Gauge Symmetry Breaking:
Higgs-less Mass Generation and Radiation Phenomena[1]

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Abstract

Gauge symmetries generally appear as a constraint algebra, under which one expects all physical states to be singlets. However, *quantum anomalies* and boundary conditions introduce central charges and change this picture, thus causing gauge/diffeomorphism modes to become physical. We expose a cohomological (*Higgs-less*) generation of mass in $U(N)$-gauge invariant Yang-Mills theories through non-trivial representations of the gauge group. This situation is also present in black hole evaporation, where the Virasoro algebra turns out to be the relevant subalgebra of surface deformations of the horizon of an arbitrary black hole.

1 Introduction

Let T be a gauge/diffeomorphism group. From a classical perspective, physical states $\Psi_{\text{phys.}}$ are expected to be singlets under T, i.e.

$$U\Psi_{\text{phys.}} = \Psi_{\text{phys.}}, \quad U = e^{\varphi^a \Phi_a} \in T. \tag{1}$$

For example, in standard Yang-Mills theory, the infinitesimal counterpart of the finite expression (1) is nothing other than the "Gauss law" condition $\Phi_a \Psi_{\text{phys.}} = 0$. However, upon quantization, anomalies and boundary conditions can change this picture, causing gauge/diffeomorphism modes φ^a to become physical, so that physical states transform non trivially under T,

$$U\Psi_{\text{phys.}} = D_T^{(\epsilon)}(U)\Psi_{\text{phys.}}, \tag{2}$$

according to a representation $D_T^{(\epsilon)}$ of T with index ϵ. Eventually, the index ϵ could represent a ϑ-angle or a mass parameter m and, in general, it labels *non-equivalent quantizations*. In fact, the possibility of non-trivial representations $D_T^{(m)}$ of a $T = U(N)$-invariant Yang-Mills theory will lead to a (*Higgs-less*) generation of mass for vector bosons. The mass parameters m show up as central charges in the Lie algebra of constraints, which transmute to second-class constraints. Some of the gauge modes become physical, i.e., they acquire dynamics outside the null-mass shell and provide the longitudinal field degrees of freedom that massless vector bosons need to form massive vector bosons (see Refs. [1] and later on Sec. 4.3). This seems to be an important and general feature of quantum gauge theories as opposite to their classical counterparts. In fact, this situation is also present in quantum gravity, where the Virasoro

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algebra turns out to be the relevant subalgebra of surface deformations of the horizon of an arbitrary black hole and constitutes the general gauge (diffeomorphism) principle that governs the density of states. Nevertheless, although surface deformations appear as a constraint algebra, under which one might expect all the physical states on the horizon to be singlets, *quantum anomalies* and boundary conditions introduce central charges and change this picture, thus causing gauge/diffeomorphism modes to become physical along the horizon (see e.g. [2] and later on Sec. 5).

In order to set the context, let us describe a simple, but illustrative, example of an abstract quantizing algebra $\tilde{\mathcal{G}}$ which eventually applies to a diversity of physical systems. After all, any consistent (non-perturbative) quantization is mostly a unitary irreducible representation of a suitable (Lie, Poisson) algebra.

2 A simple abstract quantizing algebra

Our particular algebra under study will be the following:

$$
\begin{array}{rcl}
[X_j, P_k] &=& i\delta_{jk}I\,, \\
[\Phi_a, \Phi_b] &=& if_{ab}^c\Phi_c + i\epsilon_{ab}I\,, \\
[X_j, \Phi_a] &=& i\check{f}_{ja}^k X_k, \quad [P_j, \Phi_a] = i\check{f}_{ja}^k P_k,
\end{array}
\tag{3}
$$

where X_j and P_k represent standard "position" and "momentum" operators, respectively, corresponding to the *extended phase space* $\mathcal{F} \sim \mathbb{R}^{2m}$ of the preconstrained (free-like) theory; The operators Φ_a represent the constraints which, for the moment, are supposed to close a Lie subalgebra $\tilde{\mathcal{T}}$ with structure constants f_{ab}^c and central charges ϵ_{ab}. We also consider a diagonal action of constraints Φ on X and P with structure constants $\check{f}_{ja}^k$ (non-diagonal actions mixing X and P lead to interesting "anomalous" situations which we shall not discuss here [3]). By I we simply denote the identity operator, that is, the generator of the typical phase invariance $\Psi \sim e^{i\beta}\Psi$ of Quantum Mechanics. At this stage, it is worth mentioning that we could have introduced dynamics in our model by adding a Hamiltonian operator H to $\tilde{\mathcal{G}}$. However, we have preferred not to include it because, although we could make compatible the dynamics H and the constraints Φ, the price could result in an unpleasant enlarging of $\tilde{\mathcal{G}}$, which would make the quantization procedure much more involved. Anyway, for us, the *true* dynamics (that which preserves the constraints) will eventually arise as part of the set of *good* operators (observables) of the theory (7).

