Generalized Conformal Symmetry and Extended Objects from the Free Particle[1]

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Abstract

The algebra of linear and quadratic function of basic observables on the phase space of either the free particle or the harmonic oscillator possesses a finite-dimensional anomaly. The quantization of these systems outside the critical values of the anomaly leads to a new degree of freedom which shares its internal character with spin, but nevertheless features an infinite number of different states. Both are associated with the transformation properties of wave functions under the Weyl-symplectic group $WSp(6,\Re)$. The physical meaning of this new degree of freedom can be established, with a major scope, only by analysing the quantization of an infinite-dimensional algebra of diffeomorphisms generalizing string symmetry and leading to more general extended objects.

1 Introduction

The irreducible representations of the Schrödinger group were first studied by Niederer [1] and Perroud [2], who found for some of these an unclear association with the concept of elementary systems, as a consequence of the appearance of an infinite number of internal states. These internal states correspond to the infinite-dimensional carrier space that supports the irreducible representations with Bargmann index k [3] of the (non-compact) $SL(2,\Re)$ subgroup replacing the time translation, in either the Galilei or the harmonic oscillator (Newton) group, to give the Schrödinger symmetry.

Representations of the Schrödinger group, or more generally $WSp(6,\Re)$, with non-trivial k should correspond to linear systems (free particle, harmonic oscillator, etc.) with a new internal degree of freedom, much in the same way representations of the (compact) $SO(3)$ subgroup

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with non-trivial index s are associated with elementary particles with spin degree of freedom, although in this case supporting a finite number of states ($2s+1$ indeed). However, unlike spin, which occurs in nature for any half-integer value of s, only the value $k = \frac{d}{4}$ (d stands for the spatial dimension) is currently found in physical systems such as those described in Quantum Optics [4].

The differences and analogies between k and s are clarified if we realize that the Schrödinger group is an anomalous symmetry [6] and that $k = \frac{d}{4}$ is the quantum value of the anomaly. This means that $k = \frac{d}{4}$ is the value that quantization associates with the naïvely expected, classical value $k = 0$ corresponding to a classical system without any extra internal degree of freedom (hereafter we shall call $\bar{s} = -k$ the *symplectic spin* of just the *symplin* for the sake of brevity). Thus, roughly speaking we can say that this value of k corresponds to a quantum system without symplin.

However, the possibility of a non-trivial value of k, even though anomalous, prompts us to wonder whether or not quantizing a free system for a non-critical value of k makes any sense. In this paper, we provide a detailed quantization of a linear system (three-dimensional isotropic harmonic oscillator) bearing both spin and symplin, seeking tp clarify the behaviour of the Hilbert space of the theory according to the values of k, in full agreement with the behaviour of more standard, infinite-dimensional, anomalous theories.

Unlike the spin, the physical meaning of which seems to be well described in terms of fermionic and bosonic objects, the symplin does not appear to fit any known characteristic of the elementary particle. Rather, it seems to be understood as forming part of a larger set of degrees of freedom originating in the free particle and conforming an extended object which proves to generalize other physical systems bearing conformal symmetry. More precisely, this extended object arises when trying to quantize more classical observables than those allowed by the well-known *no-go* theorems [10], the Schrödinger anomaly being the first obstruction to standard quantization. In fact, a proper choice of an infinite-dimensional basis of the classical Poisson algebra on the phase space of a linear system (the free particle or the harmonic oscillator) and an adequate treatment of Lie-algebra cohomology and central extensions, as well as anomalies, will lead to a dynamical system which should be understood as an extended object with an infinite number of degrees of freedom. This symmetry, which in particular contains string symmetry, has the Schrödinger algebra as the maximal finite-dimensional subalgebra. In this way, the particle with symplin mentioned above would appear as the simplest (and the only finite-dimensional) part of this generalized object, yet possessing one of these extended degrees of freedom.

This paper is organized as follows. Sec. II is devoted to presenting the Group Approach to Quantization (GAQ) technique, which we shall use extensively through the paper. This formalism proves especially suitable in discussing the connection between dynamical degrees of freedom and group cohomology and, therefore, the role played by anomalies. In Sec. III, we compute and fully describe the irreducible representations of the Schrödinger group with non-trivial indices s and k, as corresponding to a particle with spin and symplin. In particular, we dwell on the specific situation $k = \frac{d}{4}$, for which the Hilbert space acquires an exceptional reduction, reflecting the presence of the anomaly; we compare this situation with that of ordinary bosonic string, which is better known. Finally, in Sec. IV, and to best understand the meaning of the extra degree of freedom introduced here, we pursue in depth the ideas which have lead us to the k degree of freedom. In so doing, we seek to quantize the Poisson algebra associated with a free particle far beyond no-go theorems and naïve analyticity obstructions. We end with a generalization of the DeWitt w_∞ algebra, which contains, in particular, the full Virasoro algebra $\{L_n, n \in Z\}$ and the algebra of string modes $\{\alpha_m\}$, closing a semi-direct product. The actual quantization of the entire algebra and the analysis of the possible anomalies is only sketched.

2 Quantization on a group, (pseudo-)co-homology and anomalies

The starting point of GAQ [11] is a group $\tilde{G}$ (the quantization group) with a principal fiber bundle structure $\tilde{G}(M,T)$, having T as the structure group and M the base. The group T generalizes the phase invariance of Quantum Mechanics. Although the situation can be even more general [12], we shall start with the rather general case in which $\tilde{G}$ is a central extension of a group G by T [$T = U(1)$ or even $T = \mathbf{C}^* = \Re^+ \otimes U(1)$]. For the one-parameter group $T = U(1)$, the group law for $\tilde{G} = \{\tilde{g} = (g, \zeta)/g \in G, \zeta \in U(1)\}$ adopts the following form:

$$\tilde{g}' * \tilde{g} = (g' * g, \zeta'\zeta e^{i\xi(g',g)}) \tag{1}$$

where $g'' = g' * g$ is the group operation in G and $\xi(g', g)$ is a two cocycle of G on $\Re$ fulfilling:

$$\xi(g_2, g_1) + \xi(g_2 * g_1, g_3) = \xi(g_2, g_1 * g_3) + \xi(g_1, g_3) \quad , g_i \in G. \tag{2}$$

In the general theory of central extensions [13], two two-cocycles are said to be equivalent if they differ in a coboundary, i.e. a cocycle which can be written in the form $\xi(g', g) = \delta(g' * g) - \delta(g') - \delta(g)$, where $\delta(g)$ is called the generating function of the coboundary. However, although cocycles differing on a coboundary lead to equivalent central extensions as such, there are some coboundaries which provide a non-trivial connection on the fibre bundle $\tilde{G}$ and Lie-algebra structure constants different from that of the direct product $G \otimes U(1)$. These are generated by a function δ with a non-trivial gradient at the identity, and can be divided into equivalence Pseudo-cohomology subclasses: two pseudo-cocycles are equivalent if they differ in a coboundary generated by a function with trivial gradient at the identity [14, 15, 7]. Pseudo-cohomology plays an important role in the theory of finite-dimensional semi-simple group, as they have trivial cohomology. For them, Pseudo-cohomology classes are associated with coadjoint orbits [7].

The right and left finite actions of the group $\tilde{G}$ on itself provide two sets of mutually commuting (left- and right-, respectively) invariant vector fields:

$$\tilde{X}^L_{\tilde{g}^i} = \frac{\partial \tilde{g}''^j}{\partial \tilde{g}^i}\Big|_{\tilde{g}=e} \frac{\partial}{\partial \tilde{g}^j}, \quad \tilde{X}^R_{\tilde{g}^i} = \frac{\partial \tilde{g}''^j}{\partial \tilde{g}'^i}\Big|_{\tilde{g}'=e} \frac{\partial}{\partial \tilde{g}^j}, \quad \left[\tilde{X}^L_{\tilde{g}^i}, \tilde{X}^R_{\tilde{g}^j}\right] = 0, \tag{3}$$

where $\{\tilde{g}^j\}$ is a parameterization of $\tilde{G}$. The GAQ program continues finding the left-invariant 1-form Θ (the *Quantization 1-form*) associated with the central generator $\tilde{X}^L_\zeta = \tilde{X}^R_\zeta, \zeta \in T$, that is, the T-component $\tilde{\theta}^{L(\zeta)}$ of the canonical left-invariant 1-form $\tilde{\theta}^L$ on $\tilde{G}$. This constitutes the generalization of the Poincaré-Cartan form of Classical Mechanics (see [16]). The differential $d\Theta$ is a *presymplectic* form and its *characteristic module*, $\mathrm{Ker}\Theta \cap \mathrm{Ker}d\Theta$, is generated by a left subalgebra $\mathcal{G}_\Theta$ called *characteristic subalgebra*. The quotient $(\tilde{G}, \Theta)/\mathcal{G}_\Theta$ is a *quantum manifold* in the sense of Geometric Quantization [17, 18, 19, 20]. The trajectories generated by the vector fields in $\mathcal{G}_\Theta$ constitute the generalized equations of motion of the theory (temporal evolution, rotations, gauge transformations, etc...), and the "Noether" invariants under those equations are $F_{\tilde{g}^j} \equiv i_{\tilde{X}^R_{\tilde{g}^j}}\Theta$, that is, the contraction of right-invariant vector fields with the Quantization 1-form. Those vector fields with null Noether charge are called *gauge* [21] and the subspace expanded by all the gauge vector fields is termed *gauge subalgebra*, which proves to be an ideal of the whole algebra of $\tilde{G}$.

Let $\mathcal{B}(\tilde{G})$ be the set of complex-valued *T-functions* on $\tilde{G}$ in the sense of principal bundle theory:

$$\psi(\zeta * \tilde{g}) = D_T(\zeta)\psi(\tilde{g}), \quad \zeta \in T \tag{4}$$

where D_T is the natural representation of T on the complex numbers $\mathbf{C}$. The representation of $\tilde{G}$ on $\mathcal{B}(\tilde{G})$ generated by $\tilde{\mathcal{G}}^R = \{\tilde{X}^R\}$ is called *Bohr Quantization* and is *reducible.* The true *Quantization* is achieved when this pre-quantization is fully reduced, usually by means of the restrictions imposed by a *full polarization* $\mathcal{P}$:

$$\tilde{X}^L\psi_p = 0, \;\; \forall \tilde{X}^L \in \mathcal{P} \tag{5}$$

which is a maximal, horizontal (i.e. in $Ker\Theta$) left subalgebra of $\tilde{\mathcal{G}}^L$ which contains $\mathcal{G}_\Theta$. It should be noted, however, that the existence of a full polarization, containing the whole subalgebra $\mathcal{G}_\Theta$, is not guaranteed. In case of such a breakdown, called *anomaly*, or simply by the desire of choosing a preferred representation space, a *higher-order polarization* must be imposed [6, 7, 8]. A higher-order polarization is a maximal subalgebra of the left enveloping algebra $U\tilde{\mathcal{G}}^L$ with no intersection with the abelian subalgebra of powers of $\tilde{X}^R_\zeta$.

