PERIODICITY AND CHAOS ON A MODIFIED SAMUELSON MODEL

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Abstract

Several discrete time nonlinear growth models with complicated dynamical behavior have been introduced in the literature. In this paper we propuse a modified Samuelson model and we study its dynamical behavior depending on several parameters, which turn out to be the same as the logistic family. Moreover in the base situation the dynamical behavior only depends on the initial values of supply and demand.

Key words. Interval dynamics, attractors, chaos.

1 Introduction

Understanding the dynamic behavior of price and quantity adjustment in individual competitive markets is a natural starting point for the study of economic dynamics. The classical concept of a competitive market is one in which individual firms and households, non of which posses monopoly power, will supply an demand commodities in response to current and expected price according to their individual best interests.

Walras (see [12] p.447) formulated two kinds of price adjustment process: one called producer's tâtonnement, and another referred to as consumer's tâtonnement, which is the form to be studied in this paper. Much later Samuelson gave the model an explicitly mathematical treatment. Samuelson (see [7] p.263) actually formulated the model in continuous time. Advanced analysis of tâtonnement in discrete time will be found in [3], [8], [9] and [10].

Let $S(p)$ and $D(p)$ be the supply and demand for a given commodity where p is the commodity's price. The *excess demand* is defined to be the difference between these two functions $e(p) = D(p) - S(p)$. Samuelson tâtonnement specifies that prices change as a monotonically increasing function of excess demand. In discrete time this means that $p_{t+1} = p_t + g(e(p_t))$, where g is a monotonically increasing function.

We will consider classes of demand functions generated by a shift parameter μ. Denote these functions as $S_\mu(p) = \mu S(p)$ and $D_\mu(p) = \mu D(p)$ where $S(\cdot)$ and $D(\cdot)$ are the original functions. Excess of demand, denoted by $e_\mu(p)$ remain as $e_\mu(p) = \mu e(p)$. We can think of μ as the market "strength" or the "extent of the market" relative to the base situation when $\mu = 1$. To make sure that price is not negative, the price adjustment equation is given by

$$p_{t+1} = f(p_t) = \max\{0, p_t + g(e_\mu(p_t))\} \tag{1.1}$$

The simplest version of this relationship is obtained by assuming that $g(e_\mu(p_t)) = \lambda e_\mu(p_t)$, where λ is a positive constant called the *speed of adjustment*. This case has been studied in [5] as a standard example of the general equilibrium pure exchange economy.

In this paper we study the dynamic of f given in (1.1) in the case in which $g(e_\mu(p_t)) = \lambda p_t e_\mu(p_t)$, where λ is a positive real number. We will formulate some results concerning the regularity (i.e, periodic behavior) in the complicated dynamics and the persistence of this regularity under small smooth perturbations of the system. In particular, f has at most one stable periodic orbit, and whenever f has an attracting periodic orbit $\mathcal{P}$, then the orbits of almost all initial values will converge to $\mathcal{P}$ and the orbit of the critical point will converge to $\mathcal{P}$. Moreover $\mathcal{P}$ is a metric attractor, and a small variation of the parameter values does not change the qualitative behavior of the system.

On the other hand, there exist a positive measure set Λ, such that if the parameters values belong to Λ then the map f has no periodic attractor, f admits an absolutely continuous invariant measure η which describes the asymptotic distribution of almost all orbits under f, and f has positive Lyapunov exponent almost everywhere, in particular at the critical value. Hence f is strongly chaotic with positive probability.

The paper is organized as follows. First we introduce basic mathematical notions and results to make readable the paper. Then, we apply this results to our model, and obtain conclusions.

