A New Approach for Bounding Awards in Bankruptcy Problems

Jiménez-Gómez, José M. (Jmanuel.Jimenez@upct.es)
Dep. Economía, Univ. Politécnica de Cartagena,
Ps. Alfonso XIII, 50, 30203 Cartagena, Murcia, Spain.
Marco-Gil, M.Carmen (Carmen.Marco@upct.es)
Dep. Economía, Univ. Politécnica de Cartagena,
Ps. Alfonso XIII, 50, 30203 Cartagena, Murcia, Spain.

RESUMEN

The solution for the 'Contested Garment Problem' proposed in the Babylonic Talmud, one of the most important sources of inspiration for solving situations where demand overcomes supply of some resources, suggests that each agent should receive at least some part of the available amount when facing these situations. This idea has underlied the theoretical analysis of bankruptcy problems from its beginning (O'Neill, 1982) to present day (Dominguez and Thomson, 2006). In this context, starting from the fact that a society establishes its own set of 'Commonly Accepted Equity Principles', we propose a new lower bound on awards defined, for each agent, as the minimum amount she gets according to all the admissible rules for such a society. Moreover, we analyze the recursive application of this new bound, since it will not exhaust the resources, in general.

Palabras clave: [Bankruptcy problems, bankruptcy rules, lower bound, recursive process.]
Clasificación JEL (Journal Economic Literature): [C71, D63, D71.]
Área temática: [Aspectos cuantitativos del fenómeno económico]

1 Introduction

A bankruptcy problem reflects a situation where a group of agents claim more quantity of a good than available. According to that, a bankruptcy rule prescribes how to share out an amount of a perfectly divisible resource, called estate, among a group of agents, depending on a profile of demands whose aggregate overcomes its supply. In this context two natural questions arise: *How should the available resources be rationed among claimants? Should each agent have guaranteed a level of awards?*

The main goal of the two approaches to the study of bankruptcy problems: the axiomatic and the game theoretical methods, has been identifying bankruptcy rules by means of appealing properties. Following this line, many authors have found reasonable establishing some bound on awards. In fact, the formal definition of a solution for bankruptcy problems includes, by demanding that no agent gets more than her claim and less than zero, both an upper and a lower bounds on awards. In 1982, O'Neill [15] provides a new lower bound on awards called *Respect of Minimal Right,* which requires that each claimant receives at least the available amount of the estate after the other claimants have been fully compensated, or 0 if this amount is negative. Later, Herrero and Villar [9, 10] introduce two properties that bound awards, called *Sustainability* and *Exemption. Sustainability* says that, if we truncate all claims by an agent i's claim and the bankruptcy problem becomes feasible, then agent i will get all her claim. *Exemption* demands that agent i does not be rationed when equal division provides her more than she claims. After that, Moulin [14] defines a new restriction on awards, called *Lower Bound,* which imposes that each agent has the amount corresponding to the egalitarian division guaranteed except those who demand less, in which case their demand is met in full. Afterwards, Moreno-Ternero and Villar [12] present a weaker notion of Moulin's *Lower Bound*, named *Securement*, which says that each agent should obtain at least the nth part of her claim truncated at the amount to divide. Finally, Dominguez [7] proposes the *Min Lower Bound*, which modifies *Securement* by substituting each agent's claim by the lowest one.

Apart from *Respect of Minimal Right*, property which is implied by the formal definition of a bankruptcy rule, the rest of the proposed limits on awards have been justified by their own reasonability or appeal. Our goal is to establish restrictions on awards taking as starting point a set, P, of 'basic' requirements, called *'Commonly Accepted Equity Principles'*, on which a society could willingly agree. Then we consider the ordinary meaning of guarantee over all the bankruptcy rules satisfying properties in P as follows. By applying to a bankruptcy problem all *Socially Admissible Bankruptcy* rules we determine the agent's *P-Safety* as the lower amount she gets among those ones provided by such rules. Finally, we define the associated bound on awards, *Respect of P-Safety*, by demanding that each agent receives, at least, her *P-Safety*.

Since, in general, the aggregate guaranteed amount by means of our *P-Safety* will not exhaust the available resources, we propose and analyze its recursive application, called the *Recursive P-Safety* rule. The idea of recursivity is not new, in fact it has already been used for introducing bankruptcy rules by Alcalde et al. [2], who generalize the Ibn Ezra's proposal, and by Dominguez and Thomson [8], who propose the *Recursive* rule by using the Moreno-Ternero and Villar's concept of boundedness, among other authors. Dominguez [7] also studies the behavior of the recursive application of a generic bound.