The group $\tilde{G}$ is irreducibly represented on the space $\mathcal{H}(\tilde{G}) \equiv \{|\psi\rangle\}$ of (higher-order) polarized wave functions. If we denote by

$$\psi_p(\tilde{g}) \equiv \langle \tilde{g}_p|\psi\rangle\,, \;\; \psi'^*_p(\tilde{g}) \equiv \langle\psi'|\tilde{g}_p\rangle \tag{6}$$

the coordinates of the "ket" $|\psi\rangle$ and the "bra" $\langle\psi|$ in a representation defined through a polarization $\mathcal{P}$ (first or higher order), then, a scalar product on $\mathcal{H}(\tilde{G})$ can be naturally defined as:

$$\langle\psi'|\psi\rangle \equiv \int_{\tilde{G}} v(\tilde{g})\psi'^*_p(\tilde{g})\psi_p(\tilde{g}), \tag{7}$$

where

$$v(\tilde{g}) \equiv \theta^L_{\tilde{g}^i} \wedge \overset{\dim(\tilde{G})}{\cdots} \wedge \theta^L_{\tilde{g}^n} \tag{8}$$

is the left-invariant integration volume in $\tilde{G}$ and

$$1 = \int_{\tilde{G}} |\tilde{g}_p\rangle v(\tilde{g}) \langle\tilde{g}_p| \tag{9}$$

formally represents a *closure* relation. A direct computation proves that, with this scalar product, the group $\tilde{G}$ is unitarily represented through the *left* finite action (ρ denotes the representation)

$$\langle\tilde{g}_p|\rho(\tilde{g}')|\psi\rangle \equiv \psi_p(\tilde{g}'^{-1} * \tilde{g}) \tag{10}$$

Constraints are consistently incorporated into the theory by enlarging the structure group T (which always includes $U(1)$), i.e, through T-function conditions:

$$\rho(\tilde{t})|\psi\rangle = D^{(\epsilon)}_T(\tilde{t})|\psi\rangle\,, \;\; \tilde{t} \in T \tag{11}$$

or, for continuous transformations,

$$\tilde{X}^R_{\tilde{t}}|\psi\rangle = dD^{(\epsilon)}_T(\tilde{t})|\psi\rangle \ , \tag{12}$$

$D^{(\epsilon)}_T$ means a specific representation of T [the index ϵ parametrizes different (inequivalent) quantizations] and $dD^{(\epsilon)}_T$ is its differential.

Ofcourse, for a non-central structure group T, not all the right operators $\tilde{X}^R_{\tilde{g}}$ will preserve these constraints; a sufficient condition for a subgroup $\tilde{G}_T \subset \tilde{G}$ to preserve the constraints is (see [22, 23]):

$$\left[\tilde{G}_T, T\right] \subset \mathrm{Ker} D^{(\epsilon)}_T \tag{13}$$

[note that, for the trivial representation of T, the subgroup $\tilde{G}_T$ is in fact the *normalizer* of T]. $\tilde{G}_T$ takes part of the set of *good* operators [12], of the enveloping algebra $U\tilde{\mathcal{G}}^R$ in general, for which the subgroup T behaves as a *gauge* group (see [21] for a thorough study of gauge symmetries and constraints from the point of view of GAQ). A more general situation can be posed when the constraints are lifted to the higher-order level, not necessarily first order as in (12); that is, they are a subalgebra of the right enveloping algebra $U\tilde{\mathcal{G}}^R$. A good example of this last case is found when one selects representations labelled by a value ϵ of some Casimir operator C of a subgroup $\tilde{G}_C$ of $\tilde{G}$ [24].

In the more general case in which T is not a trivial central extension, $T \neq \check{T} \times U(1)$, where $\check{T} \equiv T/U(1)$ -i.e. T contains second-class constraints- the conditions (12) are not all compatible and we must select a subgroup $T_B = T_p \times U(1)$, where T_p is the subgroup associated with a right polarization subalgebra of the central extension T (see [12]).

For simplicity, we have sometimes made use of infinitesimal (geometrical) concepts, but all this language can be translated to their finite (algebraic) counterparts (see [12]), a desirable way of proceeding when discrete transformations are incorporated into the theory.

Before ending this section, we eish to insist a bit more on the concept of (algebraic) anomaly, which will be quite relevant to what follows. We have introduced the concept of full polarization subalgebra intended to reduce the representation obtained through the right-invariant vector fields acting on T-equivariant functions on the group. It contains "half" of the symplectic vector fields as well as the entire characteristic subalgebra. If the full reduction is achieved, the whole set of physical operators can be rewritten in terms of the basic ones, i.e. those for which the left counterpart is not in the characteristic subalgebra $\mathcal{G}_\Theta$. For instance, the energy operator for the free particle can be written as $\frac{\hat{p}^2}{2m}$, the angular momentum in 3+1 dimensions is the vector product $\hat{\mathbf{x}} \times \hat{\mathbf{p}}$, or the energy for the harmonic oscillator is $\hat{a}^\dagger \hat{a}$ (note that, since we are using first-order polarizations, all these operators are really written as first-order differential operators).

However, the existence of a full polarization is guaranteed only for semisimple and solvable groups [5]. We define an *anomalous* group [6, 7] as a central extension $\tilde{G}$ which does not admit any full polarization for certain values of the (pseudo-)cohomology parameters, called the *classical* values of the anomaly (they are called classical because they are associated with the coadjoints orbits of the group $\tilde{G}$, that is, with the classical phase space of the physical system). Anomalous groups feature another set of values of the (pseudo-)cohomology parameters, called the *quantum* values of the anomaly, for which the carrier space associated with a full polarization contains an invariant subspace. For the classical values of the anomaly, the classical solution manifold undergoes a reduction in dimension, thus increasing the number of (non-linear) relationships among Noether invariants, whereas for the quantum values the number of basic operators decreases on the invariant (reduced) subspace due to the appearance of (higher-order) relationships among the quantum operators.

We should comment that the anomalies we are dealing with in this paper are of *algebraic* character in the sense that they appear at the Lie algebra level, and must be distinguished from the *topologic anomalies* which are associated with the non-trivial homotopy of the (reduced) phase space [22].

The non-existence of a full polarization is traced back to the presence in the characteristic subalgebra, for certain values of the (pseudo-)cohomology parameters (the classical values of the anomaly), of some elements the adjoint action of which is not diagonalizable in the "$x-p$-like" algebra subspace. The anomaly problem presented here parallels that of the non-existence of invariant polarizations in the Kirillov-Kostant co-adjoint orbits method [25], and the conventional anomaly problem in Quantum Field Theory which manifests itself through the appearance of central charges in the quantum current algebra, absent from the classical (Poisson bracket) algebra [26].

The practical way in which an anomaly appears and how a higher-order polarization fully reduces the Hilbert space of the quantum theory for the particular quantum value of the anomaly will be apparent with the finite-dimensional example discussed in the next section and the comparison with other much better known infinite-dimensional cases.

3 Internal degrees of freedom associated with the elementary particle

Internal degrees of freedom of a linear, quantum system with $\Re^{2d}$ as phase space are generally associated with non-trivial transformation properties of the phase ζ of the wave function under the symplectic group $Sp(2d,\Re) \subset WSp(2d,\Re)$. Their presence is evident in the emergence of central charges in the Lie algebra $sp(2d,\Re)$ of symplectic transformations –which is isomorphic to the classical Poisson algebra of all quadratic functions of the position x_j and the conjugate momentum p_j– and their origin is cohomological. The simplest case to be considered is $d=1$ (particle on a line) for which the Lie algebra of the symplectic group $Sp(2,\Re) \simeq SL(2,\Re) \simeq SU(1,1)$, isomorphic to the Poisson algebra generated by $\{\frac{1}{2}x^2, \frac{1}{2}p^2, xp\}$, appears to be naturally extended, providing a representation with Bargmann index $k=\frac{1}{4}$. As already mentioned, there is no *a priori* physical significance for other representations carrying different values of the Bargmann index k (the symplin $\bar{s}$ for us). To construct explicitly these representations and compare them with the most usual case of the spin, we shall consider the $d=3$ case and restrict ourselves to the Schrödinger subgroup of the Weyl-symplectic $WSp(6,\Re)$ group, where we have replaced the $Sp(6,\Re)$ group by its $SL(2,\Re)\otimes SO(3)$ subgroup, for which the Lie algebra is isomorphic to the Poisson algebra generated by $\{\frac{1}{2}\mathbf{x}^2,\ \frac{1}{2}\mathbf{p}^2,\ \mathbf{x}\cdot\mathbf{p};\ \mathbf{x}\times\mathbf{p}\}$. At this juncture, it will be convenient to use a oscillator-like parametrization in terms of the usual complex combinations:

$$\mathbf{a} \equiv \frac{1}{\sqrt{2}}(\mathbf{x}+i\mathbf{p})\,, \quad \mathbf{a}^* \equiv \frac{1}{\sqrt{2}}(\mathbf{x}-i\mathbf{p})\,, \tag{14}$$

where we have settled $\hbar = 1 = m = \omega$ for simplicity. In the same manner, we will consider the complexified version $SU(1,1)$ of $SL(2,\Re)$ defined as

$$SU(1,1) \equiv \left\{ \bar{U} = \begin{pmatrix} \bar{z}_1 & \bar{z}_2 \\ \bar{z}_2^* & \bar{z}_1^* \end{pmatrix}, \bar{z}_i, \bar{z}_i^* \in C / \det(\bar{U}) = |\bar{z}_1|^2 - |\bar{z}_2|^2 = 1 \right\} \tag{15}$$

and the two-covering

$$SU(2) \equiv \left\{ U = \begin{pmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{pmatrix}, z_i, z_i^* \in C / \det(U) = |z_1|^2 + |z_2|^2 = 1 \right\} \tag{16}$$

of $SO(3)$ to account for half-integer values of the spin.

