ANTISYMMETRIC ELEMENTS IN GROUP RINGS II

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Abstract

Let R be a commutative ring, G a group and RG its group ring. Let $\varphi : RG \to RG$ denote the R-linear extension of an involution φ defined on G. An element x in RG is said to be φ-antisymmetric if $\varphi(x) = -x$. A characterization is given of when the φ-antisymmetric elements of RG commute. This is a completion of earlier work.

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1 Introduction.

Throughout this paper R is a commutative ring with identity, G is a group and φ is an involution on G. Clearly φ can be extended linearly to an involution $\varphi : RG \to RG$ of the group ring RG. Set $R_2 = \{r \in R \mid 2r = 0\}$. We denote by $(RG)^-_\varphi$ the Lie algebra consisting of the φ-antisymmetric elements of RG, that is

$$(RG)^-_\varphi = \{\alpha \in RG | \ \varphi(\alpha) = -\alpha\}.$$

For general algebras A with an involution φ, we recall some important results that show that crucial information of the algebraic structure of A can be determined by that of $(A)^-_\varphi$ and the latter has information that is determined by the φ-unitary unit group $U_\varphi(A) = \{u \in A \mid u\varphi(u) = \varphi(u)u = 1\}$. By $U(A)$ we denote the unit group of A. Amitsur in [1] proves that if A^-_φ satisfies a polynomial identity (in particular when A^-_φ is commutative) then A satisfies a polynomial identity. Gupta and Levin in [10] proved that for all $n \geq 1$ $\gamma_n(\mathcal{U}(A)) \leq 1 + L_n(A)$. Here $\gamma_n(G)$ denotes the nth term in the lower central series of the group G and $L_n(A)$ denotes the two sided ideal of A generated by all Lie elements of the form $[a_1, a_2, \ldots, a_n]$ with $a_i \in A$ and $[a_1] = a_1$, $[a_1, a_2] = a_1a_2 - a_2a_1$ and inductively $[a_1, a_2, \ldots, a_n] = [[a_1, a_2, \ldots, a_{n-1}], a_n]$. Smirnov and Zalesskii in [14], proved that, for example, if the Lie ring generated by the elements of the form $g + g^{-1}$ with $g \in \mathcal{U}(A)$ is Lie nilpotent then A is Lie nilpotent. In [5] Giambruno and Polcino Milies show that if A is a finite dimensional semisimple algebra over an algebraically closed field F with $char(F) \neq 2$ then $\mathcal{U}_\varphi(A)$ satisfies a group identity if and only if $(A)^-_\varphi$ is commutative. Furthermore, if F is a nonabsolute field then $\mathcal{U}_\varphi(A)$ does not contain a free group of rank 2 if and only if $(A)^-_\varphi$ is commutative. Giambruno and Sehgal, in [6], showed that if B is a semiprime ring with involution φ, $B = 2B$ and $(B)^-_\varphi$ is Lie nilpotent then $(B)^-_\varphi$ is commutative and B satisfies a polynomial identity of degree 4.

Special attention has been given to the classical involution $*$ on RG, that is, the R-linear map defined by mapping $g \in G$ onto g^{-1}. In case R is a field of characteristic 0 and G is a periodic group, Giambruno and Polcino Milies in [5] described when $\mathcal{U}_*(RG)$ satisfies a group identity. Gonçalves and Passman in [8] characterized when $\mathcal{U}_*(RG)$ does not contain non abelian free groups when G is a finite group and R is a nonabsolute field. Giambruno and Sehgal, in [7], characterized when $(RG)^-_*$ is Lie nilpotent provided R is a field of characteristic $p \geq 0$, with $p \neq 2$.

Motivated by all these connections, in this paper we deal with the question of when $(RG)^-_\varphi$ is commutative for an arbitrary involution φ on G. Let $G_\varphi = \{g \in G \mid \varphi(g) = g\}$ be the subset of φ-symmetric elements of G, i.e. the set of elements of G fixed by φ. The following complete answer is obtained.

Theorem 1.1 *Let R be a commutative ring. Suppose G is a non-abelian group and φ is an involution on G. Then, $(RG)^-_\varphi$ is commutative if and only if one of the following conditions holds:*

1. *$K = \langle g \in G \mid g \notin G_\varphi \rangle$ is abelian (and thus $G = K \cup Kx$, where $x \in G_\varphi$, and $\varphi(k) = xkx^{-1}$ for all $k \in K$) and $R_2^2 = \{0\}$.*
2. *$R_2 = \{0\}$ and G contains an abelian subgroup of index 2 that is contained in G_φ.*
3. *$\operatorname{char}(R) = 4$, $|G'| = 2$, $G/G' = (G/G')_\varphi$, $g^2 \in G_\varphi$ for all $g \in G$, and G_φ is commutative in case $R_2^2 \neq \{0\}$.*
4. *$\operatorname{char}(R) = 3$, $|G'| = 3$, $G/G' = (G/G')_\varphi$ and $g^3 \in G_\varphi$ for all $g \in G$.*

Clearly, as an R-module, $(RG)^-_\varphi$ is generated by the set

$$\mathcal{S} \;=\; \{g - \varphi(g) \mid \; g \in G \setminus G_\varphi\} \cup \{rg \mid \; g \in G_\varphi,\ r \in R_2\}$$

Therefore $(RG)^-_\varphi$ is commutative if and only if the elements in $\mathcal{S}$ commute.

This work is a continuation of the work started in [4] (for the classical involution), [11] and [3]. In the latter one considers the involutions η on RG introduced by Novikov in [13]: $\eta(\sum_{g\in G} \alpha_g g) = \sum_{g\in G} \alpha_g \sigma(g) g^{-1}$, where $\sigma : G \to \{\pm 1\}$ is a group homomorphism. Unfortunately, in [4, 11] the set $\mathcal{S}_1 = \{rg \mid r \in R_2,\ g \in G_\varphi\}$ was not included in the set $\mathcal{S}$. Therefore, the results given in [4, 11] only deal with commuting of elements in the set $\mathcal{S} \setminus \mathcal{S}_1$. Hence, provided $R_2 = \{0\}$, there is a complete characterization of when $(RG)^-_\varphi$ is commutative in [11] when $\operatorname{char}(R) \neq 2, 3$ and in [4] when φ is the classical involution and $\operatorname{char}(R) \neq 2$. The case $\operatorname{char}(R) = 3$ was left as an open problem in [11], and the case $\operatorname{char}(R) = 2$ has been dealt with in [2, 12] because then $(RG)^-_\varphi$ coincides with the set of φ-symmetric elements of RG.