In this paper we apply the previous methodology to different sets of *'Commonly Accepted Equity Principles'*. First of all, we propose as basic properties the set P_1, composed by *Resource Monotonicity*, *Super-Modularity* and *Midpoint Property*. In this case we find out that the *P-Safety* is the minimum of *Piniles'* rule (Piniles [17]) and its dual. Moreover, we prove that the *Recursive P-Safety* rule retrieves the *Dual of Piniles'* rule. Secondly, we only consider, as equity principles, *Resource Monotonicity* and *Order Preservation*, that is P_2. Then, we show that the associated *P-Safety* is the minimum of two different extreme bankruptcy solutions, the *Constrained Equal Awards* rule (many authors, see Thomson [20]) and its dual, the *Constrained Equal Losses* rule (Maimonides 12th Century, among others). Besides this, we demonstrate that the *Recursive P-Safety* rule retrieves the *Constrained Equal Losses* rule.

Our previous results could be written as follows: 'The recursive application of the *P-Safety* recovers, in the set of all admissible bankruptcy rules according to both P_1 and P_2, one of its extremes, that one providing more awards to the higher claimants'. Then, the analysis of the generalization of this statement arises as a natural question. With this aim, we define a new set of socially accepted requirements, P_3, consisting of *Resource Monotonicity*, *Order Preservation* and *Midpoint Property.* Surprisingly enough, since $P_2 \subset P_3 \subset P_1$, we show both that this generalization is not possible and that the rule obtained by the recursive application of the *P-Safety* does not satisfy the equity principles which this process is based on.

The paper is organized as follows: Section 2 presents the model. Section 3 proposes our new approach for bounding awards and its recursive application. Section 4 provides new basis to classical bankruptcy rules using the previous ideas, and the incompatibility of the proposed process with some 'appealing' set of equity principles. Section 5 summarizes our conclusions. Finally, all the resutls are proved in Jiménez-Gómez and Marco-Gil [11].

2 Preliminaries

A bankruptcy problem is a situation where the agents' demand of a good exceeds its supply. Formally,

Definition 1 *A* ***bankruptcy problem*** *is a vector* $(E, c) \in \mathbb{R}_+ \times \mathbb{R}^n_+$ *such that*

$$E \leq \sum_{i \in N} c_i.$$

E is known as the estate, and represents the perfectly divisible good quantity that should be distributed among the agents in $N = \{1, ..., i, ..., n\}$. Each agent $i \in N$ has a claim c_i on the estate. Claims add up to more than the available amount. Therefore, the resource should be rationed.

For notational convenience, B will denote the set of all bankruptcy problems, problems from now on; C the sum of the agents' claims, $C = \sum_{i \in N} c_i$; L the total amount of losses to distribute among the agents, $L = C - E$; and B_0 the set of problems in which claims are increasingly ordered, that is problems with $c_i \leq c_j$ for $i < j$.

A bankruptcy rule associates for each problem a distribution of the available amount among the group of claimants. Next, we present this concept formally and define the rules that will be used in the following sections, emphasizing their dual relations.

Definition 2 *A* ***bankruptcy rule****, or simply a* ***rule****, is a function,* $\varphi : B \to \mathbb{R}^n_+$, *such that for each* $(E, c) \in B$,

(a) $\sum_{i \in N} \varphi_i(E, c) = E$ *(efficiency) and*
(b) $0 \leq \varphi_i(E, c) \leq c_i$ *for each* $i \in N$ *(non-negativity and claim-boundedness).*

Constrained Equal Awards rule (Maimonides 12th Century, among others) recommends equal gains to all claimants subject to no-one receiving more than her claim.

Constrained Equal Awards rule, *CEA*: for each $(E, c) \in B$ and each $i \in N$, $CEA_i(E, c) \equiv \min\{c_i, \mu\}$, where μ is chosen so that $\sum_{i \in N} \min\{c_i, \mu\} = E$.

