QUANTUM FIELD THEORY IN A SYMMETRIC CURVED SPACE FROM A SECOND QUANTIZATION ON A GROUP*

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Abstract

In this article we propose a "second quantization" scheme especially suitable to deal with non-trivial, highly symmetric phase spaces, implemented within a more general Group Approach to Quantization, which recovers the standard Quantum Field Theory (QFT) for ordinary relativistic linear fields. We emphasize, among its main virtues, greater suitability in characterizing vacuum states in a QFT on a highly symmetric curved space-time and the absence of the usual requirement of global hyperbolicity. This can be achieved in the special case of the Anti-de Sitter universe, on which we explicitly construct a QFT.

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1 Introduction

This paper is devoted basically to generalizing the Minkowskian concept of "second quantization" to certain, non-trivial symmetric space-times. By second quantization we mean the (canonical) quantization of an infinite-dimensional classical system constituted by the linear space of wave functions associated with a quantum mechanical particle, the evolution of which is considered as a trajectory of a classical field. More precisely, the Fourier coefficients of the wave functions $a(k), a(k)^*$ are considered to be the co-ordinates of the phase space of the classical field to be quantized, and the corresponding quantum operators $\hat{a}(k), \hat{a}(k)^\dagger$ are interpreted as the annihilation and creation operators of identical copies of the particle originally (firstly) quantized. The fundamental problem in going from Minkowski space-time to any other universe is the translation of the notion of annihilation and creation operators. If the mono-particle configuration space, where wave functions are defined, is not flat, there is a problem in interpreting the field operators in terms of annihilation and creation operators. The standard decomposition of the field, in flat space, into positive and negative frequency parts, $\hat{f} = \sum_k(\phi_k \hat{a}(k) + \bar{\phi}_k \hat{a}(k)^\dagger)$, has no invariant meaning in curved space-time; that is, the choice of positive frequency modes may not be unique, hence the notion of creation and annihilation are inherently ambiguous [1, 2, 3]. This ambiguity frustrates any attempt to define uniquely the energy-momentum tensor by the usual method of normal ordering, and also implies a lack of a preferred definition of particles; the Hawking effect [4] for evaporating black holes, and the Unruh effect [5] for vacuum radiation in non-inertial reference frames, are a consequence of this last fact (see also [6] for a connection between the Unruh effect and conformal symmetry breaking).

In general terms, the crucial point in any geometrical quantization procedure of a phase space with non-trivial geometry is the *global reduction* of the prequantization mapping (in the sense of Lie representation theory [7]), which is only able to account for the semiclassical or Bohr-Sommerfeld quantization condition. This reduction is achieved in practice by restricting the arguments of the wave functions with the help of a set of partial differential equations, usually referred to in the literature as *Polarization* or *Plank* conditions [8, 9, 10, 11, 12], leading to a given "representation" space (q-space, p-space, a^*-space, etc.). The problem that then arises is to determine the extent to which those restriction conditions can be consistently (globally) written. The second quantization scheme obviously inherits the difficulty of globally stating the arguments of the first-quantized wave functions associated with the "first-quantized" problem.

Here we propose a "second quantization" scheme especially suitable to deal with non-trivial, highly symmetric phase spaces, implemented within a more general Group Approach to Quantization (GAQ) (see, for instance, Refs. [13, 14]) , which recovers the standard Quantum Field Theory (QFT) for known cases such as standard relativistic linear fields. The starting point is just the group of quantum symmetries of the model from which GAQ extracts the corresponding dynamical system.

The paper is organized as follows. In Sec. 2, we give a brief description of the GAQ formalism (Subsec. 2.1), as self-contained as possible, and particularize it for the case of Quantum Mechanics on highly symmetric, curved space-times. The simple example of the relativistic free particle, which contains most of all the essential elements of more general cases to which GAQ is applied, is worked out in Subsec. 2.2. The general approach to Quantum Mechanics on symmetric curved spaces of Subsec. 2.1 is illustrated with the specific and interesting example of the Anti-de Sitter universe, which is fully developed in Subsec. 2.3 in an unconventional picture (Fock-like picture). In Subsec. 3.1 we present the general scheme of "second quantization" on a group, and again use the case of the Anti-de Sitter universe to illustrate our procedure in Subsec. 3.2. In particular, we provide an explicit expression for propagators and an algebraic characterization of vacuum states for symmetric curved spaces. Finally, Sec. 4 is devoted to a discussion on the restriction to the standard configuration-space picture and makes some comments as to how the method of GAQ differs from previous approaches to AdS space. In particular, we discuss how the problem of a lack of global hyperbolicity for the case of a symmetric curved space, such as the AdS space, fades away in working with the corresponding symmetry group.