Note that a flexibility in the class of the constraints has being allowed by introducing arbitrary central charges ϵ_{ab} in (3). Thus, the operators $\{\Phi_a\}$ represent a mixed set of first- and second-class constraints. Let us denote by $\mathcal{T}^{(1)} = \{\Phi_n^{(1)}\}$ the subalgebra of first-class constraints, that is, the ones which do not give rise to central terms proportional to ϵ_{ab} at the right hand side of the commutators (3). The rest of constraints (second-class) will be arranged by conjugated pairs $(\Phi_{+\alpha}^{(2)}, \Phi_{-\alpha}^{(2)})$, so that $\epsilon_{+\alpha,-\alpha} \neq 0$.

The simplest ('classical') case is when $\epsilon_{ab} = 0$, $\forall a, b$, that is, when all constraints are first class $\mathcal{T}^{(1)} = \mathcal{T} = \tilde{\mathcal{T}}/u(1)$ and wave functions are singlets under $\mathcal{T}$. However, the 'quantum' case $\epsilon_{ab} \neq 0$ entails non-equivalent quantizations with important physical consequences. This possibility indicates a non-trivial response (1) of the wave function Ψ under $\tilde{\mathcal{T}}$. That is, Ψ acquires a non-trivial dependence on extra degrees of freedom $\phi_{-\alpha}^{(2)}$ ('negative modes' attached

to pairs of second-class constraints), in addition to the usual configuration space variables x_j (attached to X_j).

Let us formally outline the actual construction of the unitary irreducible representations of the group $\tilde{G}$, with Lie-algebra (3), following the Group Approach to Quantization framework [4]. Wave functions Ψ are defined as complex functions on $\tilde{G}$, $\Psi : \tilde{G} \to \mathbb{C}$, so that the (let us say) left-action

$$L_{\tilde{g}'}\Psi(\tilde{g}) \equiv \Psi(\tilde{g}'^{-1} * \tilde{g}), \ \tilde{g}', \tilde{g} \in \tilde{G} \tag{4}$$

defines a reducible (in general) representation of $\tilde{G}$. The reduction is achieved by means of that maximal set of right restrictions on wave functions

$$R_{\tilde{g}_p}\Psi = \Psi, \ \forall \tilde{g}_p \in G_p, \tag{5}$$

(which commute with the left action) compatible with the natural condition $I\Psi = \Psi$. The right restrictions (5) generalize the notion of *polarization conditions* of Geometric Quantization and give rise to a certain representation space depending on the choice of the subgroup $G_p \subset \tilde{G}$. For the algebra (3), a polarization subgroup can be $G_p^= F_P \times_s T_p$, that is, the semi-direct product of the Abelian group of translations $F_P \sim \mathbb{R}^m$ generated by $\mathcal{F}_P \equiv \{P_k\}$ (half of the symplectic generators in $\mathcal{F} \sim \mathbb{R}^{2m}$) times a polarization subalgebra $\mathcal{T}_p = \{\Phi_n^{(1)}, \Phi_{+\alpha}^{(2)}\}$ of $\tilde{T}$ consisting of first-class constraints (the *unbroken gauge subalgebra* $\mathcal{T}^{(1)}$) and half of second-class constraints (namely, the 'positive modes' $\Phi_{+\alpha}^{(2)}$). The polarization conditions (5) lead to the configuration-space representation made of wave functions $\Psi(x_j, \phi_{-\alpha}^{(2)})$ depending arbitrarily on the group coordinates on $\tilde{G}/G_p$ only. Thus, as mentioned above, wave functions transform non-trivially under the left-action $L_\phi\Psi(\tilde{g}) = D_{\tilde{T}}^{(\epsilon)}(\phi)\Psi(\tilde{g})$ of $\tilde{T}$ according to a given representation $D_{\tilde{T}}^{(\epsilon)}$ like in (1). The physical Hilbert space is made of those wave functions $\Psi_{\text{ph.}}$ that transform as *highest-weight vectors* under $\tilde{T}$, that is, they stay invariant under the left-action of first-class constraints and (let us say) negative second-class modes:

$$L_{\phi_n^{(1)}}\Psi_{\text{ph.}} = \Psi_{\text{ph.}}, \ n = 1, ..., \dim(T^{(1)}),$$
$$L_{\phi_{-\alpha}^{(2)}}\Psi_{\text{ph.}} = \Psi_{\text{ph.}}, \ \alpha = 1, ..., \dim(T/T^{(1)})/2, \tag{6}$$

which close the subgroup $T_p \subset \tilde{T}$.