Piniles' rule (Piniles [17]) assigns the *Constrained Equal Awards* rule when the available amount is less than the half-sum of the claims. Otherwise, first each agent receives her half-claim, then the *Constrained Equal Award* rule is re-applied to divide the remainder but using the agents' half-claims in the formula.

Piniles' rule, *Pin*: for each $(E, c) \in B$ and each $i \in N$,

$$Pin_i(E,c) \equiv \begin{cases} CEA_i(E, c/2) & \text{if } E \leq C/2 \\ c_i/2 + CEA_i(E - C/2, c/2) & \text{if } E \geq C/2 \end{cases}.$$

Constrained Egalitarian rule (Chun et al. [**?**]) is inspired by the *Uniform* rule (Sprumont [19]), a solution to the problem of fair division when the preferences are single-peaked. It makes the minimal adjustment in the formula of the *Uniform* rule taking the half-claims as the peak and guaranteeing that awards are ordered as claims are.

Constrained Egalitarian rule, *CE*: for each $(E, c) \in B$ and each $i \in N$,

$$CE_i(E,c) \equiv \begin{cases} CEA_i(E, c/2) & \text{if } E \leq C/2 \\ \max\{c_i/2, \min\{c_i, \delta\}\} & \text{if } E \geq C/2 \end{cases},$$

where δ is chosen so that $\sum_{i \in N} CE_i(E, c) = E$.

Given a rule φ, its dual shares losses in the same way as φ divides the available amount (Aumann and Maschler [1]).

The **dual rule** of φ, denoted by φ^d, assigns for each $(E, c) \in B$ and each $i \in N$, $\varphi_i^d(E, c) = c_i - \varphi_i(L, c)$.

It is straightforward to check that for each rule, φ, its dual is well defined since given $(E, c) \in B$, $(L, c) \in B$ and given that φ satisfies efficiency, non-negativity and claim-boundedness, φ^d will as well.

Constrained Equal Losses rule, discussed by Maimonides (Aumann and Maschler [1]), is the dual of the *Constrained Equal Awards* rule (Herrero [9]). Specifically, it chooses the awards vector at which losses from the claims vector are the same for all agents subject to no-one receiving a negative amount.

Constrained Equal Losses rule, *CEL*: for each $(E, c) \in B$ and each $i \in N$, $CEL_i(E, c) \equiv \max\{0, c_i - \mu\}$, where μ is chosen so that $\sum_{i \in N} \max\{0, c_i - \mu\} = E$.

Dual of Piniles' rule assigns the *Constrained Equal Losses* rule when the available amount is less than the half-sum of the claims. Otherwise, first each agent receives her half-claim, then the *Constrained Equal Losses* rule is re-applied to divide the remainder but only taking into account the agents' half-claims.

Dual of Piniles' rule, *DPin*: for each $(E, c) \in B$ and each $i \in N$,

$$DPin_i(E, c) = \begin{cases} c_i/2 - \min\{c_i/2, \lambda\} & \text{if } E \leq C/2 \\ c_i/2 + (c_i/2 - \min\{c_i/2, \lambda\}) & \text{if } E \geq C/2 \end{cases},$$

where λ is such that $\sum_{i \in N} DPin_i(E, c) = E$.

Dual Constrained Egalitarian rule gives the half-claims a central role and makes the minimal adjustment in the formula of the *Dual Uniform* rule to guarantee that losses are ordered as claims are.

Dual Constrained Egalitarian rule, *DCE*: for each $(E, c) \in B$ and each $i \in N$,

$$DCE_i(E, c) \equiv \begin{cases} c_i - \max\{c_i/2, \min\{c_i, \delta\}\} & \text{if } E \leq C/2 \\ c_i - \min\{c_i/2, \delta\} & \text{if } E \geq C/2 \end{cases},$$

where δ is chosen such that $\sum_{i \in N} DCE_i(E, c) = E$.

Next, we introduce some properties of rules which, subsequently, will be interpreted as *'Commonly Accepted Equity Principles'*, and we present the notion of *Self-Duality* between rules. Let φ be a generic rule.

Resource Monotonicity (Curiel et al. [5], Young [**?**] and others) demands that if the estate increases, then all individuals should receive at least as much as they did initially.

Resource Monotonicity: for each $(E, c) \in B$ and for each $E' \in \mathbb{R}_+$ such that $C \geq E' > E$, then $\varphi_i(E', c) \geq \varphi_i(E, c)$, for each $i \in N$.