2 Quantum mechanics on a symmetric curved space

In the standard approach, Quantum Mechanics on a curved space Q is explicitly built in a configuration-space picture making use of the intrinsic differentiable structure of Q (usually a globally hyperbolic pseudo-Riemanian manifold with pseudo-Riemannian metric $g_{\mu\nu}$). A natural generalization of this picture points towards the possibility of considering the space Q as embedded within a larger differentiable structure containing the phase space of the theory. Physically suitable for this is a group G, which will be the driver of the quantization procedure. We shall concentrate on the cases in which Q (or, rather, its universal covering $\bar{Q}$) can be considered as (or is diffeomorphic to) a homogeneous space $Q = G/P$ of G, where P is a subgroup of G containing momentum-like coordinates p and other non-dynamical (non-symplectic) coordinates such as rotations, gauge symmetries, etc. This case corrresponds to a highly symmetric curved space, although more general situations are being investigated in connection with some sort of "perturbative-group" quantization [15, 16].

We shall demonstrate the possibility of considering representation spaces (momentum, holomorphic, etc.) other than the standard configuration-space picture (see, for instance, the explicit example of the free relativistic particle in Sec. 2.2). A given phase space may possibly be embedded in different groups. In this case, either the physical situation is able to select the appropriate group, or each of these selections might give rise to different quantum theories having, in particular, non-equivalent vacua. This fact is partially shared with more standard approaches to QFT on curved space-time [1, 2, 17] (see also [6], where we discuss the different structure of the "Poincaré" vacuum with respect to the "conformal" vacuum for a massless quantum field theory, and the effect of radiation under relativistic accelerations).

In working on a group, the problem of first and second canonical quantization on a (symmetric) curved space (which, relies heavily on both the structure of space-time itself and the structure of the *classical* solution manifold of the field equations) is translated to the problem of finding unitary, irreducible representations of a suitable group. From this viewpoint, the traditional and difficult problem of the lack of global hyperbolicity for a curved space is diluted in a group framework. Rather, the problem that really matters is of algebraic character: that of finding an appropriate polarization intended to reduce unitarily the group representation (see below). To clarify this situation briefly, let us remark that for a non-globally hyperbolic space-time, the incompleteness of the dynamics in the standard formulation shows up as the lack of self-adjointness of the elliptic operator $\hat{K}$ associated with the classical wave equation [17, 18]

$$(\frac{\partial^2}{\partial t^2} + K)\phi = 0\,. \tag{1}$$

The description of the dynamics in our group approach, however, is not forced to adopt this form. In fact, in a non-anomalous group [19, 20] (non-anomalous groups correspond to those possessing an *admissible* subalgebra [12]), there always exists a first-order polarization (providing a first-order differential system) which leads to a unitary, irreducible representation. Only when one attempts to describe, alternatively, the system in a configuration-space-like "representation" [paralleling (1)], does a higher-order polarization become required and hermiticity problems with some operators can appear. However, this lack of hermiticity is not directly attached to the lack of global hyperbolicity of space-time but, rather, to the lack of "classical integrability" of the higher-order polarization in the sense that it does not define a proper classical submanifold where the wave functions have support (see the example of the Poincaré group in the next section and Ref. [20]).

Irrespective of the above mentioned disruptive effects of non-globally hyperbolic space-times, there is at present increased interest in QFT on these spaces —for example, QFT on *time machine universes* [21, 22], and space-times which can be backgrounds for supergravity theories [23]. In fact, we have chosen the case of the AdS universe to illustrate our method of quantization because it shares both appealing properties: it is highly symmetric and non-globally hyperbolic.

2.1 Quantization on a group $\tilde{G}$

GAQ makes use of fibre-bundle-theory concepts, which are among the most powerful tools for exploring the interplay between groups and topology, and highlight topological quantum effects (see e.g. [24] for a relevant application).

The GAQ formalism was originally conceived [13] to improve Geometric Quantization (GQ) by freeing it from several limitations and technical obstructions. Among these, we point out the impossibility of considering quantum systems without classical limit, the lack of a proper (and naturally defined) Schrödinger equation in many simple cases and the ineffectiviness in dealing with anomalous systems [19, 25].

The main ingredient which enables GAQ to avoid these limitations is a Lie group structure on the manifold $\tilde{G}$ replacing the quantum manifold Q_P of GQ. $\tilde{G}$ is also a principal bundle with structure group $U(1)$, but now $\tilde{G}/U(1)$ is not forced to wear a symplectic structure. In this way, non-symplectic parameters associated with symmetries such as time translations, rotations, gauge transformations, etc. are naturally allowed and give rise to relevant operators (Hamiltonian, angular momentum, null charges, etc.). In addition, on any Lie group, there are always two sets of mutually commuting vector fields. In fact, the sets of left- and right-invariant vector fields constitute two realizations of the Lie algebra of the group, one of which can be used to represent the group, and the other to reduce the representation in a compatible manner (see below).

Needless to say, the requirement of a group structure in $\tilde{G}$ represent some drawback, although it is lesser, in practice, than it might seem. After all, any consistent (non-perturbative) quantization is only a unitary irreducible representation of a suitable (Lie, Poisson) algebra. Also, constrained quantization (see below and Refs. [14, 24]) increases the range of applicability of the formalism.

Nonetheless, we should remark that the GAQ formalism is not meant to quantize a classical system (a phase space) but, rather, the quantizing group is the primary quantity and in some cases (anomalous groups [19, 20, 25], for instance) it is unclear how to associate a phase space with the quantum theory obtained.