Let us look at the structure of both groups as principal fibre bundles and choose a system of coordinates adapted to this fibration as follows:

$$\begin{aligned} \bar{\eta} &\equiv \frac{\bar{z}_1}{|\bar{z}_1|}, \quad \bar{\alpha} \equiv \frac{\bar{z}_2}{\bar{z}_1}, \quad \bar{\alpha}^* \equiv \frac{\bar{z}_2^*}{\bar{z}_1^*}, \qquad \bar{\eta} \in S^1, \quad \bar{\alpha}, \bar{\alpha}^* \in D_1\,; \\ \eta &\equiv \frac{z_1}{|z_1|}, \quad \alpha \equiv \frac{z_2}{z_1}, \quad \alpha^* \equiv \frac{z_2^*}{z_1^*}, \qquad \eta \in S^1, \quad \alpha, \alpha^* \in S^2\,, \end{aligned} \tag{17}$$

i.e., $SU(1,1)$ is a principal fibre bundle with fibre $U(1)$ and base the open unit disk D_1, whilst $SU(2)$ has the sphere S^2 as the base (to be precise, the coordinates α, α^*, corresponding to a local chart at the identity, are related to the stereographical projection of the sphere on the plane). The action of $SU(1,1)$ on $\mathbf{a}, \mathbf{a}^*$ can be written in matricial form as:

$$\begin{pmatrix} \mathbf{a} \\ \mathbf{a}^* \end{pmatrix} \rightarrow \bar{U} \begin{pmatrix} \mathbf{a} \\ \mathbf{a}^* \end{pmatrix} = \sqrt{\frac{1}{1 - \bar{\alpha}\bar{\alpha}^*}} \begin{pmatrix} \bar{\eta} & \alpha\bar{\eta} \\ \bar{\alpha}^*\bar{\eta}^* & \bar{\eta}^* \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{a}^* \end{pmatrix} \tag{18}$$

whereas the action of $SU(2)$ can be obtained by making use of the isometry between the 2×2 hermitian matrices with null trace $A \equiv \begin{pmatrix} a_3 & a_1 + ia_2 \\ a_1 - ia_2 & -a_3 \end{pmatrix}$ and $\Re^3$ which leads to:

$$\begin{aligned} A &\rightarrow UAU^\dagger = \frac{1}{1 + \alpha\alpha^*} \begin{pmatrix} \eta & \alpha\eta \\ -\alpha^*\eta^* & \eta^* \end{pmatrix} \begin{pmatrix} a_3 & a_1 + ia_2 \\ a_1 - ia_2 & -a_3 \end{pmatrix} \begin{pmatrix} \eta^* & -\alpha\eta \\ \alpha^*\eta^* & \eta \end{pmatrix} \Rightarrow \\ \mathbf{a} &\rightarrow R\mathbf{a}\,, \end{aligned} \tag{19}$$

where the correspondence $U \rightarrow R$ stands for the usual homomorphism between $SU(2)$ and $SO(3)$.

Now let us write, in compact form, the group law $\tilde{g}'' = \tilde{g}' * \tilde{g}$ of the 13-parameter Schrödinger quantizing group $\tilde{G}$ in $d = 3$, which consists of a semidirect product $G = \mathbf{C}^3 \otimes_s (SU(2) \otimes SU(1,1))$ suitably extended by $U(1)$ as follows:

$$\begin{aligned} \bar{U}'' &= \bar{U}'\bar{U} \\ U'' &= U'U \quad (\text{or } R'' = R'R) \\ \begin{pmatrix} \mathbf{a}'' \\ \mathbf{a}^{*\prime\prime} \end{pmatrix} &= \begin{pmatrix} \mathbf{a} \\ \mathbf{a}^* \end{pmatrix} + \bar{U}^{-1} \begin{pmatrix} R^{-1}\mathbf{a}' \\ R^{-1}\mathbf{a}^{*\prime} \end{pmatrix} \\ \zeta'' &= \zeta'\zeta \exp\frac{1}{2}\left\{ \begin{pmatrix} \mathbf{a} & \mathbf{a}^* \end{pmatrix} \Omega\bar{U}^{-1} \begin{pmatrix} R^{-1}\mathbf{a}' \\ R^{-1}\mathbf{a}^{*\prime} \end{pmatrix} \right\} \left(\eta''\eta'^{-1}\eta^{-1}\right)^{2s} \left(\bar{\eta}''\bar{\eta}'^{-1}\bar{\eta}^{-1}\right)^{2\bar{s}}\,, \end{aligned} \tag{20}$$

where we denote $\Omega \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ the central matrix in the Bargmann cocycle and $s, \bar{s}$ represent the spin and symplin indices related to both pseudo-extensions of $SU(2)$ and $SU(1,1)$ with generating functions $\delta(U) = s\theta,\, \theta \equiv -2i \log \eta$ and $\delta(\bar{U}) = \bar{s}\bar{\theta},\, \bar{\theta} \equiv -2i \log \bar{\eta}$, respectively. Note that both spin and symplin are forced to take half integer values

$$s \equiv \frac{k}{2}, \quad \bar{s} \equiv \frac{l}{2}, \; k, l \in Z \tag{21}$$

only, for globality conditions (single-valuedness), as can be seen by expressing $\eta, \bar{\eta}$ in terms of the global coordinates $z_i, \bar{z}_i$, $i = 1,2$ like in (17) (see below).

The group law (20) will be the starting point for GAQ to obtain the irreducible representations of the Schrödinger group which, afterwards, we shall see, correspond with a three-dimensional isotropic harmonic oscillator carrying two internal degrees of freedom. To this end, let us start writing the explicit expression of the left- and right-invariant vector fields:

$$\begin{aligned}
\tilde{X}^L_\zeta &= \tilde{X}^R_\zeta = \zeta \frac{\partial}{\partial \zeta} \\
\tilde{X}^L_{\mathbf{a}} &= \frac{\partial}{\partial \mathbf{a}} + \frac{1}{2}\mathbf{a}^* \zeta \frac{\partial}{\partial \zeta}\,, \quad \tilde{X}^L_{\mathbf{a}^*} = \frac{\partial}{\partial \mathbf{a}^*} - \frac{1}{2}\mathbf{a}\zeta \frac{\partial}{\partial \zeta} \\
\tilde{X}^L_{\bar{\eta}} &= \bar{\eta}\frac{\partial}{\partial \bar{\eta}} - 2\bar{\alpha}\frac{\partial}{\partial \bar{\alpha}} + 2\bar{\alpha}^* \frac{\partial}{\partial \bar{\alpha}^*} - \mathbf{a}\frac{\partial}{\partial \mathbf{a}} + \mathbf{a}^* \frac{\partial}{\partial \mathbf{a}^*} \\
\tilde{X}^L_{\bar{\alpha}} &= -\frac{1}{2}\bar{\eta}\bar{\alpha}^* \frac{\partial}{\partial \bar{\eta}} + \frac{\partial}{\partial \bar{\alpha}} - \bar{\alpha}^{*2}\frac{\partial}{\partial \bar{\alpha}^*} - \mathbf{a}^* \frac{\partial}{\partial \mathbf{a}} - \bar{s}\bar{\alpha}^* \zeta \frac{\partial}{\partial \zeta} \\
\tilde{X}^L_{\bar{\alpha}^*} &= \frac{1}{2}\bar{\eta}\bar{\alpha}\frac{\partial}{\partial \bar{\eta}} - \bar{\alpha}^2 \frac{\partial}{\partial \bar{\alpha}} + \frac{\partial}{\partial \bar{\alpha}^*} - \mathbf{a}\frac{\partial}{\partial \mathbf{a}^*} + \bar{s}\bar{\alpha}\zeta \frac{\partial}{\partial \zeta} \\
\tilde{X}^L_\eta &= \eta \frac{\partial}{\partial \eta} - 2\alpha \frac{\partial}{\partial \alpha} + 2\alpha^* \frac{\partial}{\partial \alpha^*} - 2i(a_2 \frac{\partial}{\partial a_1} - a_1 \frac{\partial}{\partial a_2}) - 2i(a_2^* \frac{\partial}{\partial a_1^*} - a_1^* \frac{\partial}{\partial a_2^*}) \\
\tilde{X}^L_\alpha &= \frac{1}{2}\eta\alpha^* \frac{\partial}{\partial \eta} + \frac{\partial}{\partial \alpha} + \alpha^{*2}\frac{\partial}{\partial \alpha^*} + s\alpha^* \zeta \frac{\partial}{\partial \zeta} \\
&+ (a_3 \frac{\partial}{\partial a_1} - a_1 \frac{\partial}{\partial a_3}) + i(a_2 \frac{\partial}{\partial a_3} - a_3 \frac{\partial}{\partial a_2}) + (a_3^* \frac{\partial}{\partial a_1^*} - a_1^* \frac{\partial}{\partial a_3^*}) + i(a_2^* \frac{\partial}{\partial a_3^*} - a_3^* \frac{\partial}{\partial a_2^*}) \\
\tilde{X}^L_{\alpha^*} &= -\frac{1}{2}\eta\alpha \frac{\partial}{\partial \eta} + \alpha^2 \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \alpha^*} - s\alpha\zeta \frac{\partial}{\partial \zeta} \\
&+ (a_3 \frac{\partial}{\partial a_1} - a_1 \frac{\partial}{\partial a_3}) - i(a_2 \frac{\partial}{\partial a_3} - a_3 \frac{\partial}{\partial a_2}) + (a_3^* \frac{\partial}{\partial a_1^*} - a_1^* \frac{\partial}{\partial a_3^*}) - i(a_2^* \frac{\partial}{\partial a_3^*} - a_3^* \frac{\partial}{\partial a_2^*}) \\
\tilde{X}^R_{\mathbf{a}} &= \bar{z}_1^* R \frac{\partial}{\partial \mathbf{a}} - \bar{z}_2^* R \frac{\partial}{\partial \mathbf{a}^*} - \frac{1}{2}(\bar{z}_2^* R\mathbf{a} + \bar{z}_1^* R\mathbf{a}^*)\zeta \frac{\partial}{\partial \zeta} \\
\tilde{X}^R_{\mathbf{a}^*} &= -\bar{z}_2 R \frac{\partial}{\partial \mathbf{a}} + \bar{z}_1 R \frac{\partial}{\partial \mathbf{a}^*} + \frac{1}{2}(\bar{z}_1 R\mathbf{a} + \bar{z}_2 R\mathbf{a}^*)\zeta \frac{\partial}{\partial \zeta} \\
\tilde{X}^R_{\bar{\eta}} &= \bar{\eta}\frac{\partial}{\partial \bar{\eta}} \\
\tilde{X}^R_{\bar{\alpha}} &= \frac{1}{2}\bar{\eta}^{-1}\bar{\alpha}^* \frac{\partial}{\partial \bar{\eta}} + \bar{\eta}^{-2}(1 - \bar{\alpha}\bar{\alpha}^*)\frac{\partial}{\partial \bar{\alpha}} + \bar{s}\bar{\eta}^{-2}\bar{\alpha}^* \zeta \frac{\partial}{\partial \zeta} \\
\tilde{X}^R_{\bar{\alpha}^*} &= -\frac{1}{2}\bar{\eta}^3 \bar{\alpha}\frac{\partial}{\partial \bar{\eta}} + \bar{\eta}^2(1 - \bar{\alpha}\bar{\alpha}^*)\frac{\partial}{\partial \bar{\alpha}^*} - \bar{s}\bar{\eta}^2 \bar{\alpha}\zeta \frac{\partial}{\partial \zeta}
\end{aligned} \tag{22}$$