*Order Preservation (*Aumann and Maschler [1]) requires respecting the claims order, i.e., if agent i's claim is at least as large as agent j's claim, she should receive and she should loss at least as such agent j does respectively.

Order Preservation: for each $(E, c) \in B$ and each $i, j \in N$ such that $c_i \geq c_j$, then $\varphi_i(E, c) \geq \varphi_j(E, c)$ and $c_i - \varphi_i(E, c) \geq c_j - \varphi_j(E, c)$.

*Super-Modularity (*Dagan et al. [6]) demands, when the estate increases, that agents with larger claims receive a greater part of the increment than those with lower claims.

Super-Modularity: for each $(E, c) \in B$, all $E' \in \mathbb{R}_+$ and each $i, j \in N$ such that $C \geq E' > E$ and $c_i \geq c_j$, then $\varphi_i(E', c) - \varphi_i(E, c) \geq \varphi_j(E', c) - \varphi_j(E, c)$.

*Midpoint Property (*Chun, Schummer and Thomson [**?**]) says that if the estate is equal to the sum of the half-claims, then every individual should get her half-claim.

Midpoint Property: for each $(E, c) \in B$ such that $E = C/2$, $\varphi_i(E, c) = c_i/2$, for each $i \in N$.

Self-Duality implies that a rule treats symmetrically the problem of dividing 'what is available' and the problem of dividing 'what is missing'.

Self-Duality: for each $(E, c) \in B$ and each $i \in N$, $\varphi_i(E, c) = c_i - \varphi_i(L, c)$.

Finally, we present the idea of duality between properties, which has been analyzed by many authors (see, for instance, Herrero and Villar [**?**] and Moulin [13]).

Two properties, $\mathcal{P}$ and $\mathcal{P}'$, are **dual** if whenever a rule, φ, satisfies $\mathcal{P}$, its dual, φ^d, satisfies $\mathcal{P}'$. A property, $\mathcal{P}$, is **Self-Dual** when it coincides with its dual.

It is straightforward to check that all the properties previously introduced, *Resource Monotonicity, Order Preservation, Super-Modularity,* and *Midpoint Property*, are *Self-Dual,* a fact that will be used later on.

3 A new approach: Bounding awards from equity principles.

As we have noted, most of the lower bounds on awards that have been proposed in the literature have been justified by their own reasonability. A clear exception is *Respect of Minimal Right,* which requires that each claimant receives at least the available amount of the estate after the other claimants have been fully compensated, or 0 if this amount is negative. This property, as Thomson [20] pointed out, is a consequence of efficiency, non-negativity and claim boundedness together (See Definition 2).

In this section we introduce a new method for bounding awards based on a set of *'Commonly Accepted Equity Principles'* by a society. With this aim and considering such a set of basic properties, next we propose the following extension of a problem.

Definition 3 *A* ***Bankruptcy Problem with Legitimate Principles*** *is a vector* (E, c, P) *where* $(E, c) \in B$ *and* P *is a set of principles on which a society has agreed.*

From now on, let $\mathbb{P}$ be the set of all subsets of properties on bankruptcy rules, and let B_P be the set of all *Problems with Legitimate Principles.*

In this context, a *Socially Admissible* bankruptcy rule is a rule satisfying all properties in P.

Definition 4 *A* ***Socially Admissible rule,*** *or simply an* ***Admissible rule,*** *is a function,* $\overline{\varphi} : B_P \rightarrow \mathbb{R}^n_+$, *such that for each* $(E, c, P) \in B_P$,

(a) $\sum_{i \in N} \overline{\varphi}_i(E, c, P) = E$,

(b) $0 \leq \overline{\varphi}_i(E, c, P) \leq c_i$ *for each* $i \in N$, *and*

(c) $\overline{\varphi}$ *satisfies all properties in* P.

Let Φ denote the set of all rules and let $\Phi(P)$ be the subset of rules satisfying P.