The starting point of GAQ is a group $\tilde{G}$ (the quantizing group) with a principal fibre bundle structure $\tilde{G}(M,T)$, having T as the structure group and M being the base. The group T generalizes the phase invariance of Quantum Mechanics. Although the situation can be more general [14], we shall start with the rather general case in which $\tilde{G}$ is a central extension of a group G by T [$T = U(1)$ or even $T = C^* = \Re^+ \times U(1)$]. For the one-parametric group $T = U(1)$, the group law for $\tilde{G} = \{\tilde{g} = (g,\zeta)/g \in G, \zeta \in U(1)\}$ adopts the following form:

$$\tilde{g}' * \tilde{g} = (g' * g, \zeta'\zeta e^{i\xi(g',g)}) \, , \tag{2}$$

where $g'' = g' * g$ is the group operation in G and $\xi(g',g)$ is a two-cocycle of G on $\Re$ fulfilling:

$$\xi(g_2, g_1) + \xi(g_2 * g_1, g_3) = \xi(g_2, g_1 * g_3) + \xi(g_1, g_3) \ \ , g_i \in G. \tag{3}$$

In the general theory of central extensions [26], two-cocycles are said to be equivalent if they differ in a two-coboundary, i.e. a two-cocycle which can be written in the form $\xi(g',g) = \delta(g' * g) - \delta(g') - \delta(g)$, where $\delta(g)$ is called the generating function of the two-coboundary (from now on we shall omit the prefix "two" when referring to both cocycles and coboundaries). However, although cocycles differing on a coboundary lead to equivalent central extensions as such, there are some coboundaries which provide a non-trivial connection on the fibre bundle $\tilde{G}$ and Lie-algebra structure constants different from that of the direct product $G \times U(1)$. These are generated by a function δ with a non-trivial gradient at the identity, and can be divided into Pseudo-cohomology equivalence subclasses: two pseudo-cocycles are equivalent if they differ in a coboundary generated by a function with trivial gradient at the identity [27, 28, 20]. Pseudo-cohomology plays an important role in the theory of finite-dimensional semi-simple groups, as they have trivial cohomology. For them, Pseudo-cohomology classes are associated with coadjoint orbits [29].

The right and left finite actions of the group $\tilde{G}$ on itself provide two sets of mutually commuting (left- and right-, respectively) invariant vector fields:

$$\tilde{X}^L_{\tilde{g}^i} = \left.\frac{\partial \tilde{g}''^j}{\partial \tilde{g}^i}\right|_{\tilde{g}=e} \frac{\partial}{\partial \tilde{g}^j}, \quad \tilde{X}^R_{\tilde{g}^i} = \left.\frac{\partial \tilde{g}''^j}{\partial \tilde{g}'^i}\right|_{\tilde{g}'=e} \frac{\partial}{\partial \tilde{g}^j}, \quad \left[\tilde{X}^L_{\tilde{g}^i}, \tilde{X}^R_{\tilde{g}^j}\right] = 0, \tag{4}$$

where $\{\tilde{g}^j\}$ is a parameterization of $\tilde{G}$. The GAQ program continues by finding the left-invariant 1-form Θ (the *Quantization 1-form*) associated with the central generator $\tilde{X}^L_\zeta = \tilde{X}^R_\zeta, \zeta \in T$; that is, the T-component $\tilde{\theta}^{L(\zeta)}$ of the canonical left-invariant 1-form $\tilde{\theta}^L$ on $\tilde{G}$. This constitutes the generalization of the Poincaré-Cartan form of Classical Mechanics (see [30]). The differential $d\Theta$ is a *presymplectic* form and its *characteristic module*, $\mathrm{Ker}\Theta \cap \mathrm{Ker}d\Theta$, is generated by a left subalgebra $\mathcal{G}_\Theta$ named the *characteristic subalgebra*. The quotient $(\tilde{G}, \Theta)/\mathcal{G}_\Theta$ is a *quantum manifold* in the sense of Geometric Quantization [8, 9, 10, 11]. The trajectories generated by the vector fields in $\mathcal{G}_\Theta$ constitute the generalized equations of motion of the theory (temporal evolution, rotations, gauge symmetries, etc.), and the Noether invariants under those equations are $F_{\tilde{g}^j} \equiv i_{\tilde{X}^R_{\tilde{g}^j}} \Theta$; that is, the contraction of right-invariant vector fields with the Quantization 1-form. Vector fields with null Noether invariant are called *gauge* and close an *horizontal ideal* of the whole Lie algebra of $\tilde{G}$ (see Ref. [31]).