The counting of *true degrees of freedom* is as follows: polarized-constrained wave functions (6) depend arbitrarily on $d = \dim(\tilde{G}) - \dim(G_p) - \dim(T_p) - 1$ reduced-space coordinates (we are subtracting the phase coordinate $e^{i\beta}$ too). The algebra of observables of the theory, $\tilde{\mathcal{G}}_{\text{good}} \subset \mathcal{U}(\tilde{\mathcal{G}})$ (a subalgebra of the universal enveloping algebra), has to be found inside the *normalizer* of constraints, that is:

$$\left[\tilde{\mathcal{G}}_{\text{good}}, \mathcal{T}_p\right] \subset \mathcal{T}_p. \tag{7}$$

From this characterization, the subalgebra of first-class constraints $\mathcal{T}^{(1)}$ become a horizontal ideal (a *gauge* subalgebra [5]) of $\tilde{\mathcal{G}}_{\text{good}}$. The Hamiltonian operator has to be found inside $\tilde{\mathcal{G}}_{\text{good}}$ by using extra physical arguments.

3 Global considerations: the quantizing group

In order to discuss some global (versus local) problems in quantization, it is necessary to translate the previous infinitesimal (algebraic) concepts to their finite counterparts. The exponentiation

of the algebra (3) leads to a Weyl-symplectic-like group $\tilde{G}$, with group law:

$$
\begin{aligned}
(g'', \zeta'') &= (g', \zeta') * (g, \zeta) = (g'g, \zeta'\zeta e^{\frac{i}{2\hbar}\xi(g,g')}), \quad g = (\vec{x}, \vec{p}, U), \\
U'' &= U'U \in T, \\
\vec{V}'' &= \vec{V}' + U'\vec{V} \in F, \quad \vec{V} = \left(\begin{array}{c} \vec{x} \\ \vec{p} \end{array} \right)_{2m \times 1}, \\
\zeta'' &= \zeta'\zeta e^{\frac{i}{2\hbar}\xi(g,g')} \in U(1), \quad \xi(g,g') = \xi(g,g')_{\mathrm{B}} + \xi(g',g)_{\epsilon}, \\
\xi(g,g')_{\mathrm{B}} &= \vec{V}'^t \left(\begin{array}{cc} 0 & I_m \\ -I_m & 0 \end{array} \right) U'\vec{V},
\end{aligned}
$$

which is a central extension $\tilde{G} \sim G \times U(1)$ of the semidirect product $G = \mathbb{R}^{2m} \times_s T$ of phase-space translations $\vec{V} \in \mathbb{R}^{2m}$ and gauge transformations $U = e^{\phi^a \Phi_a} \in T$ by $U(1) \ni \zeta$. The map $\xi : G \times G \to \mathbb{R}$ is a two-cocycle with two parts: 1) The Bargmann cocycle $\xi(g,g')_{\mathrm{B}}$ says that position $\vec{x}$ and momenta $\vec{p}$ are conjugated variables [see the first commutator of Eq. (3)], and 2) the two-cocycle $\xi(g',g)_{\epsilon}$ is meant to provide couples of second-class constraints.

Two two-cocycles are said to be equivalent if they differ by a coboundary, i.e. a two-cocycle which can be written in the form $\xi(g',g) = \eta(g' * g) - \eta(g') - \eta(g)$, where $\eta(g)$ is called the generating function of the coboundary. Although two-cocycles differing by a coboundary lead to equivalent central extensions as such, there are some coboundaries which provide a non-trivial connection

$$
\Theta = \frac{\partial}{\partial g^j}\xi(g',g)\Big|_{g'=g^{-1}} dg^j - i\hbar\zeta^{-1}d\zeta , \tag{8}
$$

on the fibre bundle $\tilde{G}$, and Lie-algebra structure constants different from those of the direct product $G \times U(1)$. These are generated by a function η with a non-trivial gradient at the identity $d\eta(g)|_{g=e} = \frac{\partial \eta(g)}{\partial g^j}\Big|_{g=e} dg^j \neq 0$, and can be divided into pseudo-cohomology equivalence subclasses (see [6] in this volume). Pseudo-cohomology plays an important role in the theory of finite-dimensional semi-simple groups, as they have trivial cohomology. For them, pseudo-cohomology classes are associated with coadjoint orbits (see [6]). Next section, we shall show how the introduction of coboundaries in some physical systems alters the corresponding quantum theory. From the mathematical point of view, pseudo-cocicles entail trivial redefinitions of some Lie-algebra generators; however, from the physical point of view, they resemble the appearance of *non-zero vacuum expectation values*:

$$
\langle 0|(\Phi_a - \epsilon_a I)|0\rangle = 0 \Rightarrow \langle 0|\Phi_a|0\rangle = \epsilon_a. \tag{9}
$$

Let us discuss inside this framework the quantization of massless and massive electromagnetism, linear Gravity, Abelian two-form and non-Abelian Yang-Mills gauge field theories, and to point out a cohomological (Higgs-less) origin of mass.