Taking extended problems as a starting point, we propose a new lower bound on awards based on the application of the ordinary meaning of guarantee. That is, each agent will receive at least her lower amount among those ones provided by all the rules satisfying the selected properties. Formally,

Definition 5 *Given (E, c, P) in B_P, the **P-Safety,** s, is for each $i \in N$,*

$$s_i(E, c, P) = \min_{\varphi \in \Phi(P)} \{\varphi_i(E, c)\}.$$

Now, using the previous idea of guarantee, our new lower bound on awards, called *Respect of P-Safety,* demands that each claimant receives at least her *P-Safety.*

Definition 6 *Given $P \in \mathbb{P}$, a rule φ satisfies **Respect of P-Safety** if for each $(E, c) \in B$ and each $i \in N$, $\varphi_i(E, c) \geq s_i(E, c, P)$.*

Since, in general, the sum of the *P-Safeties* of a problem (E, c, P) will not exhaust the available quantity of resources, properties requiring composition from such a lower bound arise in a natural way. These properties ask the awards vector to be equivalently obtainable (i) directly, or (ii) by first assigning to each agent her lower bound on awards, adjusting claims down by these amounts, and finally, applying the rule to divide the remainder. The following definition applies this idea to our bound on awards.

Definition 7 *Given $P \in \mathbb{P}$, a rule φ satisfies **P-Safety First** if for each $(E, c) \in B$ and each $i \in N$, $\varphi_i(E, c) = s_i(E, c, P) + \varphi_i(E - \sum_{i \in N} s_i(E, c, P), c - s(E, c, P))$.*

Although many of the proposed lower bound on awards are respected by most of the rules, composition from such lower bounds is quite demanding. For instance, *Respect of Minimal Right* is satisfied by any rule, however none of the *Proportional, Constrained Equal Awards* or *Minimal Overlap* rules satisfy *Minimal Right First* (See Thomson [20]). In fact, imposing this kind of composition or equivalently applying a recursive method from a lower bound on awards, has been used to propose new rules. Next, following the previous ideas, we define the recursive application of our *P-Safety*, which will be called the *Recursive P-Safety Process*.

Definition 8 *Given $m \in \mathbb{N}$, the* ***Recursive P-Safety Process*** *at the m-th step,* RS^m*, associates for each $(E, c, P) \in B_P$ and each $i \in N$,*

$$[RS^m(E,c,P)]_i = s_i(E^m, c^m, P),$$

$$\text{where } (E^1, c^1) \equiv (E, c) \text{ and for } m \geq 2,$$

$$(E^m, c^m) \equiv (E^{m-1} - \sum_{i \in N} s_i(E^{m-1}, c^{m-1}, P_t), c^{m-1} - s(E^{m-1}, c^{m-1}, P)).$$

According to this process, an agent will get at the first step her *P-Safety* of the original problem. At the second step, we redefine the residual problem, in which the estate is the remaining resources and the claims are adjusted down by the amounts just given. Then each agent receives her *P-Safety* of such a residual problem, and so on. Let us note that, in general, it can not be ensured that the sum of the amounts that agents get in each and everyone of the previous steps provides an *Admissible* rule, but when that happens, we will call it the *Recursive P-Safety* rule[1].

Definition 9 *The* ***Recursive P-Safety*** *rule, $\bar{\varphi}^R$, associates for each $(E, c, P) \in B_P$ and each $i \in N$, $\bar{\varphi}_i^R(E,c,P) = \sum_{m=1}^{\infty} [RS^m(E,c,P)]_i$, whenever*

(i) $\sum_{i \in N} \left(\sum_{m=1}^{\infty} [RS^m(E,c,P)]_i \right) = E$*, and*

(ii) $\bar{\varphi}^R$ *satisfies all properties in P.*

[1] Let us note that non-negativity and claim boundedness are satisfied by construction. Moreover, it can be checked (by adapting the proof of Remark 3 in Appendix 1) that whenever the *P-Safety* provides, in each step, a positive amount to some agent, efficiency is met.

4 Main results

In this section we consider three possible choices of '*Commonly Accepted Equity Principles'* by a society to apply the approach introduced previously for bounding awards.

Specifically,

$P_1 = \{$*Resource Monotonicity*, *Super-Modularity* and *Midpoint Property*$\}$,

$P_2 = \{$*Resource Monotonicity* and *Order-Preservation*$\}$ and

$P_3 =\{$*Resource Monotonicity*, *Order Preservation* and *Midpoint Property*$\}$.

Starting from Bosmans and Lauwers [3] and Schummer and Thomson [18], and using the concept of dual rule and the fact that all the properties considered are *Self-Dual,* we obtain our next two results which define the opposite extreme rules marking out the region of admissible path of awards for P_1 and P_2, respectively.