Let $\mathcal{B}(\tilde{G})$ be the set of complex-valued *T-functions* on $\tilde{G}$ in the sense of principal bundle theory:

$$\psi(\zeta * \tilde{g}) = D_T(\zeta)\psi(\tilde{g}), \ \ \zeta \in T \ , \tag{5}$$

where D_T is the natural representation of T on the complex numbers C. The representation of $\tilde{G}$ on $\mathcal{B}(\tilde{G})$ generated by $\tilde{\mathcal{G}}^R = \{\tilde{X}^R\}$ is called *Bohr Quantization* and is *reducible*. The reduction can be achieved by means of left restrictions on wave functions ψ compatible with (5); that is, by imposing a *full polarization* $\mathcal{P}$:

$$\tilde{X}^L \psi_{\mathcal{P}} = 0, \ \ \forall \tilde{X}^L \in \mathcal{P} \ , \tag{6}$$

which is a maximal, horizontal (excluding $\tilde{X}^L_\zeta$) left subalgebra of $\tilde{\mathcal{G}}^L$ which contains $\mathcal{G}_\Theta$. There is a one-to-one correspondence between full polarizations and classes of inequivalent irreducible representations (physical systems). This problem was firstly studied by A.A. Kirillov (Ref. [12]) under the different denomination of "admissible subalgebras".

It should be noted that the existence of a full polarization, containing the whole subalgebra $\mathcal{G}_\Theta$, is not guaranteed. In case of such a breakdown, called an *anomaly*, or simply the desire to choose a preferred representation space, a higher-order polarization $\mathcal{P}^{HO}$ must be imposed [19, 20]. A higher-order polarization is a maximal, horizontal subalgebra of the left enveloping algebra $U\tilde{\mathcal{G}}^L$ which contains $\mathcal{G}_\Theta$. The kind of theory that interests us is a particular case of this last situation; for the case of a representation space $Q = G/P \sim (t, \vec{x})$, the higher-order polarization must be made of the extended vector fields corresponding to the Lie algebra of the subgroup P (which is parametrized by the co-ordinates h^i complementary to the space-time ones, i.e., boosts, rotations, gauge symmetries, etc) and a "deformation" $\tilde{X}^{HO}_t$ of the vector field $\tilde{X}^L_t$ associated with the temporal evolution which, for most cases, can be chosen to be a Casimir operator of G. In summary,

$$\mathcal{P}^{HO} = < \tilde{X}^{HO}_t, \tilde{X}^L_{h^i} >, \ \ h^i \in P \ . \tag{7}$$

If the group contains more than one Casimir operator, the extension procedure $G \to \tilde{G}$ chooses one of them and the higher-order polarization condition $\tilde{X}^{HO}_t \psi = 0$ will represent the equation of motion of the theory (a generalization of the relativistic wave equations: Klein-Gordon, Dirac, etc., when the curved space Q is not locally Lorentzian).

The group $\tilde{G}$ is irreducibly represented on the space $\mathcal{H}(\tilde{G}) \equiv \{|\psi\rangle\}$ of polarized wave functions, and on its dual $\mathcal{H}^*(\tilde{G}) \equiv \{\langle\psi|\}$. If we denote by

$$\psi_{\mathcal{P}}(\tilde{g}) \equiv \langle \tilde{g}_{\mathcal{P}}|\psi\rangle\,,\ \psi'^*_{\mathcal{P}}(\tilde{g}) \equiv \langle \psi'|\tilde{g}_{\mathcal{P}}\rangle \tag{8}$$

the coordinates of the "ket" $|\psi\rangle$ and the "bra" $\langle\psi'|$ in a representation defined through a polarization $\mathcal{P}$ (first- or higher-order), then, a scalar product on $\mathcal{H}(\tilde{G})$ can be naturally defined as:

$$\langle\psi'|\psi\rangle \equiv \int_{\tilde{G}} \mu(\tilde{g})\psi'^*_{\mathcal{P}}(\tilde{g})\psi_{\mathcal{P}}(\tilde{g}), \tag{9}$$

where

$$\mu(\tilde{g}) \equiv \theta^L_{\tilde{g}^1}\wedge \overset{\dim(\tilde{G})}{\cdots} \wedge\theta^L_{\tilde{g}^n} \tag{10}$$

is the left-invariant integration volume in $\tilde{G}$ and

$$1 = \int_{\tilde{G}} |\tilde{g}_{\mathcal{P}}\rangle\mu(\tilde{g})\langle\tilde{g}_{\mathcal{P}}| \tag{11}$$

formally represents a *closure* relation. A direct computation proves that, with this scalar product, the group $\tilde{G}$ is unitarily represented through the *left* finite action (ρ denotes the representation)

$$\langle\tilde{g}_{\mathcal{P}}|\rho(\tilde{g}')|\psi\rangle \equiv \psi_{\mathcal{P}}(\tilde{g}'^{-1} * \tilde{g}) \tag{12}$$

The *adjoint* action is then defined as

$$\langle\psi'|\rho^\dagger(\tilde{g}')|\psi\rangle \equiv \langle\psi|\rho(\tilde{g}')|\psi'\rangle^*, \ \text{ i.e } \ \langle\tilde{g}_{\mathcal{P}}|\rho^\dagger(\tilde{g}')|\psi\rangle = \psi_{\mathcal{P}}(\tilde{g}' * \tilde{g})\,. \tag{13}$$