4 Unified quantization of massless and massive vector and tensor bosons

4.1 The electromagnetic and Proca fields:

Let us start with the simplest case of the electromagnetic field. Let us use a Fourier parametrization

$$A_\mu(x) \equiv \int \frac{d^3k}{2k^0}[a_\mu(k)e^{-ikx} + a_\mu^\dagger(k)e^{ikx}], \quad \Phi(x) \equiv \int \frac{d^3k}{2k^0}[\varphi(k)e^{-ikx} + \varphi^\dagger(k)e^{ikx}],$$

for the vector potential $A_\mu(x)$ and the constraints $\Phi(x)$ (the generators of local $U(1)(x)$ gauge transformations). The Lie algebra $\tilde{\mathcal{G}}$ of the quantizing electromagnetic group $\tilde{G}$ has the following form [1]

$$\left[a_\mu(k), a_\nu^\dagger(k')\right] = \eta_{\mu\nu}\Delta_{kk'}I, \qquad \left[\varphi(k), \varphi^\dagger(k')\right] = k^2\Delta_{kk'}I,$$

$$\left[a_\mu^\dagger(k), \varphi(k')\right] = -ik_\mu\Delta_{kk'}I, \qquad \left[a_\mu(k), \varphi^\dagger(k')\right] = -ik_\mu\Delta_{kk'}I,$$

where $\Delta_{kk'} = 2k^0\delta^3(k - k')$ is the generalized delta function on the positive sheet of the mass hyperboloid and $k^2 = m^2$ is the squared mass. Constraints are first-class for $k^2 = 0$ and constraint equations $\varphi\Psi = 0 = \varphi^\dagger\Psi$ keep 2 field degrees of freedom out of the original 4, as corresponds to a photon. For $k^2 \neq 0$, constraints are second-class and the restrictions $\varphi\Psi = 0$ keep 3 field degrees of freedom out of the original 4, as corresponds to a Proca field.

4.2 Linear gravity:

For symmetric and anti-symmetric tensor potentials $A_{\mu\nu}^{(\pm)}$, the algebra is the following [1]:

$$\left[a_{\lambda\nu}^{(\pm)}(k), a_{\rho\sigma}^{\dagger(\pm)}(k')\right] = N_{\lambda\nu\rho\sigma}^{(\pm)}\Delta_{kk'}I, \quad \left[\varphi_\rho^{(\pm)}(k), \varphi_\sigma^{\dagger(\pm)}(k')\right] = k^2 M_{\rho\sigma}^{(\pm)}(k)\Delta_{kk'}I,$$

$$\left[a_{\lambda\nu}^{\dagger(\pm)}(k), \varphi_\sigma^{(\pm)}(k')\right] = -ik^\rho N_{\lambda\nu\rho\sigma}^{(\pm)}\Delta_{kk'}I, \quad \left[a_{\lambda\nu}^{(\pm)}(k), \varphi_\sigma^{\dagger(\pm)}(k')\right] = -ik^\rho N_{\lambda\nu\rho\sigma}^{(\pm)}\Delta_{kk'}I,$$

where $M_{\rho\sigma}^{(\pm)}(k) \equiv \eta_{\rho\sigma} - \kappa_{(\mp)}\frac{k_\rho k_\sigma}{k^2}$ and $N_{\lambda\nu\rho\sigma}^{(\pm)} \equiv \eta_{\lambda\rho}\eta_{\nu\sigma} \pm \eta_{\lambda\sigma}\eta_{\nu\rho} - \kappa_{(\pm)}\eta_{\lambda\nu}\eta_{\rho\sigma}$, with $\kappa_{(+)} = 1$ and $\kappa_{(-)} = 0$. For the massless $k^2 = 0$ case, all constraints are first-class for the symmetric case, whereas the massless, anti-symmetric case possesses a couple of second-class constraints:

$$\left[\check{k}^\rho\varphi_\rho^{(-)}(k), \check{k}'^\sigma\varphi_\sigma^{(-)\dagger}(k')\right] = 4(k^0)^4\Delta_{kk'}I, \tag{10}$$

where $\check{k}^\rho \equiv k_\rho$. Thus, first-class constraints for the massless anti-symmetric case are $\mathcal{T}_{(-)}^{(1)} = \{\epsilon_\mu^\rho\varphi_\rho^{(-)}, \epsilon_\mu^\rho\varphi_\rho^{(-)\dagger}\}$, $\mu = 0, 1, 2$, where ϵ_μ^ρ is a tetrad which diagonalizes the matrix $P_{\rho\sigma} = k_\rho k_\sigma$; in particular, we choose $\epsilon_3^\rho \equiv \check{k}^\rho$ and $\epsilon_0^\rho \equiv k^\rho$. There are $2 = 10 - 8$ true degrees of freedom for the symmetric case (a massless graviton) and $1 = 6 - 5$ for the anti-symmetric case (a pseudo-scalar particle).