$$
\begin{aligned}
\tilde{X}^R_{\eta} &= \eta\frac{\partial}{\partial\eta} \\
\tilde{X}^R_{\alpha} &= -\frac{1}{2}\eta^{-1}\alpha^*\frac{\partial}{\partial\eta}+\eta^{-2}(1+\alpha\alpha^*)\frac{\partial}{\partial\alpha}-s\eta^{-2}\alpha^*\zeta\frac{\partial}{\partial\zeta} \\
\tilde{X}^R_{\alpha^*} &= \frac{1}{2}\eta^3\alpha\frac{\partial}{\partial\eta}+\eta^2(1+\alpha\alpha^*)\frac{\partial}{\partial\alpha^*}+s\eta^2\alpha\zeta\frac{\partial}{\partial\zeta}\,.
\end{aligned}
$$

The (left) commutators between these vector fields are:

$$
\begin{array}{lll}
\left[\tilde{X}^L_{\zeta},\text{all}\right]=0 & \left[\tilde{X}^L_{a_j},\tilde{X}^L_{a^*_k}\right]=-\delta_{jk}\tilde{X}^L_{\zeta} & \left[\tilde{X}^L_{\eta,\alpha,\alpha^*},\tilde{X}^L_{\bar{\eta},\bar{\alpha},\bar{\alpha}^*}\right]=0 \\
\left[\tilde{X}^L_{\bar{\eta}},\tilde{X}^L_{\bar{\alpha}}\right]=2\tilde{X}^L_{\bar{\alpha}} & \left[\tilde{X}^L_{\bar{\eta}},\tilde{X}^L_{\bar{\alpha}^*}\right]=-2\tilde{X}^L_{\bar{\alpha}^*} & \left[\tilde{X}^L_{\bar{\alpha}},\tilde{X}^L_{\bar{\alpha}^*}\right]=\tilde{X}^L_{\bar{\eta}}+2\bar{s}\tilde{X}^L_{\zeta} \\
\left[\tilde{X}^L_{\eta},\tilde{X}^L_{\alpha}\right]=2\tilde{X}^L_{\alpha} & \left[\tilde{X}^L_{\eta},\tilde{X}^L_{\alpha^*}\right]=-2\tilde{X}^L_{\alpha^*} & \left[\tilde{X}^L_{\alpha},\tilde{X}^L_{\bar{\alpha}}\right]=-\tilde{X}^L_{\eta}-2\bar{s}\tilde{X}^L_{\zeta} \\
\left[\tilde{X}^L_{\bar{\eta}},\tilde{X}^L_{\mathbf{a}}\right]=\tilde{X}^L_{\mathbf{a}} & \left[\tilde{X}^L_{\bar{\alpha}},\tilde{X}^L_{\mathbf{a}}\right]=0 & \left[\tilde{X}^L_{\bar{\alpha}^*},\tilde{X}^L_{\mathbf{a}}\right]=\tilde{X}^L_{\mathbf{a}^*} \\
\left[\tilde{X}^L_{\bar{\eta}},\tilde{X}^L_{\mathbf{a}^*}\right]=-\tilde{X}^L_{\mathbf{a}^*} & \left[\tilde{X}^L_{\bar{\alpha}},\tilde{X}^L_{\mathbf{a}^*}\right]=\tilde{X}^L_{\mathbf{a}} & \left[\tilde{X}^L_{\bar{\alpha}^*},\tilde{X}^L_{\mathbf{a}^*}\right]=0 \\
\left[\tilde{X}^L_{\eta},\tilde{X}^L_{a_3}\right]=0 & \left[\tilde{X}^L_{\eta},\tilde{X}^L_{a_2}\right]=2i\tilde{X}^L_{a_1} & \left[\tilde{X}^L_{\eta},\tilde{X}^L_{a_1}\right]=-2i\tilde{X}^L_{a_2} \\
\left[\tilde{X}^L_{\alpha},\tilde{X}^L_{a_3}\right]=-\tilde{X}^L_{a_1}+i\tilde{X}^L_{a_2} & \left[\tilde{X}^L_{\alpha},\tilde{X}^L_{a_2}\right]=-i\tilde{X}^L_{a_3} & \left[\tilde{X}^L_{\alpha},\tilde{X}^L_{a_1}\right]=\tilde{X}^L_{a_3} \\
\left[\tilde{X}^L_{\alpha^*},\tilde{X}^L_{a_3}\right]=-\tilde{X}^L_{a_1}-i\tilde{X}^L_{a_2} & \left[\tilde{X}^L_{\alpha^*},\tilde{X}^L_{a_2}\right]=i\tilde{X}^L_{a_3} & \left[\tilde{X}^L_{\alpha^*},\tilde{X}^L_{a_1}\right]=\tilde{X}^L_{a_3}
\end{array}
\tag{23}
$$

where we have omitted the commutators $\left[\tilde{X}^L_{\eta,\alpha,\alpha^*},\tilde{X}^L_{a^*_j}\right]$, which have the same form as for the $\tilde{X}^L_{a_j}$ fields. The *Quantization 1-form* is:

$$
\begin{aligned}
\Theta &= \frac{i}{2}(\mathbf{a}^*d\mathbf{a}-\mathbf{a}d\mathbf{a}^*) \qquad (24)\\
&+ \frac{i}{2}\frac{\bar{\alpha}^*\mathbf{a}\mathbf{a}^*+\mathbf{a}^{*2}}{1-\bar{\alpha}\bar{\alpha}^*}d\bar{\alpha}-\frac{i}{2}\frac{\bar{\alpha}\mathbf{a}\mathbf{a}^*+\mathbf{a}^2}{1-\bar{\alpha}\bar{\alpha}^*}d\bar{\alpha}^*+i\bar{\eta}^*\frac{(1+\bar{\alpha}\bar{\alpha}^*)\mathbf{a}\mathbf{a}^*+\bar{\alpha}\mathbf{a}^{*2}+\bar{\alpha}^*\mathbf{a}^2}{1-\bar{\alpha}\bar{\alpha}^*}d\bar{\eta} \\
&- i\frac{\alpha^*L_3-L_-}{1+\alpha\alpha^*}d\alpha+i\frac{\alpha L_3-L_+}{1+\alpha\alpha^*}d\alpha^*+2i\eta^*\frac{(1-\alpha\alpha^*)L_3+\alpha L_-+\alpha^*L_+}{1+\alpha\alpha^*}d\eta \\
&+ \Theta_{SU(1,1)}+\Theta_{SU(2)}-i\zeta^{-1}d\zeta \\
\Theta_{SU(1,1)} &= \frac{i\bar{s}}{1-\bar{\alpha}\bar{\alpha}^*}(\bar{\alpha}d\bar{\alpha}^*-\bar{\alpha}^*d\bar{\alpha}-4\bar{\alpha}\bar{\alpha}^*\bar{\eta}^*d\bar{\eta}) \\
\Theta_{SU(2)} &= \frac{is}{1+\alpha\alpha^*}(-\alpha d\alpha^*+\alpha^*d\alpha+4\alpha\alpha^*\eta^*d\eta)\,,
\end{aligned}
$$

where we have denoted $\mathbf{L}\equiv i(\mathbf{a}^*\times\mathbf{a})$ and $L_\pm\equiv L_1\pm iL_2$. The *characteristic module* $\text{Ker}\Theta\cap\text{Ker}d\Theta$ is generated by the left subalgebra:

$$
\mathcal{G}_\Theta=<\tilde{X}^L_{\bar{\eta}},\ \tilde{X}^L_{\eta}>\,, \tag{25}
$$

the trajectories of which represent the generalized Hamiltonian equations of motion on the 10-dimensional symplectic manifold $\mathbf{C}^3\otimes D_1\otimes S^2$ of the theory. The Noether invariants under these equations are:

$$
\begin{array}{ll}
F_{\mathbf{a}}\equiv i_{\tilde{X}^R_{\mathbf{a}}}\Theta=i(\bar{z}^*_1R\mathbf{a}^*+\bar{z}^*_2R\mathbf{a}) & F_{\mathbf{a}^*}\equiv i_{\tilde{X}^R_{\mathbf{a}^*}}\Theta=-i(\bar{z}_1R\mathbf{a}+\bar{z}_2R\mathbf{a}^*) \\
F_{\bar{\eta}}\equiv i_{\tilde{X}^R_{\bar{\eta}}}\Theta=\bar{s}\frac{-4i\bar{\alpha}\bar{\alpha}^*}{1-\bar{\alpha}\bar{\alpha}^*}+iF_{\mathbf{a}^*}\cdot F_{\mathbf{a}} & F_{\eta}\equiv i_{\tilde{X}^R_{\eta}}\Theta=s\frac{4i\alpha\alpha^*}{1+\alpha\alpha^*}+2(F_{\mathbf{a}^*}\times F_{\mathbf{a}})_3 \\
F_{\bar{\alpha}}\equiv i_{\tilde{X}^R_{\bar{\alpha}}}\Theta=\bar{s}\frac{-2i\bar{\eta}^{-2}\bar{\alpha}^*}{1-\bar{\alpha}\bar{\alpha}^*}-\frac{i}{2}F_{\mathbf{a}}\cdot F_{\mathbf{a}} & F_{\alpha}\equiv i_{\tilde{X}^R_{\alpha}}\Theta=s\frac{2i\eta^{-2}\alpha^*}{1+\alpha\alpha^*}+(F_{\mathbf{a}^*}\times F_{\mathbf{a}})_- \\
F_{\bar{\alpha}^*}\equiv i_{\tilde{X}^R_{\bar{\alpha}^*}}\Theta=\bar{s}\frac{2i\bar{\eta}^2\bar{\alpha}}{1-\bar{\alpha}\bar{\alpha}^*}+\frac{i}{2}F_{\mathbf{a}^*}\cdot F_{\mathbf{a}^*} & F_{\alpha^*}\equiv i_{\tilde{X}^R_{\alpha^*}}\Theta=s\frac{-2i\eta^2\alpha}{1+\alpha\alpha^*}+(F_{\mathbf{a}^*}\times F_{\mathbf{a}})_+\,.
\end{array}
\tag{26}
$$