2 Previous Definitions and Preliminary Results

2.1 Attractors and periodic trajectories

The asymptotic behavior of the orbit of a point p under a continuous interval map f is described by the set of accumulation points, the *ω–limit set* of p, $\omega(p)$, defined by

$$\omega(p) = \omega_f(p) = \{y \in I|\text{ there exist } n_j \to \infty \text{ as } j \to \infty \text{ such that } y = \lim_{j\to\infty} f^{n_j}(p)\}$$

A point p is periodic of period k, if $f^k(p) = p$. The integer k is the period of p if k is the smallest integer with this property. A periodic orbit $\mathcal{P}$ is *attracting* if there is a neighborhood U of $\mathcal{P}$ such that $\omega(y) = \mathcal{P}$ for all $y \in U$. If f is of class $\mathcal{C}^1$ we define the multiplier $\lambda(\mathcal{P}) = (f^p)'(p_0)$, where $p_0 \in \mathcal{P}$. We then classify periodic orbits as follows

- $\mathcal{P}$ is *superstable* if $\lambda(\mathcal{P}) = 0$. This is equivalent to the condition that $\mathcal{P}$ contains a critical point.
- $\mathcal{P}$ is *stable* if $0 < |\lambda(\mathcal{P})| < 1$
- $\mathcal{P}$ is *neutral* if $|\lambda(\mathcal{P})| = 1$
- $\mathcal{P}$ is *unstable* if $|\lambda(\mathcal{P})| > 1$

It is clear that stable and superstable periodic orbits are attracting. A neutral periodic orbit may or not may be attracting.

A set Γ is called forward invariant if $f(\Gamma) = \Gamma$. Let $B(\Gamma)$ denote the *basin of attraction* of a forward invariant set, that is

$$B(\Gamma) = \{p|\, \omega(p) \subset \Gamma\}$$

Definition 2.1 *A forward invariant set Ω is called a metric attractor if $B(\Omega)$ satisfies*

(1) $B(\Omega)$ has positive Lebesgue measure;

(2) If Ω' is another forward invariant set, strictly contained in Ω, then $B(\Omega) \setminus B(\Omega')$ has positive measure.

An attracting fixed point or an attracting periodic cycle are of course metric attractors.

Definition 2.2 *A continuous interval map $f : I = [a, b] \to I$ is unimodal if there is a unique maximum c in the interior of I such that f is strictly increasing in $[a, c)$ and strictly decreasing in $(c, b]$.*

Let c be a critical point. If $f''(c) = 0$ we say that the critical point c is degenerate. If $f^{(n)}(c) = 0$ for all $n > 0$, we say that c is a flat critical point.

Definition 2.3 *An interval map $f : I \to I$ has negative Schwarzian derivative if f is of class $\mathcal{C}^3$ and*

$$Sf(p) := \frac{f'''(p)}{f'(p)} - \frac{3}{2}\left(\frac{f''(p)}{f'(p)}\right)^2 < 0$$

for all $p \in I \setminus \{c \in I | f'(c) = 0\}$. A unimodal map with negative Schwarzian derivative will be referred as an S–unimodal map.

For unimodal maps with negative Schwarzian derivative, the following theorem shows how the metric attractors are.

Theorem 2.4 ([1]) *Let $f : I \to I$ be a S–unimodal map with nonflat critical points. Then f has a unique metric attractor Ω . The attractor Ω is of one of the following types:*

(1) an attracting periodic orbit;

(2) a Cantor set of measure zero:

(3) a finite union of intervals with a dense orbit.

In the first two cases, $\Omega = \omega(c)$

For S–unimodal maps with nonflat critical point, Theorem 2.4 gives three possible different types of asymptotic behavior. A fundamental problem is the following: Given a family of unimodal maps depending on one or several parameters, characterize the set of parameters for which the attractor of maps is an attracting periodic orbit or an Cantor set of measure zero or a finite union of intervals. From the point of view of topology we have the following result [4].