For $k^2 \neq 0$, all constraints are second-class for the symmetric case, whereas, for the anti-symmetric case, constraints close a Proca-like subalgebra which leads to three pairs of second-class constraints, and a pair of gauge vector fields $(k^\lambda\varphi_\lambda^{(-)}, k^\lambda\varphi_\lambda^{(-)\dagger})$. The constraint equations keep $6 = 10 - 4$ field degrees of freedom for the symmetric case (massive spin 2 particle + massive scalar field —the trace of the symmetric tensor), and $3 = 6 - 3$ field degrees of freedom for the anti-symmetric case (massive pseudo-vector particle).

4.3 $SU(N)$-Gauge Invariant Yang-Mills Theories:

Let us show how mass can enter Yang-Mills theories through central (pseudo) extensions of the corresponding gauge group. This mechanism does not involve extra (Higgs) scalar particles and could provide new clues for the better understanding of the nature of the Symmetry Breaking Mechanism. We are going to outline the essential points and refer the interested reader to the Ref. [1] for further information.

Let us denote by $A^\mu(x) = r_b^a A_a^\mu(x) T^b$, $\mu = 0, ..., 3; a, b = 1, ..., N^2 - 1 = \dim(SU(N))$ the Lie-algebra valued vector potential attached to a non-Abelian gauge group which, for simplicity, we suppose to be unitary, say $T = \mathrm{Map}(\mathbb{R}^4, SU(N)) = \{U(x) = \exp \varphi_a(x) T^a\}$, where T_a are the generators of $SU(N)$, which satisfy the commutation relations $[T_a, T_b] = C_{ab}^c T_c$, and the coupling constant matrix r_b^a reduces to a multiple of the identity $r_b^a = r\delta_b^a$. We shall also make partial use of the gauge freedom to set the temporal component $A^0 = 0$, so that the Lie-algebra valued electric field is simply $E^j(x) \equiv r_b^a E_a^j(x) T^b = -\dot{A}^j(x)$. In this case, there is still a residual gauge invariance $T = \mathrm{Map}(\mathbb{R}^3, SU(N))$.

The proposed (infinite dimensional) quantizing group for quantum Yang-Mills theories will be a central extension $\tilde{G}$ of $G = (G_A \times G_E) \times_s T$ (semi-direct product of the cotangent group of the Abelian group of Lie-algebra valued vector potentials and the non-Abelian gauge group T) by $U(1)$. More precisely, the group law for $\tilde{G}$, $\tilde{g}'' = \tilde{g} * \tilde{g}$, with $\tilde{g} = (A_a^j(x), E_a^j(y), U(z); \zeta)$, can be explicitly written as (in natural units $\hbar = 1 = c$):

$$
\begin{aligned}
U''(x) &= U'(x)U(x), \\
\vec{A}''(x) &= \vec{A}'(x) + U'(x)\vec{A}(x)U'(x)^{-1}, \\
\vec{E}''(x) &= \vec{E}'(x) + U'(x)\vec{E}(x)U'(x)^{-1}, \\
\zeta'' &= \zeta'\zeta \exp\left\{-\frac{i}{r^2}\sum_{j=1}^2 \xi_j(\vec{A}', \vec{E}', U'|\vec{A}, \vec{E}, U)\right\}; \\
\xi_1(g'|g) &\equiv \int d^3x\, \mathrm{tr}\left[\left(\begin{array}{cc}\vec{A}' & \vec{E}'\end{array}\right) W \left(\begin{array}{c}U'\vec{A}U'^{-1} \\ U'\vec{E}U'^{-1}\end{array}\right)\right], \\
\xi_2(g'|g) &\equiv \int d^3x\, \mathrm{tr}\left[\left(\begin{array}{cc}\nabla U'U'^{-1} & \vec{E}'\end{array}\right) W \left(\begin{array}{c}U'\nabla UU^{-1}U'^{-1} \\ U'\vec{E}U'^{-1}\end{array}\right)\right],
\end{aligned}
\tag{11}
$$

where $W = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right)$ is a symplectic matrix and we have split up the cocycle ξ into two distinguishable and typical cocycles ξ_j, $j = 1, 2$. The first cocycle ξ_1 (a Bargmann-like one) is meant to provide *dynamics* for the vector potential, so that the couple (A, E) corresponds to a canonically-conjugate pair of field coordinates. The second cocycle ξ_2, the *mixed* cocycle, provides a non-trivial (non-diagonal) action of the gauge subgroup $\tilde{T}$ on vector potentials and determines the number of degrees of freedom of the constrained theory; in fact, it represents the "quantum" counterpart of the "classical" unhomogeneous term $U(x)\nabla U(x)^{-1}$ we miss at the right-hand side of the gauge transformation of $\vec{A}$ (second line of (11)), that is, the vector potential $\vec{A}$ has to transform homogeneously under the action of the gauge group T in order to define a proper group law, whereas the inhomogeneous term $U(x)\nabla U(x)^{-1}$ modifies the *phase* ζ of the wave function according to ξ_2 (see [1] for a covariant form of this "quantum" transformation).