So, throughout the paper we assume $\operatorname{char}(R) \neq 2$. The center of G is denoted by $Z(G)$, the additive commutator $\alpha\beta - \beta\alpha$ of $\alpha, \beta \in RG$ is denoted $[\alpha, \beta]$, and the multiplicative commutator $ghg^{-1}h^{-1}$ of $g, h \in G$ is denoted by (g, h).

As mentioned above, the Theorem has been proved in [11] provided $R_2 = \{0\}$ and $\operatorname{char}(R) \neq 3$. Theorem 2.5 shows the result holds in case $R_2 \neq 0$ and Theorem 3.8 shows that it also holds if $\operatorname{char}(R) = 3$.

2 Rings with elements of additive order 2

We begin with recalling some technical results from [11]. The first lemma shows that the group generated by the non-fixed elements has index at most 2.

Lemma 2.1 [11, Lemma 2.3] *If φ is non-trivial then the subgroup $K = \langle g \in G \mid g \notin G_\varphi \rangle$ has index at most 2 in G.*

Lemma 2.2 [11, Lemma 1.1 and Lemma 1.2] *Let R be a commutative ring with* $\operatorname{char}(R) \neq 2, 3$. *Let $g, h \in G \setminus G_\varphi$ be two non-commuting elements. If $(RG)^-_\varphi$ is commutative then one of the following conditions holds*

1. $gh \in G_\varphi$, $hg \in G_\varphi$, $h\varphi(g) = g\varphi(h)$ *and* $\varphi(h)g = \varphi(g)h$.

2. $gh = h\varphi(g) = \varphi(h)g = \varphi(g)\varphi(h)$ *and* $\mathrm{char}(R) = 4$.

Note that, if non-commuting elements $g, h \in G \setminus G_\varphi$ satisfy condition (2) in the lemma then $h^{-1}gh = \varphi(g)$. Non-commutative groups G with an involution φ such that $h^{-1}gh \in \{g,\ \varphi(g)\}$ for all $g, h \in G$ have been described in [9, Theorem III.3.3]. These are precisely the groups G with a unique non-trivial commutator and that satisfy the *lack of commutativity property* ("LC" for short). The latter means that for any pair of elements $g, h \in G$ it is the case that $gh = hg$ if and only if either $g \in Z(G)$ or $h \in Z(G)$ or $gh \in Z(G)$. It turns out [9, Proposition III.3.6] that such groups are precisely those non-commutative groups G with $G/Z(G) \cong C_2 \times C_2$, where C_2 denotes the cyclic group of order 2.

In the next lemma we give the structure of the group generated by two elements $g, h \in G \setminus G_\varphi$ satisfying (2) of Lemma 2.2.

Lemma 2.3 [11, Lemma 3.1] *Let R be a commutative ring with* $\mathrm{char}(R) = 4$. *Suppose $g, h \in G \setminus G_\varphi$ are non-commuting elements that satisfy (2) of Lemma 2.2. If $(RG)^-_\varphi$ is commutative then the group $H = \langle g, h, \varphi(g), \varphi(h)\rangle = \langle g, h\rangle$ satisfies the LC–property and has a unique non-trivial commutator s and the involution restricted to H is given by $\varphi(h) = sh$ if $h \in H \setminus Z(H)$ and $\varphi(h) = h$ if $h \in Z(H)$.*

From the next lemma it follows that if $R_2 \neq \{0\}$ and $\mathrm{char}(R) \neq 4$ then any two elements of $G \setminus G_\varphi$ that satisfy condition (1) of Lemma 2.2 must commute.

Lemma 2.4 *Assume $R_2 \neq \{0\}$ and $(RG)^-_\varphi$ is commutative. Let $g, h \in G$ and suppose $(g, h) \neq 1$.*

1. *If $g \in G_\varphi$ and $h \notin G_\varphi$ then $gh = \varphi(h)g$ and $hg = g\varphi(h)$.*

2. *If $g, h \in G \setminus G_\varphi$ then $gh \notin G_\varphi$ (in particular, g and h do not satisfy condition (1) of Lemma 2.2).*

3. *If $g, h \in G \setminus G_\varphi$ then* $\mathrm{char}(R) = 4$, *$\langle g, h\rangle$ is LC with a unique non-trivial commutator and $\varphi(g) = (g, h)g$ and $\varphi(h) = (g, h)h$. In particular, if* $\mathrm{char}(R) \neq 4$ *then $K = \langle g \in G \mid g \notin G_\varphi\rangle$ is abelian.*

Proof. Let $0 \neq r \in R_2$.

(1) Since $(RG)^-_\varphi$ is commutative, we have that $0 = [rg, h - \varphi(h)] = r(gh + g\varphi(h) + hg + \varphi(h)g)$. As $(g, h) \neq 1$ and $h \notin G_\varphi$, it follows that $gh = \varphi(h)g$ and $hg = g\varphi(h)$.

(2) Suppose $g, h \in G \setminus G_\varphi$. Assume $gh \in G_\varphi$. Then, by (1), $\varphi(h)\varphi(g)h = ghh = \varphi(h)gh$ and thus $g \in G_\varphi$, a contradiction.

(3) This follows at once from Lemma 2.2, (2) and Lemma 2.3 . ∎

We now give a complete characterization of when $(RG)^-_\varphi$ is commutative provided $R_2 \neq \{0\}$ (and thus $\mathrm{char}(R) \neq 3$).