Theorem 2.5 *If $f_{\lambda_1,\ldots,\lambda_l}$ is an l–parameter family of real analytic S–unimodal maps of an interval, depending in a real analytic fashion on the parameter(s), such that the family contains at least one map with a stable periodic attractor and at least two non conjugated maps, then maps with a stable periodic attractor are dense in the set of parameters $\{(\lambda_1, \ldots, \lambda_l)\}$*

The above result says that from the point of view of topology periodic (or simple) maps are normal. From the point of view of probability, chaotic maps are more usual. We precise this in the next subsection.

2.2 On definition of chaos. Natural measures

There are several definitions of chaos in the setting of discrete dynamical systems which are not equivalent. In general, for unimodal maps one can think on chaos as the opposite of a periodic map (see Theorem 2.4). Keeping this idea in mind, we can talk about chaotic maps. To precise this idea we need the following notation.

A Borel Measure η is invariant for $f : I \to I$ if $\eta(f^{-1}(E)) = \eta(E)$ for every measurable set $E \subseteq I$. We look for invariant measures that describe the asymptotic distribution under iteration for a large set of initial points.

Definition 2.6 *An invariant measure η is called a* natural measure for f *if*

$$\eta = \lim_{n\to\infty} \sum_{k=0}^{n-1} \delta_{f^k(x)}$$

for all x in a set of positive Lebesgue measure. Here δ_x denotes the Dirac mass in x, and the limit is in the weak sense[1]*.*

One could think in natural measures as absorbing stationary distributions. We will also be interested in the case when the natural measure has a density, which we interpret as a sing of chaos. The following abbreviations will be used:

- *acim* stands for *absolutely continuous* (with respect to Lebesgue measure) *invariant measure.*
- *acip* stands for *absolutely continuous invariant probability measure*, a finite and normalized *acim.*

In [6], Guckenheimer made the following definition:

Definition 2.7 *An interval map f has sensitive dependence on initial conditions if there exist a set K of positive Lebesgue measure and $\delta > 0$ such that for every $x \in K$ and every interval neighborhood J of x there is an n such that $f^n(J)$ has length larger than δ.*

This means that with positive probability we find points with arbitrarily small neighborhood which sooner or later expand to macroscopic scale. If f is as in Theorem 2.4 Guckenheimer proved that

- if f has a periodic attractor, then f does not have sensitive dependence on initial condictions.
- if f has an interval attractor, then f has sensitive dependence on initial condictions.

So for S–unimodal maps with nonflat critical points, sensitive dependence on initial conditions is equivalent to the presence of an interval attractor.

[1]Recall that η is the weak limit of η_n if $\int_I f d\eta = \lim_{n\to\infty} \int_I f d\eta_n$ for any function f continuous on I

There are several definitions to say that a map f is chaotic, where sensitive dependence on initial conditions is one of the weakest. Even weaker is the type of chaos implied by the existence of a 3–cycle [2]. Such a chaos could coexist with a stable periodic attractor whose basin of attraction has full measure. Other possibility is to say that *f is chaotic if f admits an acip*. In the case of S–unimodal maps with nonflat critical points, almost all orbits distribute themselves according to this measure over entire interval. In fact it is equivalent to having a positive Lyapunov exponent almost everywhere.

The following discussion is about the well–known logistic map $g_\alpha(x) = \alpha x(1-x)$, $x \in [0,1]$, $\alpha \in [0,4]$. This is a one parameter family which has been extensively studied from the point of view of discrete dynamical systems. In [11] the following result can be found.

(P1) $\mathcal{P} := \{\alpha \mid \Omega \text{ is a periodic cycle}\}$ is dense in the parameter space and consist of countably infinitely many nontrivial intervals. Moving the parameter inside one connected component of $\mathcal{P}$ we see the period-doubling scenario, with universal scaling in parameter space.

(P2) $\mathcal{C} := \{\alpha \mid \Omega \text{ is a Cantor set}\}$ is a completely disconnected set of Lebesgue measure zero.

(P3) $\mathcal{I} := \{\alpha \mid \Omega \text{ is a union of intervals}\}$ is a completely disconnected set of positive Lebesgue measure.