We can relate the coordinates of $|\psi\rangle$ in two given representations, corresponding to two different polarizations $\mathcal{P}_1$ and $\mathcal{P}_2$, as follows

$$\psi_{\mathcal{P}_1}(\tilde{g}) = \langle\tilde{g}_{\mathcal{P}_1}|\psi\rangle = \int_{\tilde{G}} \mu(\tilde{g}')\langle\tilde{g}_{\mathcal{P}_1}|\tilde{g}'_{\mathcal{P}_2}\rangle\langle\tilde{g}'_{\mathcal{P}_2}|\psi\rangle \equiv \int_{\tilde{G}} \mu(\tilde{g}')\Delta_{\mathcal{P}_1\mathcal{P}_2}(\tilde{g},\tilde{g}')\psi_{\mathcal{P}_2}(\tilde{g}')\,, \tag{14}$$

where $\Delta_{\mathcal{P}_1\mathcal{P}_2}(\tilde{g},\tilde{g}')$ is a "polarization-changing operator" (for example, the Fourier transform $\langle x|p\rangle$, the Bargmann transform $\langle x|a\rangle$, etc). The finiteness of the integration (14) for non-compact semi-simple groups is controlled by the partial weights in the wave functions, as a consequence of the polarization equations, which damps the wave functions down for specific values of the pseudo-cohomology extension parameters —see later on Eq. (45) for the specific case of $SU(1,1)$ and Ref. [32]. For non-semisimple and non-compact (but non-anomalous) groups, $i_{\tilde{X}^L_{h^1}} i_{\tilde{X}^L_{h^2}} \dots i_{\tilde{X}^L_{h^j}} \mu(\tilde{g}),\ \tilde{X}^L_{h^i} \in \mathcal{P}^{HO}$, is an invariant restricted measure on Q. However, a minimal representation on a Cauchy hypersurface $\Sigma \subset Q$ would require a regularization procedure for each particular case.

An explicit expression of $\Delta_{\mathcal{P}_1\mathcal{P}_2}$ is possible by using a basis $\mathcal{B}(\mathcal{H}(\tilde{G})) = \{|n\rangle\}_{n\in I}$ (I is a set of index, non-necessarily discreet) of $\mathcal{H}(\tilde{G})$, as follows

$$\Delta_{\mathcal{P}_1\mathcal{P}_2}(\tilde{g},\tilde{g}') = \langle\tilde{g}_{\mathcal{P}_1}|\tilde{g}'_{\mathcal{P}_2}\rangle = \sum_{n\in I} \psi^*_{\mathcal{P}_1,n}(\tilde{g})\psi_{\mathcal{P}_2,n}(\tilde{g}')\,, \tag{15}$$

where $\psi_{\mathcal{P}_i,n}(\tilde{g}) \equiv \langle \tilde{g}_{\mathcal{P}_i}|n\rangle$ are the coordinates of $|n\rangle$ in a polarization $\mathcal{P}_i$. The basis $\{|n\rangle\}_{n\in I}$ is made of eigenvectors of the right counterpart of an Abelian subalgebra of $\mathcal{P}$; in particular, of the Hamiltonian $\tilde{X}^R_t$.

Constraints are consistently incorporated into the theory by enlarging the structure group T (which always includes $U(1)$), i.e, through T-function conditions:

$$\rho(\tilde{\tau})|\psi\rangle = D_T^{(\epsilon)}(\tilde{\tau})|\psi\rangle\,, \quad \tilde{\tau} \in T \tag{16}$$

or, for continuous transformations,

$$\tilde{X}^R_{\tilde{\tau}}|\psi\rangle = dD_T^{(\epsilon)}(\tilde{\tau})|\psi\rangle\ , \tag{17}$$

$D_T^{(\epsilon)}$ means a specific representation of T [the index ϵ parametrizes different (inequivalent) quantizations] and $dD_T^{(\epsilon)}$ is its differential. As a particular example, let us mention the case when Quantum Mechanics on Q (non-simply connected) is recovered from Quantum Mechanics on its universal covering $\bar{Q}$ by choosing $T = \Pi_1(Q) \times U(1)$ as the structure group [$\Pi_1(Q)$ means the first homotopy group of Q], thus leading to topological quantum effects commonly known as ϑ-structure [33] (see also Ref. [24]).

It is clear that, for a non-central structure group T, not all the right operators $\tilde{X}^R_{\tilde{g}}$ will preserve these constraints; a sufficient condition for a subgroup $\tilde{G}_T \subset \tilde{G}$ to preserve the constraints is (see [24]):

$$\left[\tilde{G}_T, T\right] \subset \mathrm{Ker} D_T^{(\epsilon)} \tag{18}$$

[note that, for the trivial representation of T, the subgroup $\tilde{G}_T$ is simply the *normalizer* of T]. $\tilde{G}_T$ belongs to the set of *good* operators [14], for which the subgroup T behaves as a *gauge* group (see [31] for a thorough study of gauge symmetries and constraints from the standpoint of GAQ). A more general situation can be posed in which the constraints are lifted to higher-order level, not necessarily first-order as in (17); that is, when the constraints constitute a subalgebra of the right enveloping algebra $U\tilde{\mathcal{G}}^R$. A useful example of this last case results when we select representations labelled by a value ϵ of some Casimir operator K of a subgroup $\tilde{G}_K$ of $\tilde{G}$ (see later on in Sec. 3.1 and Ref. [6] for a relevant application). The good operators have to be found, in general, in the right enveloping algebra with the only condition of preserving the constraint.