Theorem 2.5 *Let R be a commutative ring with elements of additive order* 2. *Assume G is a non-abelian group and φ is an involution on G. Then, $(RG)^-_\varphi$ is commutative if and only if one of the following conditions holds:*

(a) *$K = \langle g \in G \mid g \notin G_\varphi\rangle$ is abelian (and thus $G = K \cup Kx$, where $x \in G_\varphi$, and $\varphi(k) = xkx^{-1}$ for all $k \in K$), and $R_2^2 = \{0\}$.*

(b) $\mathrm{char}(R) = 4$, $|G'| = 2$, $G/G' = (G/G')_\varphi$, $g^2 \in G_\varphi$ *for all* $g \in G$, *and* G_φ *is commutative in case* $R_2^2 \neq \{0\}$.

Proof. Let G be a non-abelian group and φ an involution on G. Assume $(RG)^-_\varphi$ is commutative. Notice that Lemma 2.1 implies that if $K = \langle g \in G |\ g \notin G_\varphi \rangle$ is abelian (and thus $K \neq G$) then $G = K \cup Kx$ for some $x \in G_\varphi$. Furthermore, one gets that $\varphi(k) = xkx^{-1}$ for all $k \in K$. Indeed, since $x \notin K$ it follows that $xk \notin K$ and hence $xk = \varphi(xk) = \varphi(k)x$ and therefore $\varphi(k) = xkx^{-1}$. Also, since x is not central, we get that $xk \neq kx$ for some $k \in K$. Now, for any $r_1, r_2 \in R_2$ we have that $x, kx \in G_\varphi$ and thus, by assumption, $r_1r_2(xkx - kx^2) = [r_1x, r_2kx] = 0$. Since $xkx \neq kx^2$, it follows that $r_1r_2 = 0$. Consequently, $R_2^2 = \{0\}$. So, condition (a) follows.

If $\mathrm{char}(R) \neq 4$ then it follows from Lemma 2.4.(3) that K is abelian. Hence, by the above, condition (a) follows.

So, to prove the necessity of the mentioned conditions, we are left to deal with the case that $\mathrm{char}(R) = 4$ and K is not abelian. We need to prove that condition (b) holds. Because of Lemma 2.4 (3), we also know that $H = \langle x, y \rangle$ is LC with a unique non-trivial commutator and $\varphi(h) = (x, y)h$ if $h \in H \setminus Z(H)$ and $\varphi(h) = h$ if $h \in Z(H)$.

Now we claim that for all $g \in G \setminus G_\varphi$ we have that $g^2 \in G_\varphi$ and $g^{-1}\varphi(g) = (x, y)$ (in particular, $G/G' = (G/G')_\varphi$). Indeed, let $g \in G \setminus G_\varphi$. If $(g, x) \neq 1$ then by Lemma 2.4 (3) $g^2 \in G_\varphi$ and $g^{-1}\varphi(g) = x^{-1}\varphi(x) = (x, y)$. Similarly, if $(g, y) \neq 1$ then $g^{-1}\varphi(g) = y^{-1}\varphi(y) = (x, y)$. Assume now that $(g, x) = (g, y) = 1$. If $gx \in G_\varphi$ then $g^2x^2 = (gx)^2 = \varphi((gx)^2) = \varphi(g^2x^2) = \varphi(g^2)x^2$ and hence $g^2 \in G_\varphi$. Moreover, in this case, $gx = \varphi(g)\varphi(x)$ and hence $g^{-1}\varphi(g) = x\varphi(x^{-1}) = x^{-1}\varphi(x) = (x, y)$ as desired. If $gx \notin G_\varphi$ then, by Lemma 2.4 (3) and since $(gx, y) = (x, y) \neq 1$, we get that $(gx)^{-1}\varphi(gx) = (gx, y) = (x, y) = x^{-1}\varphi(x)$. Hence, $x^{-1}g^{-1}\varphi(x)\varphi(g) = x^{-1}\varphi(x)$ and thus $g^{-1}\varphi(g)\varphi(x) = g^{-1}\varphi(x)\varphi(g) = \varphi(x)$. So, $g = \varphi(g)$, a contradiction. This finishes the proof of the claim.

Next we show that $G' = \langle (x, y) \rangle = \{1, (x, y)\}$ (and thus $G' \subseteq G_\varphi$). Indeed, let $g, h \in G$ such that $(g, h) \neq 1$. If $g, h \notin G_\varphi$ then by the previous claim and Lemma 2.4 (3) it follows that $(g, h) = g^{-1}\varphi(g) = x^{-1}\varphi(x) = (x, y)$, as desired. If $g \in G_\varphi$ and $h \notin G_\varphi$ then, by Lemma 2.4 (1), $gh = \varphi(h)g$ and hence by the previous claim we get that $(x, y) = \varphi(x)x^{-1} = \varphi(h)h^{-1} = ghg^{-1}h^{-1} = (g, h)$. Finally if $g, h \in G_\varphi$ then $hg \notin G_\varphi$ (because otherwise $(g, h) = 1$), and hence by the previous claim $(x, y) = \varphi(x)x^{-1} = \varphi((hg))(hg)^{-1} = ghg^{-1}h^{-1} = (g, h)$, as desired.

To finish the prove of the necessity, we remark that if $R_2^2 \neq \{0\}$ then G_φ is commutative. Indeed, let $r_1, r_2 \in R_2$ be so that $r_1r_2 \neq 0$ and let $g_1, g_2 \in G_\varphi$. Since $(RG)^-_\varphi$ is commutative, we have that $r_1r_2(g_1g_2 - g_2g_1) = [r_1g_1, r_2g_2] = 0$. Hence $(g_1, g_2) = 1$.

In order to prove the sufficiency we need to show that the elements in

$$\mathcal{S} = \{g - \varphi(g) \mid g \in G,\ g \notin G_\varphi\ \} \cup \{rg \mid\ g \in G_\varphi,\ r \in R_2\}$$

commute.