3 The model

We assume affine demand and supply functions,

$$\begin{cases} D_\mu(p) := \mu(a-bp) \\ S_\mu(p) := \mu(c+dp) \end{cases}$$

where b, c and d are positive real numbers, and we assume that the demand is greater than supply at zero price ($a > c$). Either of these functions could become negative. The excess of demand is then given by

$$e_\mu(p) := \mu(a-c-(b+d)p)$$

Then the price adjustment process (1.1) remains:

$$p_{t+1} = f_{\lambda,\mu,a,b,c,d}(p_t) = \max\{0, p_t + g(e_\mu(p_t))\} = \max\{0, p_t + \lambda\mu p_t(a-c-(b+d)p_t)\} \tag{3.2}$$

where we have taken $g(e(p)) = \lambda p e(p)$ and λ is a positive real number called the speed of adjustment, that is, the prices change proportionally to the excess of demand, being this proportion λp.

Therefore

$$f_{\lambda,\mu,a,b,c,d}(p) = \begin{cases} p + \lambda\mu p(a-c-(b+d)p) & \text{if } p \in [0, \frac{1+(a-c)\mu\lambda}{\mu\lambda(b+d)}] := I_{\lambda,\mu,a,b,c,d} \\ 0 & \text{Otherwise} \end{cases} \tag{3.3}$$

Then $f_{\lambda,\mu,a,b,c,d}$ has an only maximum $c = \frac{1+(a-c)\mu\lambda}{2\mu\lambda(b+d)}$ in the interior of $I_{\lambda,\mu,a,b,c,d}$. The interval $I_{\lambda,\mu,a,b,c,d}$ will be $f_{\lambda,\mu,a,b,c,d}$–invariant iff $f_{\lambda,\mu,a,b,c,d}(c) \leq \frac{1+(a-c)\mu\lambda}{\mu\lambda(b+d)}$, which holds if and only if $\lambda \leq \frac{3}{(a-c)\mu}$. So, from now on, we assume that $\lambda \leq \frac{3}{(a-c)\mu}$ and therefore $I_{\lambda,\mu,a,b,c,d}$ is a $f_{\lambda,\mu,a,b,c,d}$–invariant compact interval.

As the interval $I_{\lambda,\mu,a,b,c,d}$ on which the map $f_{\lambda,\mu,a,b,c,d}$ is defined depends on λ, μ, a, b, c and d and in order to make this map to be defined in a fixed interval not depending on any parameter we are going to conjugate $f_{\lambda,\mu,a,b,c,d}$ by the homeomorphism $\varphi : I_{\lambda,\mu,a,b,c,d} \to [0,1]$ given by $\varphi(p) = \frac{\mu\lambda(b+d)}{1+(a-c)\mu\lambda} p$, obtaining the map $\widetilde{f}_{\lambda,\mu,a,c} = \varphi \circ f_{\lambda,\mu,a,b,c,d} \circ \varphi^{-1} : [0,1] \to [0,1]$ given by

$$\widetilde{f}_{\lambda,\mu,a,c}(p) = (1 + \lambda\mu(a-c))p(1-p). \tag{3.4}$$

We shall call the map $\widetilde{f}_{\lambda,\mu,a,c}$, ($\widetilde{f}$ for short), defined by (3.4) a Samuelson map.

Remark 1. As $\widetilde{f}$ is the conjugate of $f_{\lambda,\mu,a,b,c,d}$ by the homeomorphism φ, the dynamic behavior of the map $f_{\lambda,\mu,a,b,c,d}$ is the same as the dynamic behavior of $\widetilde{f}$.

Remark 2. Notice that while the map $f_{\lambda,\mu,a,b,c,d}$ depends on six parameters the conjugated Samuelson map $\widetilde{f}$ depends only on four parameters: λ, μ, a, c. That is, the dynamic behavior of $\widetilde{f}$ do not depends on the speed in which the demand decreases b neither the speed in which the supply increases d.