Theorem 1 *For each $(E, c) \in B$, the Dual of Piniles' rule is the only one in $\Phi(P_1)$ such that: (i) the gap between the smallest and the largest loss any claimant incurs is the smallest, and (ii) the variance of the losses incurred by all claimants is the smallest.*

Proof. See Section 4 in Jiménez-Gómez and Marco-Gil [11]. ■

Theorem 2 *For each $(E, c) \in B$, the Constrained Equal Losses rule is the only one in Φ such that: (i) the gap between the smallest and the largest loss any claimant incurs is the smallest, and (ii) the variance of the losses incurred by all claimants is the smallest.*

Proof. See Section 4 in Jiménez-Gómez and Marco-Gil [11]. ■

The following lemmas determine the *P-Safety* for both P_1 and P_2.

Lemma 3 *Given (E, c, P_1) in B_P, the P-Safety, s,is for each $i \in N$,*

$$s_i(E, c, P_1) = \min \{Pin_i(E, c), DPin_i(E, c)\} .$$

Proof. See Section 4 in Jiménez-Gómez and Marco-Gil [11]. ■

Lemma 4 *Given* (E, c, P_2) *in* B_P*, the P-Safety,* s*,is for each* $i \in N$*,*

$$s_i(E, c, P_2) = \min \{ CEA_i(E, c), CEL_i(E, c) \}.$$

Proof. See Section 4 in Jiménez-Gómez and Marco-Gil [11]. ■

Next theorems prove that the recursive application of the *P-Safety* retrieves the *Dual of Piniles'* rule for P_1 and the *Constrained Equal Losses* rule for P_2.

Theorem 5 *For each* $(E, c, P_1) \in B_P$*, the Recursive P-Safety rule is the Dual of Piniles' rule,* $\bar{\varphi}^R(E, c, P_1) = DPin(E, c)$*.*

Proof. See Appendices 1 and 2 in Jiménez-Gómez and Marco-Gil [11]. ■

Theorem 6 *For each* $(E, c, P_2) \in B_P$*, the Recursive P-Safety rule is the Constrained Equal Losses rule,* $\bar{\varphi}^R(E, c, P_2) = CEL(E, c)$*.*

Proof. See Appendices 1 and 3 in Jiménez-Gómez and Marco-Gil [11]. ■

These rules represent the extreme and opposite ways of sharing awards among conflicting claims in the set of *Admissible* rules according to the imposed requirements. Moreover we have proved, contrary to the first intuition which would be to get something in the middle of these extreme rules when applying the recursive procedure, that the corresponding *Recursive P-Safety* rule retrieves one of these extremes; the extreme favoring the largest claims when focusing on awards or the opposite one when sharing 'what is missing'. In this sense, our results can be interpreted as new basis for old rules. So that, a natural question comes up:

'*For any appealing equity principles set, Would its P-Safety recursive application recover one of the extremes which define the area of all the Admissible rules?*'

To answer this question let us consider the set of equity principles P_3 which is an 'intermediate' situation more permissive than P_1, since we require *Order Preservation* instead of *Super-Modularity*, but more restrictive than P_2, since we add the Midpoint Property.

Starting from Chun, Schummer and Thomson [4], and using the concept of dual rule and the fact that all the properties considered are *Self-Dual,* we obtain our next result which define the opposite extreme rules marking out the region of admissible path of awards for P_3.

Theorem 7 *For P ={Resource Monotonicity and Midpoint Property} and for each $(E, c) \in B$, the Dual Constrained Egalitarian rule is the only one in $\Phi(P)$ such that: (i) the gap between the smallest and the largest loss any claimant incurs is the smallest, and (ii) the variance of the losses incurred by all claimants is the smallest*

Proof. See Section 5 in Jiménez-Gómez and Marco-Gil [11]. ■

The following lemma determine the *P-Safety* for both P_3.

Lemma 8 *Given (E, c, P_3) in B_p, the P-Safety, s, is for each $i \in N$,*

$$s_i(E, c, P_3) = \min \{CE_i(E, c), DCE_i(E, c)\}.$$

Proof. See Section 5 in Jiménez-Gómez and Marco-Gil [11]. ■

In this context, we show that, although for the two-person problems the recursive application of the *P-Safety* for P_3 retrieves the *Dual Constrained Egalitarian* rule, this fact can not be generalized.