In a more general case, in which T is not a trivial central extension, $T \neq \check{T} \times U(1)$, where $\check{T} \equiv T/U(1)$ —i.e. T contains second-class constraints— the conditions (17) are not all compatible and we must select a subgroup $T_B = T_p \times U(1)$, where T_p is the subgroup associated with a right polarization subalgebra of the central extension T (see [14]).

For simplicity, we have sometimes made use of infinitesimal (geometrical) concepts, but all of this language can be translated to their finite (algebraic) counterparts (see [14]), a desirable way of proceeding when discrete transformations are incorporated into the theory.

2.2 Quantization of the free relativistic particle

Let us illustrate the abstract construction above with the help of the simple example of the free relativistic particle. Our starting point is the group law for the ordinary (non-extended) Poincaré group G in 1+1D (see [34] and references therein for the 1+3D case). It is easily derived from its action on the 1+1D Minkowski space-time parametrized by $\{a^\mu\} \equiv \{a^0, a^1 = a\}$: $a'^\mu = \Lambda^\mu_{.\nu}(p^0, p)a^\nu + x^\mu$, where $\{x^\mu\}$ are the translations and Λ, the boosts, are parametrized by either p or

$$\chi \equiv \sinh^{-1}\frac{p}{mc} \equiv \sinh^{-1}(\gamma\frac{V}{c})\,. \tag{19}$$

χ is the hyperpolar co-ordinate parametrizing the (upper sheet of the) hyperboloid $p^{02} - p^2 = m^2c^2$, often referred to as the *Lobachevsky space* (see [35] and references therein). In terms of p, we have

$$\Lambda = \begin{pmatrix} \frac{p^0}{mc} & \frac{p}{mc} \\ \frac{p}{mc} & 1 + \frac{p^2}{mc(p^0+mc)} \end{pmatrix}. \tag{20}$$

As a manifold, the group can be seen as the direct product of Minkowski space-time and the mass hyperboloid.

The consecutive action of two Poincaré transformations leads to the composition law

$$\begin{aligned} x^{0\prime\prime} &= x^{0\prime} + \frac{p^{0\prime}}{mc}x^0 + \frac{p'}{mc}x\,, \\ x'' &= x' + \frac{p^{0\prime}}{mc}x + \frac{p'}{mc}x^0\,, \\ p'' &= \frac{p^0}{mc}p' + \frac{p^{0\prime}}{mc}p\,. \end{aligned} \tag{21}$$

The Poincaré group admits only trivial central extensions by $U(1)$, i.e. extensions of the form (2) where the cocycle ξ is a coboundary generated by a function δ on G, $\xi(g', g) = \delta(g' * g) - \delta(g') - \delta(g)$. We choose $\delta(g) = mcx^0$, so that the $U(1)$ law to be added to (21) is

$$\zeta'' = \zeta'\zeta e^{imc(x^{0\prime\prime} - x^{0\prime} - x^0)}\,. \tag{22}$$

¿From (21) and (22) we immediately derive both left- and right-invariant vector fields:

$$\begin{aligned} \tilde{X}^L_{x^0} &= \frac{p^0}{mc}\frac{\partial}{\partial x^0} + \frac{p}{mc}\frac{\partial}{\partial x} + \frac{p^0(p^0 - mc)}{mc}i\zeta\frac{\partial}{\partial\zeta}\,, \\ \tilde{X}^L_x &= \frac{p^0}{mc}\frac{\partial}{\partial x} + \frac{p}{mc}\frac{\partial}{\partial x^0} + \frac{p(p^0 - mc)}{mc}i\zeta\frac{\partial}{\partial\zeta}\,, \\ \tilde{X}^L_p &= \frac{p^0}{mc}\frac{\partial}{\partial p} + \frac{p^0}{mc}xi\zeta\frac{\partial}{\partial\zeta}\,, \\ \tilde{X}^L_\zeta &= \tilde{X}^R_\zeta = \zeta\frac{\partial}{\partial\zeta}\,, \end{aligned} \tag{23}$$

$$
\begin{aligned}
\tilde{X}^R_{x^0} &= \frac{\partial}{\partial x^0}\,, \\
\tilde{X}^R_{x} &= \frac{\partial}{\partial x} + pi\zeta\frac{\partial}{\partial \zeta}\,, \\
\tilde{X}^R_{p} &= \frac{p^0}{mc}\frac{\partial}{\partial p} + \frac{x^0}{mc}\frac{\partial}{\partial x} + \frac{x}{mc}\frac{\partial}{\partial x^0} + \left(\frac{p}{mc}\,x^0 + \frac{p^0}{mc}\,x - x\right) i\zeta\frac{\partial}{\partial \zeta}\,.
\end{aligned}
$$