These Noether invariants parametrize the classical manifold of the corresponding quantum system [note that the non-dynamical (non-basic) Noether invariants F_η and $F_{\bar{\eta}}$, coming from the vector fields the left version of which are in the characteristic subalgebra (25), are expressed in terms of the rest (the basic ones)]. One can naturally define a Poisson braket as

$$\left\{F_{\tilde{g}^j}, F_{\tilde{g}^k}\right\} \equiv i_{\left[\tilde{X}^R_{\tilde{g}^j}, \tilde{X}^R_{\tilde{g}^k}\right]}\Theta \tag{27}$$

which, according to the Lie algebra (23), reproduces the standard expressions in terms of

$$\{1,\, \mathbf{a},\, \mathbf{a}^*,\, -\frac{i}{2}\mathbf{a}^2,\, \frac{i}{2}\mathbf{a}^{*2},\, i\mathbf{a}^*\mathbf{a},\, \mathbf{a}^* \times \mathbf{a}\} \tag{28}$$

for $\bar{s} = s = 0$ only. Also, for these particular (classical) values, all Noether invariants are expressed in terms of the basics $F_{\mathbf{a}}$ and $F_{\mathbf{a}^*}$ (as it can be easily seen in (26)), obtaining a new reduction of the symplectic manifold

$$\mathbf{C}^3 \otimes D_1 \otimes S^2 \to \mathbf{C}^3 \tag{29}$$

from 10 to 6 dimensions (i.e., losing internal degrees of freedom). From the quantum point of view, this reduction is due to the enlarging of the characteristic subalgebra (25) which, now, incorporates the whole $su(2)$ and $su(1,1)$ subalgebras.

Until now, the way to address both groups $SU(2)$ and $SU(1,1)$ has been rather parallel. The difference starts when we look for a full-polarization subalgebra (5) intended to reduce the representation (4) for the $s = \bar{s} = 0$ case, i.e., when we try to represent irreducibly and unitarily the classical Poisson algebra (28) on a Hilbert space of wave functions depending arbitrarily on half of the symplectic variables, let us say, the $\mathbf{a}^*$ coordinates only. As can be easily checked in (23), whereas the $su(2)$ subalgebra of $\mathcal{G}_\Theta$ is diagonal under commutation with either $\tilde{X}^L_{\mathbf{a}}$ or $\tilde{X}^L_{\mathbf{a}^*}$ (closing a horizontal subalgebra separately), the $su(1,1)$ subalgebra is not; i.e., it mixes $\tilde{X}^L_{\mathbf{a}}$ and $\tilde{X}^L_{\mathbf{a}^*}$ and precludes a full-polarization subalgebra for this case. This obstruction is a particular example of what we have already defined as an algebraic anomaly and shares with the more conventional characterization the appearance of central charges in the quantum algebra of operators. The standard quantization solves this problem by imposing normal order by hand, leading to a quantum algebra differing from the classical (28) Poisson algebra by central (normal order) terms and providing an irreducible representation of the metaplectic group $Mp(2,\Re)$ (two-cover of $Sp(2,\Re) \simeq SU(1,1)$) with Bargmann index $k = 3/4$ ($= d/4$ in d dimensions). This situation can be seen as a "weak" (avoidable) violation of the *no-go* theorems, and we shall show in Sec. IV that one can, in fact, go further.

Let us show how GAQ solves this obstruction (reduction of the quantum representation) by means of higher-order polarizations, the existence of which will be guaranteed only for the particular (quantum) value of $\bar{s} = -\frac{3}{4}$ [$\bar{s} = -\frac{d}{4}$ in d dimensions], as opposed to the classical value of $\bar{s} = 0$ (for which the counterpart classical reduction (29) is achieved). To this end, let us firstly calculate the irreducible representations of the Schrödinger group with arbitrary spin and symplin and then show how the aforementioned reduction takes place.

A full-polarization subalgebra exists for arbitrary (non-zero) s and $\bar{s}$ which is:

$$\mathcal{P} =< \tilde{X}^L_\eta,\, \tilde{X}^L_{\bar{\eta}},\, \tilde{X}^L_\alpha,\, \tilde{X}^L_{\bar{\alpha}},\, \tilde{X}^L_{\mathbf{a}} > \,. \tag{30}$$

The general solution to the polarization equations (5) leads to a Hilbert space $\mathcal{H}^{(s,\bar{s})}(\tilde{G})$ of wave functions of the form:

$$\begin{aligned}\psi^{(s,\bar{s})}(\zeta,\eta,\alpha,\alpha^*,\bar{\eta},\bar{\alpha},\bar{\alpha}^*,\mathbf{a},\mathbf{a}^*) &= \zeta(1+\alpha\alpha^*)^{-s}(1-\bar{\alpha}\bar{\alpha}^*)^{-\bar{s}}e^{-\frac{1}{2}(\mathbf{a}^*\mathbf{a}+\bar{\alpha}\mathbf{a}^{*2})}\phi(\chi,\bar{\chi},\mathbf{b})\\ \chi &\equiv \eta^{-2}\alpha^*, \quad \bar{\chi}\equiv\bar{\eta}^{-2}\bar{\alpha}^*, \quad \mathbf{b}\equiv(1-\bar{\alpha}\bar{\alpha}^*)^{\frac{1}{2}}\bar{\eta}^*R\mathbf{a}^*\,. \end{aligned} \tag{31}$$

A scalar product can be given through the invariant integration volume (8) of $\tilde{G}$:

$$\begin{aligned} v(\tilde{g}) &= i\frac{1}{(1+\alpha\alpha^*)}\frac{1}{(1+\bar{\alpha}\bar{\alpha}^*)}\left[\prod_{j=1}^{3}d\mathrm{Re}(a_j)\wedge d\mathrm{Im}(a_j)\right]\\ &\wedge \left[d\mathrm{Re}(\alpha)\wedge d\mathrm{Im}(\alpha)\wedge\eta^{-1}d\eta\right]\wedge\left[d\mathrm{Re}(\bar{\alpha})\wedge d\mathrm{Im}(\bar{\alpha})\wedge\bar{\eta}^{-1}d\bar{\eta}\right]\wedge\zeta^{-1}d\zeta\,. \end{aligned} \tag{32}$$

Let us call

$$\psi^{(m,\bar{m})}_{\mathbf{n}}\equiv\zeta(1+\alpha\alpha^*)^{-s}(1-\bar{\alpha}\bar{\alpha}^*)^{-\bar{s}}e^{-\frac{1}{2}(\mathbf{a}^*\mathbf{a}+\bar{\alpha}\mathbf{a}^{*2})}(\chi)^m(\bar{\chi})^{\bar{m}}(b_1)^{n_1}(b_2)^{n_2}(b_3)^{n_3} \tag{33}$$

a basic wave function where m and $\bar{m}$ stand for the third components of spin and symplin, respectively, and n_j represents the oscillator quanta in the j direction. The requirement of analyticity of these basic wave functions when expressed in terms of global coordinates $z_i, \bar{z}_i$, $i=1,2$ (as in (17)) leads to integrality conditions $2s, 2\bar{s}, m, \bar{m}\in Z$, where we recover the conditions in (21). The action of the right-invariant vector fields (operators) on $\mathcal{H}^{(s,\bar{s})}(\tilde{G})$ can be given on these basic functions as follows (we write $\hat{\mathbf{e}}_j\equiv(\delta_{1,j},\delta_{2,j},\delta_{3,j})$):

$$\begin{aligned}
\tilde{X}^R_{a_j}\psi^{(m,\bar{m})}_{\mathbf{n}} &= -\psi^{(m,\bar{m})}_{\mathbf{n}+\hat{\mathbf{e}}_j}-n_j\psi^{(m,\bar{m}+1)}_{\mathbf{n}-\hat{\mathbf{e}}_j}\\
\tilde{X}^R_{a^*_j}\psi^{(m,\bar{m})}_{\mathbf{n}} &= n_j\psi^{(m,\bar{m})}_{\mathbf{n}-\hat{\mathbf{e}}_j}\\
\tilde{X}^R_{\bar{\eta}}\psi^{(m,\bar{m})}_{\mathbf{n}} &= -(2\bar{m}+\sum_j n_j)\psi^{(m,\bar{m})}_{\mathbf{n}}\\
\tilde{X}^R_{\bar{\alpha}}\psi^{(m,\bar{m})}_{\mathbf{n}} &= -(\bar{m}-2\bar{s}+\sum_j n_j)\psi^{(m,\bar{m}+1)}_{\mathbf{n}}-\frac{1}{2}\sum_j\psi^{(m,\bar{m})}_{\mathbf{n}+2\hat{\mathbf{e}}_j}\\
\tilde{X}^R_{\bar{\alpha}^*}\psi^{(m,\bar{m})}_{\mathbf{n}} &= \bar{m}\psi^{(m,\bar{m}-1)}_{\mathbf{n}}\\
\tilde{X}^R_{\eta}\psi^{(m,\bar{m})}_{\mathbf{n}} &= -2m\psi^{(m,\bar{m})}_{\mathbf{n}}+2i(n_1\psi^{(m,\bar{m})}_{\mathbf{n}+\hat{\mathbf{e}}_2-\hat{\mathbf{e}}_1}-n_2\psi^{(m,\bar{m})}_{\mathbf{n}+\hat{\mathbf{e}}_1-\hat{\mathbf{e}}_2})\\
\tilde{X}^R_{\alpha}\psi^{(m,\bar{m})}_{\mathbf{n}} &= -(2s-m)\psi^{(m+1,\bar{m})}_{\mathbf{n}}-(n_1\psi^{(m,\bar{m})}_{\mathbf{n}+\hat{\mathbf{e}}_3-\hat{\mathbf{e}}_1}-n_3\psi^{(m,\bar{m})}_{\mathbf{n}+\hat{\mathbf{e}}_1-\hat{\mathbf{e}}_3})\\
&\quad -i(n_3\psi^{(m,\bar{m})}_{\mathbf{n}+\hat{\mathbf{e}}_2-\hat{\mathbf{e}}_3}-n_2\psi^{(m,\bar{m})}_{\mathbf{n}+\hat{\mathbf{e}}_3-\hat{\mathbf{e}}_2})\\
\tilde{X}^R_{\alpha^*}\psi^{(m,\bar{m})}_{\mathbf{n}} &= m\psi^{(m-1,\bar{m})}_{\mathbf{n}}-(n_1\psi^{(m,\bar{m})}_{\mathbf{n}+\hat{\mathbf{e}}_3-\hat{\mathbf{e}}_1}-n_3\psi^{(m,\bar{m})}_{\mathbf{n}+\hat{\mathbf{e}}_1-\hat{\mathbf{e}}_3})+i(n_3\psi^{(m,\bar{m})}_{\mathbf{n}+\hat{\mathbf{e}}_2-\hat{\mathbf{e}}_3}-n_2\psi^{(m,\bar{m})}_{\mathbf{n}+\hat{\mathbf{e}}_3-\hat{\mathbf{e}}_2})\,.
\end{aligned} \tag{34}$$