Theorem 9 *For each two-person Problem with Legitimate Principles in B_P with $P = P_3$ and each $i \in \{1, 2\}$, the Recursive P-Safety rule is the Dual Constrained Egalitarian rule, $\bar{\varphi}_i^R(E, c, P_3)$ =$DCE_i(E, c)$.*

Proof. See Appendices 1 and 4 in Jiménez-Gómez and Marco-Gil [11]. ■

Proposition 10 *There is a problem,* $(E, c) \in B$, *for which the sum of all the amounts that agents get by the recursive application of her P-Safety for* P_3 *does not coincide with the Dual Constrained Egalitarian rule,*

$$\sum_{m=1}^{\infty} [RS^m(E, c, P_3)] \neq DCE(E, c).$$

Proof. See Appendices 1 and 5 in Jiménez-Gómez and Marco-Gil [11]. ■

Our next proposition points out that the composition of 'appealing' equity principles and 'natural' processes for finding solutions does not always guarantee desirable results. Particularly, it emphasizes both the need of being very careful when establishing the equity principles of the society if the procedure seems appropriated, and the need of searching processes which respect these principles, if they are considered irremovable.

Proposition 11 *For* P_3*, the rule obtained by adding up all the amounts provided by the Recursive P-Safety Process does not satisfy Resource Monotonicity.*

Proof. See Appendices 1 and 5 in Jiménez-Gómez and Marco-Gil [11]. ■

Finally, let us note that the previous analysis can be applied on losses by using the idea of duality. When focusing on losses, the starting point will be the same sets of *'Commonly Accepted Equity Principles'*, P_1 and P_2, since all the considered properties are *Self-Dual.* Moreover, defining for each $(E, c) \in B$, the *P-Safety* for the associated problem (L, c) and applying it recursively, it can be shown that *Piniles',* the *Constrained Equal Awards* rules are retrieved for P_1 and P_2, respectively, and, for P_3, the *Constrained Egalitarian* rule for two-person problems but without guaranteeing an *Admissible* rule for the n-person case.

Let us conclude this section noting that, probably, it would not be difficult finding a society which accepts *Resource Monotonicity*, *Order Preservation* and *Midpoint Property,* willingly, and which considers fairly 'natural' our *Recursive P-Safety Process.* However, we are sure that the result of this puzzle would not be accepted by any member of such a society, since it provides a rule which does not satisfy one of the equity principles upon which the society initially agreed to found its decisions; that is, *Resource Monotonicity*, one of the properties considered unquestionable in the literature.

5 Conclusions

We have taken up again a research line which has underlied the theoretical analysis of bankruptcy problems from its beginning: the search of a 'fair' minimum amount that each agent should receive when facing these situations. In this context, our main contribution is a new method for bounding awards based on a set of *'Commonly Accepted Equity Principles'* by a society. Starting from this set, our proposal, called *P-Safety*, is obtained by assigning each agent the lower amount she gets according to all admissible rules for such a society. The fact that some part of the resources will be still available once we allocate each agent this amount has led us to introduce the *Recursive P-Safety* rule, which lies in the recursive application of our new bound.

Our main results are obtained by particularizing the previous methodology to different equity principle sets which can be interpreted, from our point of view, as 'basic' requirement. We have retrieved, respectively, for two possible societies, restrictive and permissive, the *Dual of Piniles'* and the *Constrained Equal Losses* rules when focussing on awards; and *Piniles'* and the *Constrained Equal Awards* rules when sharing losses. Next, we have ascertained that the composition of both 'reasonable' principles and recursivity, a 'standard' way of exhausting the resources, does not always provide desirable distributions. To show this fact we have not defined an artificial set of legitimate properties. Rather the contrary, by considering a society 'in the middle', we have shown that the *Recursive P-Safety* rule does not satisfy one of the equity principles upon which such a society initially agreed to found its decisions. So that, the necessity of studying in depth the consequences of the social agreements on both principles and procedure has been emphasized, since when putting them together could become meaningless.

Summarizing, this paper: (i) offers the understanding of old bankruptcy rules from a new angle, (ii) warns of the dangers that may involve the composition of 'a priori' appropriate pieces of a puzzle, and (iii) strengthens and complements the noncooperative support of the *Constrained Equal Losses* rule provided by Herrero [9], since from totally different starting points, although under somehow similar mathematical modelization, retrieve the same bankruptcy rule, that is, axiomatic and strategic methods converge.