The pseudo-extended Poincaré algebra become

$$
\begin{aligned}
\left[\tilde{X}^R_{x^0}, \tilde{X}^R_{x}\right] &= 0\,, \\
\left[\tilde{X}^R_{x^0}, \tilde{X}^R_{p}\right] &= \frac{1}{mc}\tilde{X}^R_{x}\,, \\
\left[\tilde{X}^R_{x}, \tilde{X}^R_{p}\right] &= \frac{1}{mc}\tilde{X}^R_{x^0} - i\tilde{X}^R_{\zeta}\,.
\end{aligned}
\tag{24}
$$

Notice the appearance of the central generator $\tilde{X}^R_\zeta$ in the third commutator above, making the extension by $U(1)$ less trivial and justifying the name of pseudo-extension.

The Quantization 1–form and the Characteristic Module are:

$$
\begin{aligned}
\Theta &= -(p^0 - mc)dx^0 - xdp - i\zeta^{-1}d\zeta\,, \\
\mathcal{G}_\Theta &= <\tilde{X}^L_{x^0}>\;.
\end{aligned}
\tag{25}
$$

The pseudo-extended Poincaré group admits a first-order full polarization $\mathcal{P}_p$, which is generated by

$$
\mathcal{P}_p =<\tilde{X}^L_{x^0}, \tilde{X}^L_{x}>\,, \tag{26}
$$

leading to the momentum representation. The corresponding polarized $U(1)$-functions (5), $\tilde{X}^R_\zeta\psi = \psi$, are $\psi_{\mathcal{P}_p} = \zeta\exp[-i(p^0 - mc)x^0]\phi(p)$, and the right generators act on them as quantum operators:

$$
\begin{array}{rlll}
\hat{p}^0\,\psi_{\mathcal{P}_p} \equiv & i(\tilde{X}^R_{x^0} + imc\tilde{X}^R_\zeta)\psi_{\mathcal{P}_p} & = p^0\psi_{\mathcal{P}_p} & \Rightarrow \hat{p}^0\,\phi(p) = p^0\,\phi(p)\,, \\
\hat{p}\,\psi_{\mathcal{P}_p} \equiv & -i\tilde{X}^R_x\psi_{\mathcal{P}_p} & = p\psi_{\mathcal{P}_p} & \Rightarrow \hat{p}\,\phi(p) = p\phi(p)\,, \\
\hat{k}\,\psi_{\mathcal{P}_p} \equiv & i\tilde{X}^R_p\psi_{\mathcal{P}_p} & = \zeta i\frac{p^0}{mc}\,e^{-i(p^0-mc)x^0}\frac{\partial\phi}{\partial p} & \Rightarrow \hat{k}\,\phi(p) = i\frac{p^0}{mc}\frac{\partial\phi(p)}{\partial p}\,.
\end{array}
\tag{27}
$$

In this identification of quantum operator with right generators the rest mass energy has been added to the time generator to obtain the true energy operator $\hat{p}^0$. This is a consequence of the fact that the pseudo-extension is simply a redefinition of the $U(1)$ parameter. The representation (27) is unitary with the natural measure (10) restricted to the momentum space:

$$
\mu(\tilde{g}) = -i\frac{mc}{p^0}dx^0\wedge dx\wedge dp\wedge\zeta^{-1}d\zeta \rightarrow i_{\tilde{X}^L_\zeta} i_{\tilde{X}^L_x} i_{\tilde{X}^L_{x^0}}\mu(\tilde{g}) = \frac{mc}{p^0}dp\,. \tag{28}
$$

We should realize, however, that the boost operator $\hat{k}$ is not a true position operator, i.e. it is not $i\frac{\partial}{\partial p}$ and does not generate ordinary translations in the spectrum of the momentum operator $\hat{p}$.

Unfortunately, there is no first-order full polarization leading to the configuration-space representation. In fact, $\tilde{X}^L_p$ and $\tilde{X}^L_{x^0}$ do not close a proper horizontal subalgebra. However, we can resort to a higher-order polarization (7) including the $\tilde{X}^L_p$ generator:

$$\mathcal{P}^{HO}_x =< \tilde{X}^{L}_{x^0}{}^{HO} \equiv \tilde{X}^L_{x^0} + i\left[\sqrt{m^2c^2 - (\tilde{X}^L_x)^2} - mc\right]\tilde{X}^L_\zeta,\; \tilde{X}^L_p > \;. \tag{29}$$

We should note in passing that the infinite-order character of $\tilde{X}^{L}_{x^0}{}^{HO}$ is due to the restriction to the upper sheet of the mass hyperboloid and that a second-order operator exists,

$$\tilde{X}^{L\,2\mathrm{nd}}_{a^0} = \tilde{X}^L_{x^0} + \frac{i\hbar}{2mc}\left[\left(\tilde{X}^L_{x^0}\right)^2 - \left(\tilde{X}^L_x\right)^2\right], \tag{30}$$

which leads to the Klein-Gordon equation. We can say that the polarization containing $\tilde{X}^{L}_{x^0}{}^{HO}$ gives a highest-weight representation, whereas that containing $\tilde{X}^{L\,2\mathrm{nd}}_{a^0}$ gives a (reducible, since it contains the two signs of the energy) representation characterized by a given value (the mass) of the Casimir operator of the group. This duality is also valid for more general non-compact Lie groups.