From these expressions we conclude that the third component of spin m is restricted to take the values $m=0,...,2s$ (finite dimensional subspace), whilst $\bar{m}$ can take any value from 0 to ∞ (infinite dimensional subspace), the difference being the compact and non-compact character of the corresponding subgroups $SU(2)$ and $SU(1,1)$. Thus, any wave function in $\mathcal{H}^{(s,\bar{s})}(\tilde{G})$ can be expressed as an arbitrary linear combination of these basic wave functions:

$$\psi^{(s,\bar{s})}=\sum_{j=1}^{3}\sum_{n_j=0}^{\infty}\sum_{\bar{m}=0}^{\infty}\sum_{m=0}^{2s}c^{(m,\bar{m})}_{\mathbf{n}}\psi^{(m,\bar{m})}_{\mathbf{n}}\,. \tag{35}$$

Note that the set of wave functions $\{\psi_{\mathbf{n}}^{(m,\bar{m})}\}$ is not orthogonal under the scalar product (32,7), but can be expressed in terms of an orthogonal set $\{\breve{\psi}_{\mathbf{k}}^{(m,\bar{m})}\}$ as follows:

$$\begin{aligned}
\psi_{\mathbf{n}}^{(m,\bar{m})} &= \sum_{l_1=0}^{\left[\frac{n_1}{2}\right]} \sum_{l_2=0}^{\left[\frac{n_2}{2}\right]} \sum_{l_3=0}^{\left[\frac{n_3}{2}\right]} \left(\prod_{j=1}^{3} (-1)^{n_j+l_j} \Gamma_{n_j,l_j} \right) \breve{\psi}_{\mathbf{n}-2\mathbf{l}}^{(m,\bar{m}+l_1+l_2+l_3)} \\
\Gamma_{n_j,l_j} &\equiv \frac{1}{2^{l_j}} \frac{(n_j+2l_j)!}{n_j! l_j!} \\
\breve{\psi}_{\mathbf{k}}^{(m,\bar{m})} &\equiv \left(\tilde{X}^R_{a_1}\right)^{k_1} \left(\tilde{X}^R_{a_2}\right)^{k_2} \left(\tilde{X}^R_{a_3}\right)^{k_3} \psi_{\mathbf{0}}^{(m,\bar{m})} \,,
\end{aligned} \tag{36}$$

where $\left[\frac{n_j}{2}\right]$ stands for the integer part of $\frac{n_j}{2}$.

Let us define the *intrinsic* (internal) higher-order operators:

$$\begin{array}{ll}
\tilde{X}^{R(HO)}_{\bar{\eta}} \equiv \tilde{X}^R_{\bar{\eta}} - \tilde{X}^R_{\mathbf{a}} \cdot \tilde{X}^R_{\mathbf{a}^*} & \tilde{X}^{R(HO)}_{\eta} \equiv \tilde{X}^R_{\eta} + 2i \left(\tilde{X}^R_{\mathbf{a}} \times \tilde{X}^R_{\mathbf{a}^*}\right)_3 \\
\tilde{X}^{R(HO)}_{\bar{\alpha}} \equiv \tilde{X}^R_{\bar{\alpha}} + \frac{1}{2}\tilde{X}^R_{\mathbf{a}} \cdot \tilde{X}^R_{\mathbf{a}} & \tilde{X}^{R(HO)}_{\alpha} \equiv \tilde{X}^R_{\alpha} + i \left(\tilde{X}^R_{\mathbf{a}} \times \tilde{X}^R_{\mathbf{a}^*}\right)_- \\
\tilde{X}^{R\,(HO)}_{\bar{\alpha}^*} \equiv \tilde{X}^R_{\bar{\alpha}^*} - \frac{1}{2}\tilde{X}^R_{\mathbf{a}^*} \cdot \tilde{X}^R_{\mathbf{a}^*} & \tilde{X}^{R\,(HO)}_{\alpha^*} \equiv \tilde{X}^R_{\alpha^*} - i \left(\tilde{X}^R_{\mathbf{a}} \times \tilde{X}^R_{\mathbf{a}^*}\right)_+
\end{array} \tag{37}$$

which close a Lie subalgebra of the right enveloping algebra of the Schrödinger group $\tilde{G}$, isomorphic to the Lie algebra $su(2) \oplus su(1,1)$ with a particular pseudo-extension:

$$\left[\tilde{X}^{R(HO)}_{\alpha}, \tilde{X}^{R\,(HO)}_{\alpha^*}\right] = \tilde{X}^{R(HO)}_{\eta} + 2s \,, \quad \left[\tilde{X}^{R(HO)}_{\bar{\alpha}}, \tilde{X}^{R\,(HO)}_{\bar{\alpha}^*}\right] = -\tilde{X}^{R(HO)}_{\bar{\eta}} - (2\bar{s} + \frac{3}{2}) \,. \tag{38}$$

They represent the observables corresponding to the (pure) internal degrees of freedom: symplin and spin. Even more, this subalgebra proves to be, in general, an ideal (under commutation) of the right enveloping algebra $U\tilde{\mathcal{G}}^R$ of $\tilde{G}$, and a horizontal ideal for the particular values

$$\bar{s} = -\frac{3}{4}, \quad \text{and} \quad s = 0 \tag{39}$$

(as can be partially checked in (38)). This last situation requires special attention. In fact, the existence of a non-trivial (non zero) horizontal ideal –gauge subalgebra (see the paragraph after (3))– is a sign of reducibility; indeed, according to general settings [21], the right-invariant vector fields in a gauge subalgebra must be a linear combination of left-invariant vector fields in the characteristic module and, therefore, they have to be trivially represented (zero). Then, the representation (34) is reducible for the particular *quantum* values (39), as opposed to the *classical* values (concerning the symplin but not the spin) for which the classical reduction (29) was achieved. Nevertheless, whereas the (normal) reduction for $s = 0$ is reached simply by a new full polarization consisting of (30) enlarged by $\tilde{X}^L_{\alpha^*}$ (i.e. containing the whole $su(2)$ subalgebra of the characteristic algebra), the (anomalous) reduction for $\bar{s} = -3/4$ requires to use higher-order polarization techniques. The *modus operandi* to construct a higher-order polarization subalgebra for these anomalous cases usually consists in deforming the generators in the characteristic subalgebra corresponding to the classical reduction of the symplectic manifold ($su(1,1)$ in our case), by adding terms in the left enveloping algebra $U\tilde{\mathcal{G}}^L$. Also, when there are non-trivial higher-order gauge operators, their (equivalent) left counterparts are candidates for further reducing the representation. In our case, a new higher-order restriction on wave functions (31) can be consistently added to the set of first-order restrictions given by (30) for the anomalous case $\bar{s} = -3/4$ only. The candidate for this reduction process is the deformation $\tilde{X}^{L\,(HO)}_{\bar{\alpha}^*}$ of $\tilde{X}^{L}_{\bar{\alpha}^*}$, which is precisely the left counterpart of $\tilde{X}^{R\,(HO)}_{\bar{\alpha}^*}$ defined in (37). Its gauge character makes indifferent if the higher-order polarization condition

$$\tilde{X}^{L\,(HO)}_{\bar{\alpha}^*}\psi^{(s,-\frac{3}{4})} = 0 \;\;\Rightarrow \frac{\partial\phi}{\partial\bar{\chi}} - \frac{1}{2}\sum_j \frac{\partial^2\phi}{\partial b_j^2} = 0 \tag{40}$$

is imposed as a left or as a right restriction on wave functions (31), the solution of which is expressed in terms of an orthogonal and complete set of the form:

$$\breve{\psi}^{(m)}_{\mathbf{k}} \equiv \sum_{i=1}^{3}\sum_{l_i=0}^{\infty}\sum_{n_i=0}^{\infty}\left(\prod_{j=1}^{3}\Gamma_{n_j,l_j}\delta_{n_j+2l_j,k_j}\right)\psi^{(m,l_1+l_2+l_3)}_{\mathbf{n}} = \left(\prod_{j=1}^{3}(-1)^{k_j}\left(\tilde{X}^R_{a_j}\right)^{k_j}\right)\psi^{(m,0)}_{\mathbf{0}}\,, \tag{41}$$

i.e., the orbit of the creation operators $\hat{a}^\dagger_j \equiv -\tilde{X}^R_{a_j}$ through the vacuum $\psi^{(0,0)}_{\mathbf{0}}$ (when $s = 0$). In this way, the whole set of physical operators $\tilde{X}^R_{\tilde{g}^i}$ are expressed in terms of basic ones $\hat{a}^\dagger_j = -\tilde{X}^R_{a_j}$ and $\hat{a}_j \equiv \tilde{X}^R_{a^*_j}$ as in (37) taking into account that now (internal) higher-order operators $\tilde{X}^{R(HO)}_{\tilde{g}^i}$ are trivially zero (gauge). For example, the energy operator is:

$$\hat{E} \equiv -X^R_{\bar{\eta}} \equiv -\tilde{X}^R_{\bar{\eta}} - 2\bar{s}\tilde{X}^R_{\zeta} = \sum_{j=1}^{3}(\hat{a}^\dagger_j\hat{a}_j + \frac{1}{2})\,, \tag{42}$$

where the last redefinition of the $\tilde{X}^R_{\bar{\eta}}$ generator is intended to render the commutation relation $[\tilde{X}^R_{\bar{\alpha}}, \tilde{X}^R_{\bar{\alpha}^*}] = \tilde{X}^R_{\bar{\eta}} + 2\bar{s}\tilde{X}^R_{\zeta}$ in (23) to the usual $su(1,1)$ one: $[X^R_{\bar{\alpha}}, X^R_{\bar{\alpha}^*}] = X^R_{\bar{\eta}}$. Note that the zero-point energy $E_0 = \frac{1}{2}d$ of the harmonic oscillator is precisely $-2\bar{s}$ for the quantum value of the anomaly $\bar{s} = -\frac{d}{4}$. This anomalous value is obtained in the standard approach by the naïve "symmetrization rule" and proves to have important physical consequences in the experimentally observed *Casimir effect* (see [27] and references therein).