Therefore, the following questions remain open: the study of the *Dual of Piniles'* and the *Dual Constrained Egalitarian* rules from the strategic point of view; the search of new procedures which ensure the compatibility with socially accepted equity principles; and the analysis of conditions on the legitimate principle sets for guaranteeing their fulfillment when applying our recursive process.

References

[1] AUMANN, R., MASCHLER, M. (1985). "Game theoretic analysis of a bankruptcy problem from the Talmud." Journal of Economic Theory, Vol. 36, pp. 195–213.

[2] ALCALDE, J., MARCO, M.C., and SILCA. J.A. (2005). "Bankruptcy games and the Ibn Ezra's proposal." Economic Theory Vol. 26, pp. 103-114.

[3] BOSMANS, K., LAUWERDS, L. (2007). "Lorenz comparisons of nine rules for the adjudication of conflicting claims." CES Discussion Paper 07.05, Katholieke Universiteit Leuven.

[4] CHUN, Y., SCHUMMER, J., THOMSON, W. (2001). "Constrained Egalitarianism: a new solution to bankruptcy problems." Seoul Journal of Economics Vol. 14, pp. 269-297.

[5] CURIEL, I., MASCHLER, M., TIJS, S.H. (1987). "Bankruptcy games." Zeitschrift fu¨r Operations Research Vol. 31, pp. A143–A159.

[6] DAGAN, N., SERRANO, R. VOLIJ, O. (1997). "A non-cooperative view of consistent bankruptcy rules." Games and Economic Behavior Vol. 18, pp. 55-72.

[7] DOMINGUEZ, D. (2007). "Lower bounds and recursive methods for the problem of adjudicating conflicting claim." CIE Discussion Paper Series 07-05, Instituto Tecnológico Autónomo de México.

[8] DOMINGUEZ, D. and THOMSON, W. (2006). "A new solution to the problem of adjudicating conflicting claims." Economic Theory Vol. 28, pp. 283-307.

[9] HERRERO, C., VILLAR, A., (2001). "The three musketeers: four classical solutions to bankruptcy problems." Mathematical Social Sciences Vol. 39, pp. 307–328.

[10] HERRRERO, C., VILLAR, A. (2001). "Sustainability in bankruptcy problems." TOP Vol. 10, pp. 261-273.

[11] JIMENEZ-GOMEZ, J.M., MARCO-GIL, M.C. (2008). "A new approach for bounding awards in bankruptcy problems." A discussion WP-AD 2008-07, Instituto Valenciano de Investigaciones Económicas.

[12] MORENEO–TERNERO, J. and VILLAR, A. (2004). "The Talmud rule and the securement of agents' awards." Mathematical Social Sciences Vol. 47, pp. 245-257.

[13] MOULIN, H. (2000). Priority rules and other asymmetric rationing methods. Econometrica 68, 643-684.

[14] MOULIN, H. (2002). "Axiomatic cost and surplus-sharing." Arrow, K., Sen, A., Suzumura, K. (Eds.), The Handbook of Social Choice and Welfare, Vol. 1. Elsevier, Amsterdam, pp. 289– 357.

[15] O'NEILL, B. (1982). "A problem of rights arbitration from the Talmud." Mathematical Social Sciences Vol. 2, pp. 345–371.

[16] HOKARI, T. and THOMSON, W. (2003). "Claims problems and weighted generalizations of the Talmud rule." Economic Theory Vol. 21, pp. 241-261.

[17] PINILES, H.M. (1861). "Drkah shel Torah." Forester, Viena.

[18] SCHUMMER, J., THOMSON, W. (1997). "Two derivations of the uniform rule." Economics Letters Vol. 55, pp. 333-337.

[19] SPRUMONT, Y. (1991). "The division problem with single-peaked preferences: a characterization of the uniform allocation rule." Econometrica Vol. 49, pp. 509–519.

[20] THOMSON, W., 2003. "Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey." Mathematical Social Sciences Vol. 45, pp. 249-297.

[21] YOUNG, P. (1988). "Distributive justice in taxation." Journal of Economic Theory Vol. 43, pp. 321-335.