First assume G satisfies condition (a). So $G = K \cup Kx$ with $x \in G_\varphi$ and K abelian. We need to show that $[g - \varphi(g), r_1h_1] = 0$ and $[r_1h_1, r_2h_2] = 0$ for $g \in G \setminus G_\varphi$, $h_1, h_2 \in G_\varphi$ and $r_1, r_2 \in R_2$ with $(g, h_1) \neq 1$ and $(h_1, h_2) \neq 1$. The later equality is obviously satisfied because of the assumptions. To prove the former equality, we note that, by Lemma 2.4 (1), $h_1g = \varphi(g)h_1$ and $gh_1 = h_1\varphi(g)$. Hence,

$$\begin{aligned}[][g - \varphi(g), r_1h_1] &= r_1gh_1 - r_1\varphi(g)h_1 - r_1h_1g + r_1h_1\varphi(g) \\ &= r_1gh_1 + r_1h_1g + r_1h_1g + r_1gh_1 \\ &= 2r_1gh_1 + 2r_1h_1g \\ &= 0,\end{aligned}$$

as desired.

Second, assume G satisfies (b) and that $\text{char}(R) = 4$. Notice that in this case if $g \notin G_\varphi$ then $g^{-1}\varphi(g) = g\varphi(g^{-1})$ is central and equal to the unique commutator of G. Let $g, h \in G$ with $(g,h) \neq 1$ and let $r_1, r_2 \in R_2$. If $g, h \in G_\varphi$, then the assumptions imply that $R_2^2 = \{0\}$ and thus $[r_1 g, r_2 h] = 0$, as desired. If $g \notin G_\varphi$ and $h \in G_\varphi$ then

$$\begin{aligned}
[g - \varphi(g), r_1 h] &= r_1 gh - r_1\varphi(g)h - r_1 hg + r_1 h\varphi(g) \\
&= r_1 gh + r_1 hg\varphi(g^{-1})\varphi(g) + r_1 hg - r_1\varphi(g)\varphi(g^{-1})gh \\
&= r_1 gh + r_1 hg + r_1 hg + r_1 gh \\
&= 2r_1 hg + 2r_1 gh = 0.
\end{aligned}$$

Finally, if $g, h \in G \setminus G_\varphi$ then

$$\begin{aligned}
[g - \varphi(g), h - \varphi(h)] &= gh - g\varphi(h) - \varphi(g)h + \varphi(g)\varphi(h) - hg + h\varphi(g) + \varphi(h)g - \varphi(h)\varphi(g) \\
&= gh - hg - hg + gh - hg + gh + gh - hg = 4gh - 4hg = 0
\end{aligned}$$

which finishes the proof of the theorem. ∎

For the classical involution $*$ on G we get the following consequence.

Corollary 2.6 *Let R be a commutative ring with elements of additive order 2. Let G be a non-abelian group. Denote by * the classical involution. Then $(RG)^-_*$ is commutative if and only if one of the following conditions holds:*

1. *$G = K \rtimes \langle x \rangle$ where $K = \langle g \mid g^2 \neq 1 \rangle$, K is abelian, $x^2 = 1$, $xkx = k^{-1}$ for all $k \in K$ and $R_2^2 = \{0\}$.*
2. *$\text{char}(R) = 4$, G has exponent 4, G' is a cyclic group of order 2, G/G' is an elementary abelian 2-subgroup and elements of order 2 commute if $R_2^2 \neq \{0\}$.*

3 Rings of characteristic three

In this section we determine when $(RG)^-_\varphi$ is commutative if $\text{char}(R) = 3$ (and thus $R_2 = \{0\}$). Again we begin by recalling two technical lemmas from [11].

Lemma 3.1 [11, Lemma 1.3] *Let R be a commutative ring with $\text{char}(R) \neq 2$ and let $g \in G \setminus G_\varphi$. If $(RG)^-_\varphi$ is commutative then one of the following conditions holds:*

1. *$g\varphi(g) = \varphi(g)g$.*
2. *$g^2 \in G_\varphi$.*

Lemma 3.2 [11, Lemma 1.1] *Let R be a commutative ring with $\text{char}(R) = 3$. Let $g, h \in G \setminus G_\varphi$ be two non-commuting elements. If $(RG)^-_\varphi$ is commutative then one of the following conditions holds*

1. *$gh \in G_\varphi$, $hg \in G_\varphi$, $h\varphi(g) = g\varphi(h)$ and $\varphi(h)g = \varphi(g)h$.*
2. *$gh \in G_\varphi$, $hg \in G_\varphi$, $h\varphi(g) = \varphi(g)h$.*
3. *$gh \in G_\varphi$, $hg = \varphi(g)h = g\varphi(h)$.*
4. *$gh = h\varphi(g) = \varphi(g)\varphi(h)$, $\varphi(h)g = \varphi(g)h$.*

5. $gh = \varphi(h)g = \varphi(g)\varphi(h)$, $h\varphi(g) = g\varphi(h)$.

6. $hg \in G_\varphi$, $gh = \varphi(h)g = h\varphi(g)$.

The following lemma was proved in [11] in the case when char(R) is distinct from both 2 and 3.

Lemma 3.3 *Let R be a commutative ring. Let $g, h \in G \setminus G_\varphi$ be non-commuting elements that satisfy* (2) *of Lemma 3.2. If char$(R) = 3$ then g, h also satisfy* (1) *of Lemma 3.2.*

Proof. Consider the element $\varphi(g)hg \in G$. Since $h \notin G_\varphi$ we have that $\varphi(g)hg \notin G_\varphi$. Also $\varphi(g)hg$ and h do not commute because, by assumption, $h\varphi(g) = \varphi(g)h$ and $gh \neq hg$. Assume that g and h satisfy (2) of Lemma 3.2. We claim that then $\varphi(h)g = \varphi(g)h$.

We deal with two mutually exclusive cases. First, assume that $\varphi(g)hgh \in G_\varphi$, i.e., $\varphi(g)hgh = \varphi(\varphi(g)hgh) = \varphi(h)\varphi(g)\varphi(h)g = \varphi(h)hg^2$ since $hg \in G_\varphi$. If $(g^2, h) = 1$ we obtain that $\varphi(g)h = \varphi(h)g$, as desired. So, to deal with this case, we may assume that $(g^2, h) \neq 1$. If $g^2 \in G_\varphi$ then, using (2), we observe that $\varphi(h)hg^2 = \varphi(g)hgh = \varphi(g^2)\varphi(h)h = g^2\varphi(h)h = \varphi(h)g^2h$. Hence, we get that $(g^2, h) = 1$, a contradiction. In the rest of the proof we will several times use (without referring to this) that gh, $hg \in G_\varphi$ and $(g, \varphi(h)) = 1 = (h, \varphi(g))$. Moreover, since $g^2 \notin G_\varphi$, we also have that $g\varphi(g) = \varphi(g)g$, by Lemma 3.1.