Remark 3. Notice that the Samuelson map has the form of the well known logistic family $g_\alpha(p) = \alpha p(1-p)$ with $0 < \alpha \leq 4$ for $\alpha = 1 + \lambda\mu(a-c)$.

Lemma 3.1 *Let $\widetilde{f} : [0,1] \to [0,1]$ be a Samuelson map. Then $\widetilde{f}$ is S–unimodal.*

Proof. By the above paragraphs, $\widetilde{f}$ is unimodal. To see that $\widetilde{f}$ has negative Schwarzian derivative, it is enough to notice that $\widetilde{f}'''(p) = 0$ for all $p \in [0,1] \setminus \{c \in [0,1] | \widetilde{f}'(c) = 0\}$. ∎

As a consequence of Lemma 3.1 and Theorem 2.4 we obtain our first main result for Samuelson maps.

Theorem 3.2 *Each Samuelson map, $\widetilde{f}$, has a unique metric attractor Ω attracting almost all initial conditions. The attractor Ω is a periodic cycle, an attracting Cantor set or a finite union of interval with a dense orbit.*

The fixed points of $\widetilde{f}$ are $p = 0$ and $p = \frac{\lambda\mu(a-c)}{1+\lambda\mu(a-c)}$. The derivative of $\widetilde{f}$ is $\widetilde{f}'(p) = (1 + \lambda\mu(a - c))(1-2p)$. Then $|\widetilde{f}'(0)| = 1 + \lambda\mu(a-c) > 1$, and hence $p = 0$ is a repulsive fixed point for any value of the parameters λ, μ, a and c. On the other hand, $|\widetilde{f}'(\frac{\lambda\mu(a-c)}{1+\lambda\mu(a-c)})| = |1 - \lambda\mu(a-c)|$. Then the fixed point $p = \frac{\lambda\mu(a-c)}{1+\lambda\mu(a-c)}$ is attractive, that is, $|\widetilde{f}'(\frac{\lambda\mu(a-c)}{1+\lambda\mu(a-c)})| < 1$, if and only if $\lambda\mu(a-c) < 2$.

There also exist a 2–cycle given by:

$$\mathcal{P} = \{p_1 = \frac{2 + \lambda\mu(a-c) + \sqrt{-4 + \lambda^2\mu^2(a-c)^2}}{2(1 + \lambda\mu(a-c))},\ p_2 = \frac{2 + \lambda\mu(a-c) - \sqrt{-4 + \lambda^2\mu^2(a-c)^2}}{2(1 + \lambda\mu(a-c))}\}$$

if and only if $\lambda\mu(a-c) > 2$, because otherwise $\sqrt{-4+\lambda^2\mu^2(a-c)^2}$ is not a real number and therefore p_1 and p_2 would not be real numbers.

Besides the periodic orbit $\mathcal{P}$ is stable, that is, $|(\widetilde{f}^2_{\lambda,\mu})'(p_1)| < 1$, if and only if $2 < \lambda\mu(a-c) < \sqrt{6}$.

So we have shown that if $\lambda\mu(a-c) \in (0,2)$ there exist an attracting fixed point and if $\lambda\mu(a-c) \in (2,\sqrt{6})$ there is a stable 2–cycle. So we remain to study when $\lambda\mu(a-c) \in (\sqrt{6},3)$. As a consequence of Theorem 2.5 we immediately obtain the following theorem, which extends the result (P1) to our four–parameters family.