To make more explicit the intrinsic significance of these two quantities ξ_j, $j = 1, 2$, let us compute the non-trivial Lie-algebra commutators of the right-invariant vector fields (that is, the

generators of the left-action $L_{\tilde{g}'}(\tilde{g}) = \tilde{g}' * \tilde{g}$ of $\tilde{G}$ on itself) from the group law (11). They are explicitly:

$$
\begin{aligned}
\left[\hat{A}_a^j(x), \hat{E}_b^k(y)\right] &= i\delta_{ab}\delta^{jk}\delta(x-y)\hat{I}, \\
\left[\hat{A}_a^j(x), \hat{\varphi}_b(y)\right] &= -iC_{ab}^c\delta(x-y)\hat{A}_c^j(x) - \frac{i}{r}\delta_{ab}\partial_x^j\delta(x-y)\hat{I}, \quad\quad (12)\\
\left[\hat{E}_a^j(x), \hat{\varphi}_b(y)\right] &= -iC_{ab}^c\delta(x-y)\hat{E}_c^j(x) \\
\left[\hat{\varphi}_a(x), \hat{\varphi}_b(y)\right] &= -iC_{ab}^c\delta(x-y)\hat{\varphi}_c(x).
\end{aligned}
$$

The unitary irreducible representations of $\tilde{G}$ with structure subgroup $\tilde{T} = T \times U(1)$ (a direct product for this case) represent a quantum theory of $n = N^2 - 1 = \dim(SU(N))$ interacting massless vector bosons. Indeed, we start with $f = 3n$ field degrees of freedom, corresponding to the basic operators $\{\hat{A}_a^j(x), \hat{E}_a^j(x)\}$ (the ones that have a conjugated counterpart); the constraints (6) provide $c = n$ independent restrictions $\hat{\varphi}_a(x)\psi = 0$, $a = 1, \ldots, n$ (the quantum implementation of the non-Abelian Gauss law), since they are first-class constraints and we choose the trivial representation $D_{\tilde{T}}^{(\epsilon)}(\tilde{g}_t) = 1$, $\forall \tilde{g}_t = (0, 0, U(x); 1) \in T$, restrictions which lead to $f_c = f - c = 2n$ field degrees of freedom corresponding to an interacting theory of n massless vector bosons.

However, more general representations $D_{\tilde{T}}^{(\epsilon)}(U) = e^{i\epsilon_U}$ can be considered when we impose additional boundary conditions like $U(x) \overset{x \to \infty}{\longrightarrow} \pm I$, that is, when we compactify the space $\mathbb{R}^3 \to S^3$ so that the gauge group T falls into disjoint homotopy classes $\{U_l, \epsilon_{U_l} \equiv l\vartheta\}$ labeled by integers $l \in \mathbb{Z} = \pi_3(SU(N))$ (the third homotopy group). The index ϑ (the ϑ-angle) parametrizes non-equivalent quantizations, as the Bloch momentum ϵ does for particles in periodic potentials, where the wave function acquires a phase $\psi(q + 2\pi) = e^{i\epsilon}\psi(q)$ after a translation of, let us say, 2π. The phenomenon of non-equivalent quantizations can also be reproduced by keeping the constraint condition $D_{\tilde{T}}^{(\epsilon)}(U) = 1$ unchanged at the price of introducing a new (pseudo) cocycle ξ_ϑ which is added to the previous cocycle $\xi = \xi_1 + \xi_2$ in (11). The generating function η_ϑ of ξ_ϑ is

$$
\eta_\vartheta(g) = \vartheta \int d^3x\, \mathcal{C}^0(x), \quad \mathcal{C}^\mu = -\frac{1}{16\pi^2}\epsilon^{\mu\alpha\beta\gamma}\text{tr}(\mathcal{F}_{\alpha\beta}\mathcal{A}_\gamma - \frac{2}{3}\mathcal{A}_\alpha\mathcal{A}_\beta\mathcal{A}_\gamma), \quad\quad (13)
$$

where $\mathcal{A} \equiv A + \nabla U U^{-1}$ and $\mathcal{C}^0$ is the temporal component of the *Chern-Simons secondary characteristic class* $\mathcal{C}^\mu$, which is the vector whose divergence equals the Pontryagin density $\mathcal{P} = \partial_\mu \mathcal{C}^\mu = -\frac{1}{16\pi^2}\text{tr}(^*\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu})$. Like some total derivatives (namely, the Pontryagin density), which do not modify the classical equations of motion when added to the Lagrangian but have a non-trivial effect in the quantum theory, pseudo-cocycles like ξ_ϑ give rise to non-equivalent quantizations when the topology of the space is affected by the imposition of certain boundary conditions ("compactification of the space"), even though they are trivial cocycles of the "unconstrained" theory. The phenomenon of non-equivalent quantizations can be also sometimes understood as a *Aharonov-Bohm-like effect* (an effect experienced by the quantum particle but not by the classical particle) and $d\eta(g) = \frac{\partial\eta(g)}{\partial g^j}dg^j$ can be seen as an *induced gauge connection* (see [8] for the example of a super-conducting ring threaded by a magnetic flux) which modifies momenta according to the minimal coupling.