So, if $\varphi(g)hgh \in G_\varphi$ then we may assume that $g^2 \notin G_\varphi$ and $(g^2, h) \neq 1$. Hence, g^2 and h satisfy one of the six conditions of Lemma 3.2. We now show that this situation can not occur. Assume first that $g^2h \in G_\varphi$. Then, $g^2h = \varphi(g^2h) = \varphi(h)\varphi(g^2) = gh\varphi(g) = g\varphi(g)h$ and thus $g \in G_\varphi$, a contradiction. Therefore g^2 and h do not satisfy conditions $(1) - (3)$ of Lemma 3.2. If g^2 and h satisfy (4) of Lemma 3.2 then $g^2h = h\varphi(g^2) = \varphi(g^2)h$ and hence $g^2 \in G_\varphi$, a contradiction. Finally, if g^2 and h satisfy either (5) or (6) of Lemma 3.2 then $g^2h = \varphi(h)g^2 = g^2\varphi(h)$ and thus $h \in G_\varphi$, a contradiction. This finishes the proof of the first case.

Second, assume that $\varphi(g)hgh \notin G_\varphi$. Then $\varphi(g)hg$ and h satisfy one of the conditions $(4) - (6)$ of Lemma 3.2. We show that all these lead to a contradiction and hence that this case also can not occur. If $\varphi(g)hg$ and h satisfy (4) of Lemma 3.2, then $\varphi(g)hgh = h\varphi(\varphi(g)hg) = h\varphi(g)\varphi(h)g = h\varphi(hg)g = h^2g^2$ and thus $\varphi(g)gh = hg^2 = \varphi(g)\varphi(h)g$; so $gh = \varphi(h)g$ and hence $g \in G_\varphi$, a contradiction.

Suppose that $\varphi(g)hg$ and h satisfy (5) or (6) of Lemma 3.2. Then

$$\varphi(g)hgh = \varphi(h)\varphi(g)hg \tag{1}$$

First assume that $g^2 \in G_\varphi$ then we have that $\varphi(h)hg^2 = g^2\varphi(h)h = \varphi(g^2)\varphi(h)h = \varphi(g)hgh$. On the other hand $\varphi(h)\varphi(g)hg = \varphi(h)h\varphi(g)g$. Thus, by (1) we get that $g \in G_\varphi$, a contradiction. Therefore $g^2 \notin G_\varphi$ and hence, by Lema 3.1 we get that $(g, \varphi(g)) = 1$. If also $(h, \varphi(h)) = 1$ then $\varphi(g)hgh = hg\varphi(g)h$ and on the other hand, $\varphi(h)\varphi(g)hg = hg\varphi(h)\varphi(g)$. Then, by (1), we get that $\varphi(g)h = \varphi(h)\varphi(g) = gh$ and thus $g \in G_\varphi$, a contradiction. So, again by Lemma 3.1 we have that $h^2 \in G_\varphi$. Therefore $\varphi(h)\varphi(g)hg = gh^2g = h^2g^2$ and on the other hand $\varphi(g)hgh = h\varphi(g)gh$. Thus, by (1), we have that $\varphi(g)gh = hg^2 = \varphi(g)\varphi(h)g$. Therefore $\varphi(h)\varphi(g) = gh = \varphi(h)g$ and hence $g \in G_\varphi$, again a contradiction. So $\varphi(g)hg$ and h do not satisfy neither (5) nor (6) of Lemma 3.2.

So, we have proved that if (2) of Lemma 3.2 holds for non-commuting elements $g, h \in G \setminus G_\varphi$ then $\varphi(h)g = \varphi(g)h$. Since $(h, \varphi(g)) = 1 = (g, \varphi(h))$ it also follows that $h\varphi(g) = g\varphi(h)$. Consequently, we have shown that (1) of Lemma 3.2 holds for g and h. ∎

Lemma 3.4 *Let R be a commutative ring. Let $g, h \in G \setminus G_\varphi$ be non-commuting elements.*

1. *If g and h satisfy (1) (or (2)) of Lemma 3.2 then $g^3, h^3 \notin G_\varphi$*

2. *If g and h satisfy one of the conditions $(3)-(6)$ of Lemma 3.2 then $g^3, h^3 \in G_\varphi$ and $g^3, h^3 \in Z(\langle g, h, \varphi(g), \varphi(h)\rangle)$.*

Proof. 1. Let $g, h \in G \setminus G_\varphi$ be non-commuting elements. Assume that g and h satisfy (1) of Lemma 3.2. We prove by contradiction that $g^3 \notin G_\varphi$. So, suppose that $g^3 \in G_\varphi$. Since $g \notin G_\varphi$, it follows that $g^2 \notin G_\varphi$. Also by (1) of Lemma 3.2 we have that

$$g^3 h = g^2\varphi(h)\varphi(g) = gh\varphi(g^2) = \varphi(h)\varphi(g^3) = \varphi(h)g^3 \tag{2}$$

Notice that by (2) it follows that $(g^2, h) \neq 1$, because otherwise $gh = \varphi(h)g$, a contradiction. Therefore g^2 and h satisfy one of the conditions $(1)-(6)$ of Lemma 3.2. Assume first that $g^2h \in G_\varphi$. Then by (2) we have that $g\varphi(h)\varphi(g^2) = gg^2h = \varphi(h)\varphi(g^3)$. Hence $g\varphi(h) = \varphi(h)\varphi(g) = gh$ and thus $h \in G_\varphi$, a contradiction. Therefore g^2 and h do not satisfy conditions $(1)-(3)$ of Lemma 3.2. Second, assume that $g^2h = \varphi(h)g^2$. Then, by (2), it follows that $\varphi(h)g^3 = gg^2h = g\varphi(h)g^2$ and thus $(g, \varphi(h)) = 1$. Therefore, again by (2), we get that $h \in G_\varphi$, a contradiction. So, g^2 and h do not satisfy conditions $(5)-(6)$. Hence, g^2 and h satisfy (4). Then, since $(g, \varphi(g)) = 1$ by Lemma 3.1, we have that $\varphi(g^3)h = gg^2h = gh\varphi(g^2) = g\varphi(g^2)\varphi(h) = \varphi(g^2)g\varphi(h)$. Hence $\varphi(g)h = g\varphi(h) = h\varphi(g)$. Consequently, by (2), we get that $h \in G_\varphi$, a contradiction. This finishes the proof of the fact that $g^3 \notin G_\varphi$. Because of the symmetry in g and h in condition (1) of Lemma 3.2, we thus also obtain that $h^3 \notin G_\varphi$.