The half-half integer character of the symplin $\bar{s} = -\frac{1}{2}(\frac{3}{2})$ indicates, according to (21), that the representation of $SU(1,1)$ is *bi-valuated*; i.e. it is the two-cover $Mp(2,\Re)$ (metaplectic group) which is in fact faithfully represented.

At this stage, a comparison of the fundamentals of this finite-dimensional anomalous system with the more conventional one (infinite-dimensional) of the bosonic string [28] is opportune. The role played by the Virasoro group, acting on string modes $\{\alpha^\mu_m\}$,

$$\left[\hat{L}_n, \hat{\alpha}^\mu_m\right] = m\hat{\alpha}^\mu_{n+m} \tag{43}$$

is here played by $SU(1,1)$ acting on oscillator modes $\mathbf{a}, \mathbf{a}^*$. Like the $su(1,1)$ algebra, the Virasoro algebra

$$\left[\hat{L}_n, \hat{L}_m\right] = (n-m)\hat{L}_{n+m} + \frac{1}{12}(cn^3 - c'n)\delta_{n,-m}\hat{1} \tag{44}$$

appears also centrally extended, although, this time, by both a pseudo-cocycle (with parameter c' generalizing the symplin $\bar{s}$) and a true cocycle (with parameter c); the latter is permitted by the infinite-dimensional character of the Virasoro group, which violate the Whitehead Lemma

[29]. The (anomalous) reduction which allows the Virasoro operators to be written in terms of the string modes (Sugawara's construction [30]) for $c = c' = d$ (the dimension of the space-time)

$$\hat{L}_k = \frac{1}{2} g_{\mu\nu} : \sum \hat{\alpha}^{\mu}_{k-n} \hat{\alpha}^{\nu}_n : \tag{45}$$

is essentially equivalent (in nature) to the anomalous reduction which allows the $su(1,1)$ operators to be written in terms of the oscillator modes for $\bar{s} = -d/4$. In fact, formula (41) expressing the states of the invariant, reduced subspace of the Hilbert space $\mathcal{H}^{(s,\bar{s}=-\frac{3}{4})}(\tilde{G})$ as generated by the action of the creation operators corresponding to the harmonic oscillator only, parallels the construction of the reduced Hilbert space of the string by the action of just string mode operators on the vacuum (see e.g. [28])

$$\hat{\alpha}^{\mu_1}_{n_1} \hat{\alpha}^{\mu_2}_{n_2} ... \hat{\alpha}^{\mu_i}_{n_i} |0\rangle \, . \tag{46}$$

To proceed further in this comparison, we could simulate the constraints in string theory by restricting our oscillator to the sphere; more precisely, the role played by the Virasoro group generators, acting as constraints in string theory, can be played here by part of the $su(1,1)$ operators, for example, those which restrict the particle to move on the sphere. Indeed, making use of the expression (14) we can write the square of the vector position operator and its "time derivative" as:

$$\begin{aligned} \hat{\mathbf{x}}^2 &\equiv \frac{1}{2}((\tilde{X}^R_{\mathbf{a}^*})^2 + (\tilde{X}^R_{\mathbf{a}})^2 - \tilde{X}^R_{\mathbf{a}^*}\tilde{X}^R_{\mathbf{a}} - \tilde{X}^R_{\mathbf{a}}\tilde{X}^R_{\mathbf{a}^*}) = \tilde{X}^R_{\tilde{\alpha}^*} - \tilde{X}^R_{\tilde{\alpha}} - \tilde{X}^R_{\tilde{\eta}} + \frac{3}{2} \qquad (47) \\ \frac{1}{2}(\hat{\mathbf{x}}\hat{\mathbf{p}} + \hat{\mathbf{p}}\hat{\mathbf{x}}) &\equiv -\frac{i}{2}((\tilde{X}^R_{\mathbf{a}^*})^2 - (\tilde{X}^R_{\mathbf{a}})^2) = -i(\tilde{X}^R_{\tilde{\alpha}^*} + \tilde{X}^R_{\tilde{\alpha}}) \, . \end{aligned}$$

The constrained theory can be formulated by looking at the Schrödinger group as a principal fibre bundle where the structure group $T = U(1)$ has been replaced by $T = \tilde{A}(1)$, a central extension of the Affine group in 1D; more precisely, the Lie algebra of T is now:

$$\begin{aligned} \left[\tilde{X}^R_{\tilde{t}_1}, \tilde{X}^R_{\tilde{t}_2}\right] &= 2i\tilde{X}^R_{\tilde{t}_1} + 2ir^2\tilde{X}^R_{\zeta} \qquad (48) \\ \tilde{X}^R_{\tilde{t}_1} \equiv \hat{\mathbf{x}}^2 - r^2\tilde{X}^R_{\zeta} \, , &\quad \tilde{X}^R_{\tilde{t}_2} \equiv \frac{1}{2}(\hat{\mathbf{x}}\hat{\mathbf{p}} + \hat{\mathbf{p}}\hat{\mathbf{x}}) \, , \end{aligned}$$

which takes part of a subalgebra of $su(1,1)$ pseudo-extended by $U(1)$ with parameter r (radius of the sphere). The constraint on the sphere can be achieved through T-equivariant conditions (12) on arbitrary combinations $\check{\psi}^{(s)}$ of the basic wave functions (41), either as

$$\tilde{X}^R_{\tilde{t}_1}\check{\psi}^{(s)} = 0 \quad \text{or} \quad \tilde{X}^R_{\tilde{t}_2}\check{\psi}^{(s)} = 0 \, , \tag{49}$$

since the conjugate character of these two constraints (see the commutator in (48)) prevents fixing both at a time, i.e. T_p is generated by either $\tilde{X}^R_{\tilde{t}_1}$ or $\tilde{X}^R_{\tilde{t}_2}$.

With regard to the good operators of the theory, there are some sets of operators which preserve one option of $T_p \subset T_B \subset T$, but not the other. We shall restrict ourselves to the intersection of these sets to define our good operators, i.e. those operators which preserve any of the possible choices of $T_B \subset T$. This set of good operators is enough to reproduce the constrained quantum system of the particle on the sphere; it is:

$$\begin{aligned} \tilde{\mathcal{G}}_T &= \left\{\tilde{X}^R_{\mathbf{a}} \times \tilde{X}^R_{\mathbf{a}^*}, \, \tilde{X}^R_{\eta}, \, \tilde{X}^R_{\alpha}, \, \tilde{X}^R_{\alpha^*}, \, \hat{u}_1, \, \hat{u}_2, \, \hat{u}_3\right\} \qquad (50) \\ \hat{\mathbf{u}} &\equiv \hat{r}^{-1}\hat{\mathbf{x}} \, , \quad \hat{r}^{-1} \equiv \frac{1}{r}\left(1 + \sum_{n=1}^{\infty}(-1)^n \frac{(2n-1)!!}{2n!!}\left(\frac{\tilde{X}^R_{\tilde{t}_1}}{r^2}\right)^n\right) = \frac{1}{\sqrt{r^2 + \tilde{X}^R_{\tilde{t}_1}}} \end{aligned}$$

which commute with both $\tilde{X}^R_{\tilde{t}_1}$ and $\tilde{X}^R_{\tilde{t}_2}$ and close a subalgebra isomorphic to the Euclidean algebra in 3D for the case of spin $s = 0$ [note that the operators $\hat{u}_j$ live in the enveloping algebra of the Schrödinger group]. When we choose the second option in (49), the constrained Hilbert space turns out to be made up of T_B-equivariant functions constructed by taking the orbit of $\hat{u}_j$ through the only $2s + 1$ states that are "rotationally invariant" and annihilated by $\tilde{X}^R_{\tilde{t}_2}$. These prove to be:

$$Y^{(0)}_{m_s,0} \equiv \sum_{q=0}^{\infty} \frac{K}{(2q)!!(2q - N_0)!!} \left(\tilde{X}^R_\alpha\right)^{2q} \check{\psi}^{(m_s)}_{\mathbf{0}}, \quad m_s = 0, ..., 2s + 1 \tag{51}$$

where K is an arbitrary constant and $N_0 = 2\bar{s} + 1 = -\frac{1}{2}$. For $s = 0$, the state $Y^{(0)}_{0,0}$ simply represents the spherical harmonic of zero angular momentum [note that this state is an infinite linear combination of harmonic oscillator wave functions]. States with higher values of angular momentum $Y^{(l)}_{m_s,m_l}$ correspond to the repeated action of $\hat{u}_j$ on these "vacua" $Y^{(0)}_{m_s,0}$. For example, the state $Y^{(1)}_{m_s,0} \equiv \hat{u}_3 Y^{(0)}_{m_s,0}$ has (orbital) angular momentum $l = 1$ and third component $m_l = 0$. The whole set of states obtained in this way represent the Hilbert space of a spinning point particle living on the sphere. Different values of s parametrize non-equivalent quantizations.

We have preferred to maintain the internal degree of spin in order to make comparisons with other approaches to *Quantum mechanics on* S^D as Ref. [31], where the $(D + 1)$-dimensional Euclidean group was used to study the point particle on S^D, or Refs. [32, 33], where S^D is seen as a coset space $G/H = SO(D+1)/SO(D)$ of $SO(D+1)$. An important basic difference of our procedure with respect to other approaches is that the sphere S^2 where the "free particle" of [31, 32, 33] lives, seems to correspond with our *internal* sphere S^2 immersed on the symplectic manifold $T^*S^2 \times S^2$ [T^*S^2 is the cotangent of S^2], resulting from the original $\mathbf{C}^3 \otimes D_1 \otimes S^2$ (see (29)) after reducing (40) [$\rightarrow \mathbf{C}^3 \otimes S^2$] and constraining (49); i.e., there are *two* different (in nature) spheres for us, a "real" sphere immersed in $\Re^3$, where the particle lives, and an "internal" (symplectic) sphere $S^2 = SU(2)/U(1)$ corresponding to the spin degree of freedom. This situation can lead to confusions in interpretation when they quantize on coset spaces $Q = G/H$ and parametrize Q as immersed in $\Re^n$; in fact, a embedding of our $Q = S^2 = SU(2)/U(1)$ in $\Re^3 = \{y_1, y_2, y_3\}$ according to a standard stereographical projection map:

$$\alpha\eta^2 = \frac{y_1}{\rho + y_3} + i\frac{y_2}{\rho + y_3}, \quad \text{with} \quad \mathbf{y}^2 = \rho^2 \tag{52}$$

could lead us to believe that "a monopole is present" if we interpret the 1-form connection $\Theta_{SU(2)}$ in (24) as a $U(1)$-gauge potential [it is called the *H-connection* or *cannonical connection* in [32, 33]], but we know that "this monopole does not live in our world".