Theorem 3.3 *Maps with a stable periodic attractor form a dense subset:*

(1) in (λ,μ,a,c)–space for the Samuelson family $\{\widetilde{f}\}_{(\lambda,\mu,a,c)}$;

(2) in λ–space for the family $\{\widetilde{f}\}_{\lambda}$ obtained when $\mu=\mu_0, a=a_0$ and $c=c_0$ are fixed;

(3) in μ–space for the family $\{\widetilde{f}\}_{\mu}$ obtained when $\lambda=\lambda_0, a=a_0$ and $c=c_0$ are fixed;

(4) in a–space for the family $\{\widetilde{f}\}_{a}$ obtained when $\lambda=\lambda_0, \mu=\mu_0$ and $c=c_0$ are fixed, with $a > c_0$;

(5) in c–space for the family $\{\widetilde{f}\}_{c}$ obtained when $\lambda=\lambda_0, \mu=\mu_0$ and $a=a_0$ are fixed, with $a_0 > c$.

Proof. All we have to do to apply Theorem 2.5 is to find a map with a stable periodic orbit and two non conjugated maps within the family. This fact is easily proved in all the cases since the interior fixed point start out as a stable fixed point and losses its stability when a 2–cycle is born. ∎

From Theorem 3.3 we have that maps with periodic attractors, and hence no sensitive dependence on initial conditions, are predominant from a topological point of view. On the other hand, next we prove a result which allows us to state that chaotic maps are common in the sense of Lebesgue measure. It can be easily proved from (P3) and [11].

Theorem 3.4 *The Samuelson family $\widetilde{f}_{\lambda,\mu,a,c}$ is strongly chaotic with positive probability. There exists a nonempty set $\mathcal{M}$ such that for each $\mu,a,c \in \mathcal{M}$ $(a > c)$, there exists a positive measure set $\Lambda_{\mu,a,c}$ such that if $\lambda \in \Lambda_{\mu,a,c}$ then:*

(1) $\widetilde{f}$ has no periodic attractor and the unique metric attractor is a transitive interval attractor.

(2) $\widetilde{f}$ admits an absolutely continuous invariant probability measure η_λ with the following properties:

(a) η_λ describes the asymptotic distribution of almost all orbits under $\widetilde{f}$.

(b) η_λ has density in L^1.

(3) $\widetilde{f}$ has positive Lyapunov exponent almost everywhere, in particular at the critical value.

4 Conclusions

We have presented some results concerning the regularity in the complicated dynamics in discrete time processes, and the persistence of this regularity under small smooth perturbations. In particular if the map $\widetilde{f}$ has an asymptotically stable periodic orbit $\mathcal{P}$, then almost all the orbits will converge to $\mathcal{P}$. The opposite situation, which is called as chaos, is also discused in our model.

Moreover, the existence of an asymptotically stable periodic orbit also implies that the qualitative behavior of the dynamics does not change when the parameter are varied slightly. Therefore if one can determine the parameters and the initial value sufficiently accurately, and if one can show that the system has an asymptotically stable periodic orbit, then long term prediction is possible.

Since we also expect asymptotic distributions (natural measures) as a function of the parameter to behave in a singular way close to chaotic maps, we could have difficulties when using the Samuelson map $\widetilde{f}$ to model real life. If our estimated parameters put us close to the set of strongly chaotic maps, the asymptotic dynamics will behave in a extremely sensitive way on the parameter, making even statistical predictions of the long time behavior impossible. This should not be viewed as a weakness of the model, it is the way these systems, and may be nature herself, behave.

Notice that our model depends on six parameters, but two of them "dissapears" when we conjugate with the logistic family. Moreover, when b and d are zero, the model is linear. This corresponds to a model where the demand and the supply do not change. Since the model is linear, the dynamical behaviour in this case is very simple. However, even when b and d are very close to zero, one can find chaotic maps. Even more, these chaotic maps may be supported on a big compact interval (notice that the interval is $[0, \frac{1+(a-c)\mu\lambda}{\mu\lambda(d+b)}]$ and then, when b and d goes to zero, the right endpoint of the interval tends to infinite).

Finally, notice that in the base situation $\mu = 1$ and $\lambda = 1$, the model gives us a two–parameter family, depending on a and c, that is, the initial supply and demand. The difference between these quantities will gives us a one–parameter family of logistic maps.

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