2. Notice that if in (3) of Lemma 3.2 we interchange the roles of g and h then we obtain (6), if we change h by $\varphi(h)$ we have (5) and finally if we change g by $\varphi(g)$ we have (4). Therefore it is enough to show the result for (3).

So, assume that $g, h \in G \setminus G_\varphi$ are non-commuting elements that satisfy (3) of Lemma 3.2. Then $g^3h = g^2\varphi(h)\varphi(g) = g\varphi(g)h\varphi(g) = g\varphi(g^2)\varphi(h)$ and therefore, since $h \notin G_\varphi$, it follows that $g^2 \notin G_\varphi$. Thus, by Lemma 3.1, we get that $(g, \varphi(g)) = 1$. Consequently, $g^3h = \varphi(g^2)g\varphi(h) = \varphi(g^3)h$ and therefore $g^3 \in G_\varphi$. Analogously we obtain that $h^3 \in G_\varphi$. Moreover, $g^3h = g^2\varphi(h)\varphi(g) = ghg\varphi(g) = \varphi(h)\varphi(g)g\varphi(g) = \varphi(h)g\varphi(g)\varphi(g) = h\varphi(g^3) = hg^3$. So, $g^3h = hg^3$ and thus also $\varphi(h)g^3 = g^3\varphi(h)$, as desired. Similarly we get that $h^3 \in Z(\langle g, h, \varphi(g), \varphi(h)\rangle)$. ∎

Remark 3.5 *Notice that Lemma 3.4 implies that if $g, h, x, y \in G \setminus G_\varphi$ are such that g and h are non-commuting elements satisfying condition (1) of Lemma 3.2 and x and y are non-commuting elements satisfying one of the conditions $(3)-(6)$ of Lemma 3.2 then x and y commute with both g and h.*

Lemma 3.6 *Let R be a commutative ring with $\mathrm{char}(R) = 3$ and assume RG_φ^- is commutative. If there exist non-commuting $g, h \in G \setminus G_\varphi$ so that g and h satisfy (1) of Lemma 3.2 then all $x, y \in G \setminus G_\varphi$ satisfy (1) of Lemma 3.2.*

Proof. Let $g, h \in G \setminus G_\varphi$ be non-commuting elements so that g and h satisfy (1) of Lemma 3.2, that is, gh, hg, $g\varphi(h)$ and $\varphi(g)h$ are elements of G_φ. Also, by Lemma 3.4, we have that g^3, $h^3 \notin G_\varphi$.

Let $x \in G \setminus G_\varphi$. Then, by Lemma 3.2 and Lemma 3.4, it follows that $(g, x) = 1$ or $gx \in G_\varphi$. We claim that $gx \in G_\varphi$. In order to prove this claim suppose that $gx \notin G_\varphi$ and thus $(g, x) = 1$. Again, by Lemma 3.2 and Lemma 3.4, it follows that $(h, x) = 1$ or $xh \in G_\varphi$; and $(gx, h) = 1$ or $gxh \in G_\varphi$. Assume first that $(gx, h) = 1$, that is, $gxh = hgx$. Since $gh \neq gh$ we get that $hx \neq xh$ and thus $xh \in G_\varphi$. Therefore, $hgx = gxh = g\varphi(h)\varphi(x) = h\varphi(g)\varphi(x) = h\varphi(gx)$ and thus $gx \in G_\varphi$, a contradiction. Second assume that $gxh \in G_\varphi$. Then $gxh = \varphi(h)\varphi(x)\varphi(g) = \varphi(h)\varphi(g)\varphi(x) = gh\varphi(x)$ and hence $xh = h\varphi(x)$. Therefore, and since $x \notin G_\varphi$, we get that $xh \neq hx$ and thus $xh \in G_\varphi$. Then $\varphi(h)\varphi(x) = xh = h\varphi(x)$ and hence $h \in G_\varphi$, again a contradiction. This finishes the proof of the claim.

Now, let $x, y \in G \setminus G_\varphi$. We need to prove that x and y satisfy (1) of Lemma 3.2. First we deal with the case that $(x, y) \neq 1$. Because of Lemma 3.3, we only have to show that it is impossible that x and y satisfy one of the conditions $(3) - (6)$ of Lemma 3.2. So suppose the contrary. Then, by Remark 3.5, $(g, x) = 1 = (g, y)$. Also, by Lemma 3.4, x^3, $y^3 \in G_\varphi$. By the previous claim we have that $gx \in G_\varphi$. Consequently, $g^3x^3 = (gx)^3 = \varphi(gx)^3 = \varphi(g^3)\varphi(x^3) = \varphi(g^3)x^3$, and thus $g^3 \in G_\varphi$, a contradiction. So, if $xy \neq yx$ then x and y satisfy (1) of Lemma 3.2.