4 Breaking through no-go theorems: extended objects from the elementary particle

As already mentioned, the Schrödinger algebra can be viewed as the maximal Poisson subalgebra on the solution manifold of the free particle and/or the harmonic oscillator that can be quantized in a more or less canonical way. This means that the quantization mapping "ˆ" representing the Poisson subalgebra $< 1, x, p, \frac{1}{2}x^2, \frac{1}{2}p^2, xp >$ by $< \hat{1}, \hat{x}, \hat{p}, \frac{1}{2}\hat{x}^2, \frac{1}{2}\hat{p}^2, \hat{x}\hat{p} >$ is not a Lie algebra homomorphism due to the (anomalous) term $-\frac{i}{2}$ in the commutator $[\frac{1}{2}\hat{x}^2, \frac{1}{2}\hat{p}^2]$, with regard to its associated Poisson bracket. Fortunately, this anomaly can easily be hidden simply by symmetrizing the operator $\widehat{(xp)}$.

Standard canonical quantization fails to go beyond any Poisson subalgebra containing polynomials in x, p of degree greater than two [9, 10]. From the point of view of group quantization, however, we can proceed further, provided that we are able to close a definite Poisson subalgebra that, although necessarily infinite-dimensional, has a controlled growing (finite growth; see for instance [34]). Then a group law can be found, at least, by exponentiating the Lie algebra order by order, as in the case, for instance, of Kac-Moody algebras [35], and by considering all possible (pseudo-)extensions (and associated deformations) with arbitrary parameters γ_k.

Needless to say, in the quantization process many anomalies will eventually appear, requiring the use of the higher-order polarization technique. These anomalies are really obstructions to the quantization of given functions of x, p in terms of $\hat{x}, \hat{p}$. The quantum values $\gamma_k^{(0)}$ of the anomalies are precisely those for which such a task can be achieved even though the quantization morphism "$\hat{\ }$" is somewhat distorted (central terms for operators representing quadratic functions and more general terms for higher-order polynomials on x, p). Far from the quantum values of the anomalies, however, new (purely) quantum degrees of freedom must enter the theory as associated with those operators which cannot be expressed in terms of $\hat{x}, \hat{p}$. Moreover, it could well happen that no quantum values of γ_k exist for some cases, thus leading to "essentially anomalous" (inescapable) situations.

To construct such an infinite-dimensional Poisson algebra, generalizing the Schrödinger algebra, let us start with the solution manifold of the elementary particle in two dimensions parametrized by x, p. For simplicity, we shall asume that the particle is non-relativistic, although we could think of the relativistic situation so long as x really represents the classical analogue of the Newton-Wigner position operator [36, 37]; or, we could even also consider the time parameter x^0, provided that it is given a dynamical character with canonically conjugate momentum p^0 ($\{x, p\} = 1 \rightarrow \{x^\mu, p^\nu\} = g^{\mu\nu}$), and then impose the mass-shell constraint [38]. Let us continue to use an oscillator-like parametrization of the phase space, as in (14), and choose the following set of classical functions of a^*, a:

$$L_n^\alpha = \frac{1}{2} a^{2n} (a^* a)^{-\alpha - n + 1}, \quad L_{-m}^\beta = \frac{1}{2} a^{*2m} (a^* a)^{-\beta - m + 1} \qquad (53)$$
$$n, m = 0, \frac{1}{2}, 1, \frac{3}{2}, ... \quad \alpha, \beta = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, ...$$

which generalize the Virasoro algebra (as generating diffeomorphisms of the plane) and contain the Schrödinger algebra as the largest finite-dimensional subalgebra.

A straightforward computation from the basic Poisson bracket $\{a^*, a\} = i$ provides the following formal, Poisson algebra:

$$\{L_n^\alpha, L_m^\beta\} = -i[(1-\beta)n - (1-\alpha)m] L_{n+m}^{\alpha+\beta} \quad n, m, \alpha, \beta \in Z/2 \qquad (54)$$

which should not be confused with that introduced in [39]. It contains some interesting subalgebras:

Schrödinger algebra:

$$a^* = 2L_{-\frac{1}{2}}^{\frac{1}{2}}, \quad a = 2L_{\frac{1}{2}}^{\frac{1}{2}}, \quad 1 \equiv 2L_0^1, \quad \tfrac{1}{2}a^{*2} = L_{-1}^0, \quad \tfrac{1}{2}a^2 = L_1^0, \quad a^* a = 2L_0^0\,. \qquad (55)$$

Virasoro algebra:

$$L_n \equiv L_n^0 \Rightarrow \{L_n, L_m\} = -i(n-m)L_{n+m} \tag{56}$$

Unextended "string" algebra: The already identified Virasoro subalgebra can be enlarged by $\alpha_m \equiv L_m^1,\ m \in Z$. They close the following semi-direct algebra:

$$\begin{aligned} \{L_n, L_m\} &= -i(n-m)L_{n+m} \\ \{L_n, \alpha_m\} &= im\alpha_{n+m} \\ \{\alpha_n, \alpha_m\} &= 0\,, \end{aligned} \tag{57}$$

corresponding to the (classical) underlying symmetry of string theory (one for each value of the the μ index in α_m^μ), i.e., the symmetry before extending by $U(1)$.

The subalgebra of (54) corresponding to integer, positive powers of x, p, denoted in the literature by w_∞, has been considered many times and very recently in connection with the Geroch group [40]. The traditional restriction to integer, positive indices is based on analyticity grounds. However, applied to the quantum world, the analyticity requirement makes sense for only those operators which are not basic, i.e. are not directly associated with any degree of freedom and must accordingly be written in terms of the basic quantum operators ($\hat{a}^*, \hat{a}$ in our case). Conversely, Poisson algebra elements that generate Lie algebra cohomology (and, therefore, central extensions) can be kept as generators of the true quantum symmetry, as they do not have to be expressed, in principle, as fucntions depending on $\hat{a}^*, \hat{a}$. They will be referred to as an "essential anomaly" and extend the system in the sense that they generate new (independent) quantum degrees of freedom. Only the presence of anomalies will require a further reduction of the quantum representation, which is achieved in a way that permits some *a priori* basic operators to be written in terms of others effectively basic. The quantum values of the anomalies are in general those values of the central charges for which the effective extent of the extended system reduces to a minimum. In any case, and as a minor harm, if we wish to put the motivation (53) to the algebra (54) in a proper mathematical ground, we could just eliminate the point $a = a^* = 0$ of our original phase space thus restoring the analyticity of the combinations (53) [note also that the quantum analogue $\widehat{a^*a} = \hat{a}^\dagger\hat{a} + \frac{1}{2}$ of the classical function a^*a is never zero because of the anomalous value ($\neq 0$) of the symplin (zero-point energy)].

To understand fully the interplay among (a certain degree of) classical non-analyticity, group cohomology and the extent of a quantum system, let us restrict ourselves to the unextended "string" algebra (57). The generators of the classical algebra of symmetries are written as non-analytical functions [in the "weak" (avoidable) sense specified in the previous paragraph] of a^*, a:

$$\begin{aligned} L_n &= \frac{1}{2}a^{2n}(a^*a)^{1-n} \quad , \quad L_{-n} = \frac{1}{2}a^{*2n}(a^*a)^{1-n} \\ \alpha_m &= \frac{1}{2}a^{2m}(a^*a)^{-m} \quad , \quad \alpha_{-m} = \frac{1}{2}a^{*2m}(a^*a)^{-m}\,. \end{aligned} \tag{58}$$

Centrally extending this algebra in the form:

$$\left[\hat{L}_n, \hat{L}_m\right] = (n-m)\hat{L}_{n+m} + \frac{1}{12}(cn^3 + c'n)\delta_{n+m,0}\hat{1} \tag{59}$$

$$\begin{aligned} \left[\hat{L}_n, \hat{\alpha}_m\right] &= \hat{\alpha}_{n+m} \\ \left[\hat{\alpha}_n, \hat{\alpha}_m\right] &= am\delta_{n+m}\hat{1}\,, \end{aligned} \tag{60}$$

we can proceed with group quantization, finding the characteristic subalgebra as well as the canonically conjugate pairs. The precise calculations can be found in [41] and references therein (for the actual string algebra, i.e. for generators $L_n, \alpha^{\mu}_m, \mu = 0, 1, 2, ...d$, although the results are formally equivalent). We arrive at the results given in Sec. 3: for $a = 1, c = c' = 1$, the whole set of Virasoro generators can be expressed, after quantization, as quadratic (hence analytical) functions of the quantum operators $\hat{\alpha}_m$ (see (45)). These operators, however, need not be (nor indeed can be) expressed in terms of any operator since they are basic, independent operators, as a consequence of the central extension (60) (the central term in the Virasoro commutator is due to an anomaly, which is destroyed for the values of c, c', a above), giving an infinite extent to the physical system. The same clearly applies to the case $\bar{s} = -\frac{d}{4}$ of the elementary particle with symplin studied in Sec. 3 where, the pseudo-extension of $SU(1,1)$ (which redefines the generator $\tilde{X}^R_{\bar{\eta}}$) with parameter $\bar{s}$, is exactly the same as the pseudo-extension of the Virasoro algebra (which redefines the generator $L_0 = a^* a$) with parameter c'.

Our suggestion, finally, is then to consider the central extensions of the entire (formal abstract) algebra (54), as being the quantizing algebra for the minimal infinite-dimensional system extending the free particle in such a way that string itself is naturally included. In this quite extended object, the free particle with symplin appears as the only and biggest finite-dimensional subsystem. Also along these lines, (1+1D) quantum gravity could arise in a general attempt to get a full quantization of the phase space of the free particle. A general study of (54), its central extensions and quantization, will require a quite big effort and deserves a separate work.

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