We can also go further and consider more general representations $D_{\tilde{T}}^{(\epsilon)}$ of $\tilde{T}$ (in particular,

non-Abelian representations) by adding extra pseudo-cocycles to ξ. This is the case of

$$\xi_\lambda(g'|g) \equiv -2 \int d^3x \operatorname{tr}[\lambda \left(\log(U'U) - \log U' - \log U\right)], \qquad (14)$$

which is generated by $\eta_\lambda(g) = -2 \int d^3x \operatorname{tr}[\lambda \log U]$, where $\lambda = \lambda^a T_a$ is a (mass) matrix carrying some parameters λ^a which actually characterize the representation of $\tilde{G}$. In fact, this pseudo-cocycle alters the gauge group commutators and leads to the appearance of new central terms at the right-hand side of the last equation in (12), more explicitly:

$$[\hat{\varphi}_a(x), \hat{\varphi}_b(y)] = -iC_{ab}^c\delta(x-y)\hat{\varphi}_c(x) - iC_{ab}^c\frac{\lambda_c}{r^2}\delta(x-y)\hat{I}. \qquad (15)$$

Let us denote by $c \equiv \dim(T^{(1)})$ and $\tau \equiv N^2 - 1$ the dimensions of the rigid subgroup of first-class constraints and $SU(N)$, respectively. Unpolarized wave functions $\Psi(A_a^j, E_a^j, \phi_a)$ depend on $n = 2 \times 3\tau + \tau$ field coordinates in $d = 3$ dimensions; polarization equations (5) introduce $p = c + \frac{n-c}{2}$ independent restrictions on wave functions, corresponding to c non-dynamical coordinates in $T^{(1)}$ and half of the dynamical ones; finally, constraints (6) impose $q = c + \frac{\tau-c}{2}$ additional restrictions which leave $f = n - p - q = 2c + 3(\tau - c)$ field degrees of freedom (in $d = 3$). These fields correspond to c massless vector bosons (2 polarizations) attached to $T^{(1)}$ and $\tau - c$ massive vector bosons. In particular, for the massless case, we have $c = \tau$, since constraints are *first-class* (that is, we can impose $q = \tau$ restrictions) and constrained wave functions have support on $f_{m=0} = 3\tau - \tau = 2\tau \leq f_{m\neq 0}$ arbitrary fields corresponding to τ massless vector bosons. The subalgebra $\mathcal{T}^{(1)}$ corresponds to the unbroken gauge symmetry of the constrained theory. There are distinct symmetry-breaking patterns $\tilde{\mathcal{T}} \to \mathcal{T}^{(1)}$ according to the different choices of mass-matrices $\lambda_{ab} = f_{ab}^c\lambda_c$ in (12).

As already stated in (9), pseudo-cocycle parameters such as λ_c are usually hidden in a redefinition of the generators involved in the pseudo-extension $\hat{\varphi}_c(x) + \frac{\lambda_c}{r^2} \equiv \hat{\varphi}_c'(x)$, as it happens for example with the parameter c' in the Virasoro algebra (17), which is a redefinition of L_0. However, whereas the vacuum expectation value $\langle 0_\lambda|\hat{\varphi}_c(x)|0_\lambda\rangle$ is zero, the vacuum expectation value $\langle 0_\lambda|\hat{\varphi}_c'(x)|0_\lambda\rangle = \lambda_c/r^2$ of the redefined operators $\hat{\varphi}_c'(x)$ is non-zero and proportional to the cubed mass $\lambda_c \sim m_c^3$ in the 'direction' c of the unbroken gauge symmetry $T^{(1)}$, which depends on the particular choice of the mass matrix λ. Thus, the effect of the pseudo-extension manifests also in a different choice of a vacuum in which some gauge operators have a non-zero expectation value. This fact reminds us of the Higgs mechanism in non-Abelian gauge theories, where the Higgs fields point to the direction of the non-zero vacuum expectation values. However, the spirit of the Higgs mechanism, as an approach to supply mass, and the one discussed here are different, even though they share some common features. In fact, we are not making use of extra scalar fields in the theory to provide mass to the vector bosons, but it is the gauge group itself that acquires dynamics for the massive case and transfers field degrees of freedom to the vector potentials to form massive vector bosons. Thus, the appearance of mass seems to have a *cohomological origin*, beyond any introduction of extra scalar particles (Higgs bosons). The full physical implications of this alternative approach deserve further study, although some important steps have been already done (see [1]).