Finally, assume $x, y \in G\setminus G_\varphi$ and $(x, y) = 1$. Then $xy \in G_\varphi$. Indeed, suppose the contrary, that is assume $xy \notin G_\varphi$. Hence, by the above claim, $gxy \in G_\varphi$. Thus $gyx = gxy = \varphi(y)\varphi(x)\varphi(g) = \varphi(y)gx$, because $gx \in G_\varphi$. Therefore $gy = \varphi(y)g$. Since $gy \in G_\varphi$, it follows that $\varphi(y)g = gy = \varphi(y)\varphi(g)$ and thus $g \in G_\varphi$, a contradiction. Hence, indeed $yx = xy \in G_\varphi$. Replacing y by $\varphi(y)$ we thus also get that $x\varphi(y) \in G_\varphi$ if $(x, \varphi(y)) = 1$. If, on the other hand, $(x, \varphi(y)) \neq 1$ then the previous implies that again $x\varphi(y) \in G_\varphi$. Similarly, $\varphi(x)y \in G_\varphi$. Consequently, we have shown that x and y satisfy (1) of Lemma 3.2. ∎

Lemma 3.7 *Let R be a commutative ring with* $\mathrm{char}(R) = 3$*. Let $g, h \in G \setminus G_\varphi$ be non-commuting elements satisfying any of the conditions $(3) - (6)$ of Lemma 3.2. Then $\langle g^{-1}\varphi(g)\rangle = \langle h^{-1}\varphi(h)\rangle = \langle (g, h)\rangle$ and $(g, h)^3 = 1$.*

Proof. Let $g, h \in G\setminus G_\varphi$ be as in the statement of the Lemma. Because of Lemma 3.4, $g^3, h^3 \in G_\varphi$. Therefore $g^2, h^2 \notin G_\varphi$, because $g, h \notin G_\varphi$. Hence, by Lemma 3.1, it follows that $(g, \varphi(g)) = 1 = (h, \varphi(h))$.

First, assume that g and h satisfy (3) of Lemma 3.2. Hence, $\varphi(g) = hgh^{-1}$ and $\varphi(h) = g^{-1}hg$. Therefore $g^{-1}\varphi(g) = g^{-1}hgh^{-1} = \varphi(h)h^{-1} = h^{-1}\varphi(h)$. Thus, $h^{-1}\varphi(h) = g^{-1}\varphi(g) = \varphi(g)g^{-1} = hgh^{-1}g^{-1} = (g, h)^{-1}$, as desired.

Second, assume that g and h satisfy (4) of Lemma 3.2. Then $\varphi(g) = h^{-1}gh$ and $\varphi(h) = \varphi(g^{-1})gh = h^{-1}g^{-1}hgh$. Therefore $(g, \varphi(g)) = (g, h^{-1}gh) = 1 = (h, \varphi(h)) = (h, g^{-1}hg)$. Thus, $g^{-1}\varphi(g) = g^{-1}h^{-1}gh = h^{-1}ghg^{-1} = ghg^{-1}h^{-1} = (g, h)$ and $h^{-1}\varphi(h) = h^{-1}g^{-1}hg = (g^{-1}\varphi(g))^{-1}$, again as desired.

Third, assume that g and h satisfy (5) of Lemma 3.2. Then $\varphi(h) = ghg^{-1}$ and $\varphi(g) = gh\varphi(h)^{-1} = ghgh^{-1}g^{-1}$. Therefore $(h, \varphi(h)) = (h, ghg^{-1}) = 1 = (g, \varphi(g)) = (g, hgh^{-1})$. Thus, $g^{-1}\varphi(g) = hgh^{-1}g^{-1} = (g, h)^{-1}$ and $h^{-1}\varphi(h) = h^{-1}ghg^{-1} = ghg^{-1}h^{-1} = (g^{-1}\varphi(g))^{-1}$, again as desired.

Fourth, assume that g and h satisfy (6) of Lemma 3.2. Then $\varphi(h) = ghg^{-1}$ and $\varphi(g) = h^{-1}gh$. Therefore $(h, \varphi(h)) = (h, ghg^{-1}) = 1 = (g, \varphi(g)) = (g, h^{-1}gh)$. Thus, $g^{-1}\varphi(g) = g^{-1}h^{-1}gh = h^{-1}ghg^{-1} = ghg^{-1}h^{-1} = (g, h)$ and $h^{-1}\varphi(h) = h^{-1}ghg^{-1} = ghg^{-1}h^{-1} = g^{-1}\varphi(g)$, as desired.

To finish the proof of the lemma notice that since $g^3 \in G_\varphi$ and $(g, \varphi(g)) = 1$ it follows that $(g^{-1}\varphi(g))^3 = 1$ and therefore $(g, h)^3 = 1$. ∎

Theorem 3.8 *Let R be a commutative ring with* $\mathrm{char}(R) = 3$*. Suppose G is a non-abelian group and φ is an involution on G. Then $(RG)^-_\varphi$ is commutative if and only if one of the following conditions holds:*

(a) $K = \langle g \in G |\ g \notin G_\varphi\rangle$ is abelian, $G = K \cup Kx$ where $x \in G_\varphi$ and $\varphi(k) = xkx^{-1}$ for all $k \in K$.

(b) G contains an abelian subgroup of index 2 that is contained in G_φ.

(c) $|G'| = 3$, $(G/G') = (G/G')_\varphi$ and $g^3 \in G_\varphi$ for all $g \in G$.

Proof. Assume that there exist non-commuting elements $g, h \in G \setminus G_\varphi$ so that g and h satisfy (1) of Lemma 3.2. Then, by Lemma 3.6, all $x, y \in G \setminus G_\varphi$ satisfy (1) of Lemma 3.2. Because of all the stated Lemmas, one now obtains, exactly as in the proof of Theorem 2.1 in [11] that condition (a) or (b) holds.

So now suppose that there do not exist non-commuting elements $g, h \in G \setminus G_\varphi$ satisfying condition (1) (and thus not (2), by Lemma 3.3) of Lemma 3.2. Then, all non-commuting elements $x, y \in G \setminus G_\varphi$ satisfying one of the conditions $(3) - (6)$ of Lemma 3.2. In particular, by Lemma 3.4, $x^3, y^3 \in G_\varphi$. Since $x \notin G_\varphi$, we thus have that $x^2 \notin G_\varphi$ and thus, by Lemma 3.1, $(\varphi(x), x) = 1$.

