.

Some results are presented, assuming an exponential autocorrelation function that produces a 0·1 dB/km average increase of TE_{01} attenuation constant. For a minimum step L_{min}, the distance of 10 m is fixed. Computations are performed for a waveguide link of 30 km, with $\Delta\alpha = 0$ for no mode

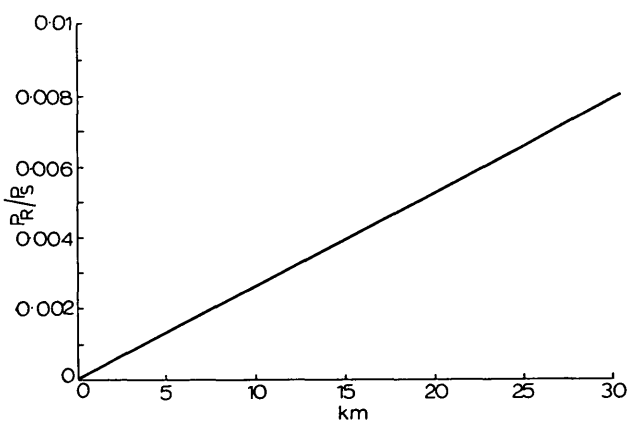

Fig. 2 *Linear increase of P_R/P_s in waveguide link with ideal mode filters*

filters or for ideal mode filters equally spaced. Neglecting the exp $(-2\alpha_0 L)$ decay for $\Delta\alpha = 0$, the solution has two components: the first is a 'steady-state' solution, the second vanishes as n approaches infinity. These results are valid for discrete sections, but are like those obtained in a continuous form for white spectra. Numerical solutions have been computed for variable L, and, before reaching L_{min}, no significant difference has been found by reducing the solution step. It is possible to conclude that the differential equations we obtain as limits of eqns. 7 and 8 are satisfactory in practice, and can be applied to verify the perturbation solution. The case considered is typical of circular-waveguide problems. The results show that the perturbation solution, if correctly interpreted, is well approximated to.

In Fig. 1, P_R and P_R/P_s are shown and compared with the perturbation results, which closely follow P_R/P_s. In Fig. 2, the result of an ideal filter every kilometre is shown; the results corroborate the linear growth of P_R/P_s for every mode-filter section, and the numerical results confirm a reduction whose ratio is the number of mode filters. Eqn. 15 is very general, and can be derived for the general case of the waveguide link with $\Delta\alpha \neq 0$ and nonideal mode filters, as well for the discrete or the continuous case. Numerical results of real situations for the Italian circular-waveguide system will be given later. All the computations, even in the more general case, can be performed on a desk calculator, and the procedure appears to be very handy. The method does not suffer any essential limitations, and its degree of approximation is readily determined in the process of solution by reducing the solution step before reaching the allowed minimum.

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APPLICATION OF DIGITAL LINEARISER WITH CYCLE COUNTER TO THERMOCOUPLES

Indexing terms: Digital instrumentation, Linearisation techniques, Thermocouples, Thermometers

The lineariser corrects the reading of digital thermometer by omitting, in cycles, some pulses entering the counter of the thermometer, and hence decreasing the reading by one digit each time a pulse is omitted. The lengths of cycles in which a pulse is omitted are shorter, as the number accumulated in the counter increases, and correct the non-linearity of the thermocouple.

Because of the nonlinear characteristics of thermocouples, a linearisation of the voltage obtained from a thermocouple is necessary in digital thermometers. There are several types of linearisation, and digital linearising techniques have some advantages over analogue ones, owing to their driftfree operation.

A digital thermometer consists of thermocouple, a conversion amplifier, a digital voltmeter (d.v.m.) and a lineariser. The voltage at the thermocouple is related nonlinearly to the measured temperature. The voltage obtained from the thermocouple is amplified in the conversion amplifier, with properly chosen amplification, giving a converted voltage whose numerical value is equal to the numerical value of the measured temperature. By measuring this converted voltage by a d.v.m., the temperature will be obtained. Owing to the nonlinearity of a thermocouple, the converted voltage will be numerically equal to the measured temperature only for limited range of the temperature, depending on the thermocouple used, the amplification of the conversion amplifier and the permitted error. As an example, for a J-type thermocouple, and an amplification of 19, the converted voltage will be numerically equal to the measured temperature ± 1 deg C within the temperature range 0–120 °C. For higher temperatures, the error will be higher, giving an indication on the d.v.m. of 743 °C when the measured temperature is only 700 °C. To reduce this error, linearisation should be performed.

A digital thermometer employing a digital lineariser with a cycle counter works as shown in Fig. 1. The straight line P represents the indication of temperature with no error, and lines P^+ and P^- are the permitted limits of error. Curve B shows the indication of the thermometer without the lineariser, which, after point Q, is outside the error limits. The lineariser reduces the reading of the thermometer at the point Q to Q',