Also, it would be worth exploring the richness of the case $SU(\infty)$ (infinite number of colours), the Lie-algebra of which is related to the (infinite-dimensional) Lie-algebra of area preserving diffeomorphisms of the torus $SDiff(T^2)$:

$$[L_{\vec{m}}, L_{\vec{n}}] = (\vec{m} \times \vec{n})L_{\vec{m}+\vec{n}} + \vec{\lambda} \cdot \vec{m}\delta_{\vec{m}+\vec{n},0}\hat{I}, \quad \vec{m}, \vec{n} \in \mathbb{Z} \times \mathbb{Z} \qquad (16)$$

which here appears centrally extended ($\vec{\lambda} = (\lambda_1, \lambda_2)$ stands for the central extension parameter)

5   String theory and radiation phenomena

We find also in String Theory that the appearance of central terms in the constraint (Virasoro) subalgebra

$$[L_n, L_m] = \hbar(n - m)L_{n+m} + \frac{\hbar^2}{12}(cn^3 - c'n)\delta_{n,-m}I\,. \tag{17}$$

does not spoil gauge invariance but forces us to impose a polarization subgroup T_p of $\tilde{T}$ only (namely, the 'positive modes' $L_{n\geq 0}$) as restrictions $L_n\Psi_{phys} = 0$ on physical wave functions; for this case, constraints are said to be of second-class.

Moreover, this situation shows up too in black hole thermodynamics. A statistical mechanical explanation of black hole thermodynamics in terms of counting of microscopic states has been recently given in [2]. According to this reference, there is strong evidence that conformal field theories provide a universal (independent of details of the particular quantum gravity model) description of low-energy black hole entropy, which is only fixed by symmetry arguments. The Virasoro algebra turns out to be the relevant subalgebra of surface deformations of the horizon of an arbitrary black hole and constitutes the general gauge (diffeomorphism) principle that governs the density of states. As already said, although surface deformations appear as a constraint algebra, under which one might expect all the physical states on the horizon to be singlets, quantum anomalies and boundary conditions introduce central charges and change this picture, thus causing gauge/diffeomorphism modes to become physical along the horizon —this situation resembles the above Higgs-less mechanism of mass generation in Yang-Mills theories. In this way, the calculation of thermodynamical quantities, linked to the statistical mechanical problem of counting microscopic states, is reduced to the study of the representation theory and central charges of a relevant symmetry algebra.

Unruh effect (vacuum radiation in uniformly accelerated frames) is another interesting physical phenomenon linked to the previous one. A statistical mechanical description (from first principles) of it has also been given in [9] and related to the dynamical breakdown of part of the conformal symmetry $SO(4,2)$: the special conformal transformations (usually interpreted as transitions to a uniformly relativistic accelerated frame), in the context of a $SO(4,2)$ conformally invariant quantum field theory. Unruh effect can be considered as a "first order effect" that gravity has on quantum field theory, in the sense that transitions to uniformly accelerated frames are just enough to account for it. To account for higher-order effects one should consider more general diffeomorphism algebras. In Refs. [10], the author has introduced higher-$U(N_+, N_-)$-spin extensions and higher-dimensional analogies of the infinite two-dimensional conformal symmetry (17), generalizing the standard W_∞ algebra (a higher-conformal-spin extension of the Virasoro algebra), viewed as a tensor operator algebra of $SU(1,1)$ in a group-theoretic framework. These centrally-extended infinite-dimensional Lie algebras could be useful as potential gauge guiding principles towards the formulation of gravity models in realistic dimensions.

References

[1] M. Calixto and V. Aldaya, J. Phys. (Math.& Gen.) **A32**, 7287 (1999).
M. Calixto and V. Aldaya, Int. J. Mod. Phys. **A15**, 1661 (2000).

M. Calixto and V. Aldaya, Nucl. Phys. (Proc. Suppl.) **88** 361 (2000).

[2] S. Carlip, Class. Quant. Grav. **16**, 3327 (1999)

[3] M. Calixto, V. Aldaya and J. Guerrero, Int. J. Mod. Phys. **A13**, 4889 (1998).

[4] V. Aldaya and J. de Azcárraga, J. Math. Phys. **23** (1982) 1297.

[5] M. Navarro, V. Aldaya and M. Calixto, J. Math. Phys. **38** (1997) 1454.

[6] J. Guerrero, J.L. Jaramillo, V. Aldaya, Group-cohomology refinement to classify G-symplectic manifolds, math-ph/0307041

[7] V. Aldaya, M. Calixto and M. Navarro, Int. J. Mod. Phys. **A12** (1997) 3609.

[8] V. Aldaya, M. Calixto and J. Guerrero, Commun. Math. Phys. **178**, 399 (1996).

[9] V. Aldaya, M. Calixto and J.M. Cerveró, Commun. Math. Phys. **200**, 325 (1999).

[10] M. Calixto, J. Phys. (Math.& Gen.) **A33**, L69 (2000).
M. Calixto, Mod. Phys. Lett. **A15**, 939 (2000).
M. Calixto, Class. Quant. Grav. **18**, 3857 (2001).
M. Calixto, "Oscillator Realization of Higher-$U(N_+, N_-)$-Spin Lie Algebras of W_∞-type and Quantized Symplectic Diffeomorphisms", hep-th/0301200