We claim that $g^3 \in G_\varphi$ for all $g \in G$. So, let $g \in G$. In case $(g, x) \neq 1$ then it follows at once from Lemma 3.4 that $g^3 \in G_\varphi$. If, on the other hand, $(g, x) = 1$, then we consider two mutually exclusive cases. First, assume $gx \notin G_\varphi$. Then, again by Lemma 3.4, $g^3x^3 = (gx)^3 \in G_\varphi$ and thus $g^3x^3 = \varphi(g^3)\varphi(x^3) = \varphi(g^3)x^3$ and thus $g^3 \in G_\varphi$. Second, assume $gx \in G_\varphi$. Thus $xg = gx = \varphi(x)\varphi(g) = \varphi(g)\varphi(x)$. Hence, by Lemma 3.7, $g^{-1}\varphi(g) = x\varphi(x)^{-1} = \varphi(x)^{-1}x = \varphi(g)g^{-1}$ is an element of order 3. Thus $1 = (g^{-1}\varphi(g))^3 = g^{-3}\varphi(g^3)$. So $g^3 \in G_\varphi$, as claimed.

Next, we claim that if $g \notin G_\varphi$ then $g^{-1}\varphi(g) \in \langle (x, y) \rangle$ (in particular $G/G' = (G/G')_\varphi$). Indeed, if $(g, x) \neq 1$, $(g, \varphi(x)) \neq 1$, $(g, \varphi(y)) \neq 1$ or $(g, y) \neq 1$ the result follows from Lemma 3.7. So assume that $(g, x) = (g, \varphi(x)) = (g, y) = (g, \varphi(y)) = 1$. If $gx \in G_\varphi$ then $gx = \varphi(gx) = \varphi(x)\varphi(g) = \varphi(g)\varphi(x)$ and hence $g^{-1}\varphi(g) = x\varphi(x)^{-1} \in \langle (x, y) \rangle$ by Lemma 3.7 as desired. Finally if $gx \notin G_\varphi$ then $(gx, y) = (x, y) \neq 1$. Then, by Lemma 3.7, we have that $(gx)^{-1}\varphi(gx) = g^{-1}\varphi(g)x^{-1}\varphi(x) \in \langle (x, y) \rangle$ and therefore, since $x^{-1}\varphi(x) \in \langle (x, y) \rangle$ we get that $g^{-1}\varphi(g) \in \langle (x, y) \rangle$, which finishes the proof of the claim.

To finish the proof of the necessity, we have to prove that $G' = \langle (x, y) \rangle$ and thus, by Lemma 3.7, $|G'| = 3$. In order to prove this, let $g, h \in G$ with $(g, h) \neq 1$. If $g, h \notin G_\varphi$ then by Lemma 3.7 and the previous claim we have that $(g, h) \in \langle g^{-1}\varphi(g) \rangle = \langle (x, y) \rangle$. Next assume that $g \in G_\varphi$ and $h \notin G_\varphi$. If $gh \notin G_\varphi$ then, by Lemma 3.7 and the previous claim, we have that $(g, h) = (gh, h) \in \langle h^{-1}\varphi(h) \rangle = \langle (x, y) \rangle$, as desired. If $gh \in G_\varphi$ we have that $gh = \varphi(h)g$ and thus, by the previous claim, $ghg^{-1}h^{-1} = \varphi(h)h^{-1} \in \langle (x, y) \rangle$, as desired. Finally, assume that $g, h \in G_\varphi$. Then, since $(g, h) \neq 1$, it follows that $h^{-1}g^{-1} \notin G_\varphi$. Therefore, by the previous claim, $(g, h) = ghg^{-1}h^{-1} = (h^{-1}g^{-1})^{-1}\varphi(h^{-1}g^{-1}) \in \langle (x, y) \rangle$ and the proof of the necessity concludes.

In order to prove the sufficiency, we need to show that the elements in $\mathcal{S} = \{g - \varphi(g) \mid g \in G,\ g \notin G_\varphi\}$ commute. If G satisfies conditions (a) or (b), then the proof is the same as the sufficiency proof of Theorem 2.1 in [11]. So, assume that G satisfies condition (c). Then, since $g^3 \in G_\varphi$, we have that $g^2 \notin G_\varphi$ for all $g \in G \setminus G_\varphi$ and hence, by Lemma 3.1, $(g, \varphi(g)) = 1$. Moreover, it follows from (c) that $\varphi(g) = t^i g$, where $G' = \langle t \rangle$, $i \in \{\pm 1\}$ for $g \notin G_\varphi$. Then, clearly, $(g, t) = 1$.

Let now $g, h \in G \setminus G_\varphi$. Then $\varphi(g) = t^i g$ and $\varphi(h) = t^j h$ with $i, j \in \{\pm 1\}$ and

$$\begin{aligned} [g - \varphi(g), h - \varphi(h)] &= gh - g\varphi(h) - \varphi(g)h + \varphi(g)\varphi(h) - hg + h\varphi(g) + \varphi(h)g - \varphi(h)\varphi(g) \\ &= gh - t^j gh - t^i gh + t^i t^j gh - hg + t^i hg + t^j hg - t^i t^j hg. \end{aligned}$$

If $(g,h)=1$ then clearly $[g-\varphi(g),h-\varphi(h)]=0$. So, assume that $1\neq (g,h)=t$. If $i=j$ then, since $\mathrm{char}(R)=3$,

$$\begin{aligned}[g-\varphi(g),h-\varphi(h)] &= gh-t^igh-t^igh+t^{-i}gh-hg+t^ihg+t^ihg-t^{-i}hg\\ &= (1-2t^i+t^{-i})gh-(1-2t^i+t^{-i})hg\\ &= (1+t^i+t^{-i})(t-1)hg\\ &= (1+t+t^2)(t-1)gh\\ &= 0\end{aligned}$$

On the other hand if $i\neq j$ then, again since $\mathrm{char}(R)=3$,

$$\begin{aligned}[g-\varphi(g),h-\varphi(h)] &= gh-t^jgh-t^igh+t^it^jgh-hg+t^ihg+t^jhg-t^it^jhg\\ &= 2gh-t^jgh-t^igh-2hg+t^ihg+t^jhg\\ &= (2-t^j-t^{-i})gh-(2-t^i-t^j)hg\\ &= (2-t^j-t^i)(t-1)hg\\ &= 2(1+t+t^2)(t-1)hg\\ &= 0\end{aligned}$$

Similarly, if $(g,h)=t^{-1}$ one gets that $[g-\varphi(g),h-\varphi(h)]=0$, which finishes the proof of the theorem. ▮

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