By solving the polarization equations associated with (29), we arrive to the general expression for wave functions

$$\psi_{\mathcal{P}_x} = \zeta \exp(-ixp)\exp\left\{-imcx^0\left(\sqrt{m^2c^2 - \frac{\partial^2}{\partial x^2}} - mc\right)\right\}\phi(x)\,. \tag{31}$$

Since the form of the polarized wave functions (31) is preserved by the right-invariant vector fields, it makes sense to restrict the operators to the arbitrary functions $\phi(x)$. The resulting operators are:

$$\begin{aligned} \hat{p}^0\phi(x) &= \sqrt{m^2c^2 - \frac{\partial^2}{\partial x^2}}\,\phi(x) \\ \hat{p}\phi(x) &= -i\frac{\partial}{\partial x}\,\phi(x) \\ \hat{k}\phi(x) &= \left[\frac{x}{mc}\sqrt{m^2c^2 - \frac{\partial^2}{\partial x^2}}\right]\phi(x)\,. \end{aligned} \tag{32}$$

Unlike the non-relativistic configuration-space representation, this *minimal* realization (restricted to x and $\frac{\partial}{\partial x}$) is not unitary with the trivially regularized restriction of the measure (28), $\mu(\tilde{g}) \to dx$, even though the representation is unitary on the complete wave functions, because the $\hat{k}$ operator is not hermitian. This breakdown is a direct consequence of the *weak* closure of the higher-order polarization (29). Although it closes on polarized wave functions, thus giving a well-defined, irreducible carrier subspace, the polarization itself is not integrable in the classical sense. The transverse space to the polarization subalgebra, that which should be the x-space, is not properly defined in the weak case.

Nonetheless, the transformation $\phi(x) \equiv e^{-ix^0\hat{p}^0}\sqrt{\hat{p}^0}\varphi(x,x^0)$ restores unitarity, taking the boost operator to the symmetrized form

$$\frac{1}{\sqrt{\hat{p}^0}}\,\hat{k}\sqrt{\hat{p}^0} = \frac{1}{2}\left(\frac{x}{mc}\sqrt{m^2c^2 - \frac{\partial^2}{\partial x^2}} + \sqrt{m^2c^2 - \frac{\partial^2}{\partial x^2}}\frac{x}{mc}\right) . \qquad (33)$$

In fact, this transformation agrees with the standard prescription for the scalar product of relativistic fields: $\int dx\phi^*(x)\phi'(x) = \frac{1}{2}\int dx\varphi^*(x,x^0)\overleftrightarrow{\partial}_0\,\varphi'(x,x^0)$.

Another way of looking at the unitarity problem of minimal configuration-space representations consists of finding a *strongly* closing higher-order polarization. This can be achieved, in the case of the Poincaré group, by adding a new momentum operator $\hat{\pi}$ and a new central extension reproducing a *canonical* commutation relation $[\hat{k},\hat{\pi}] = i\hat{1}$, thus leading to a finite-dimensional enlarged Poincaré algebra: the S-Poincaré algebra [36]. The operator $\hat{\pi} \equiv i\tilde{X}^R_\kappa$ generates true translations on the spectrum of $\hat{k}$, κ, the extra parameter of the group. The spectrum of $\hat{\pi}$, $\pi \equiv mc\chi$, is related to p through (19). The explicit expression for the abovementioned higher-order polarization is:

$$\mathcal{P}^{HO}_\kappa =< \tilde{X}^L_{x^0} + imc\left[\cosh(\frac{i}{mc}\tilde{X}^L_\kappa) - 1\right]\tilde{X}^L_\zeta,\ \tilde{X}^L_x + imc\sinh(\frac{i}{mc}\tilde{X}^L_\kappa)\,\tilde{X}^L_\zeta,\ \tilde{X}^L_p > . \qquad (34)$$

The resulting minimal configuration-space representation proves to be:

$$\begin{aligned}
\hat{p}^0\phi(\kappa) &= mc\cosh(\frac{-i}{mc}\frac{\partial}{\partial\kappa})\,\phi(\kappa)\,,\\
\hat{p}\phi(\kappa) &= mc\sinh(\frac{-i}{mc}\frac{\partial}{\partial\kappa})\,\phi(\kappa)\,,\\
\hat{k}\phi(\kappa) &= \kappa\,\phi(\kappa)\,,\\
\hat{\pi}\phi(\kappa) &= -i\frac{\partial}{\partial\kappa}\,\phi(\kappa)\,,
\end{aligned} \qquad (35)$$

which is unitary with respect to the restricted measure $d\kappa$ (see [36] for more details).

2.3 Quantum mechanics on the Anti-de Sitter space-time

Anti-de Sitter space-time (AdS) [37, 38] in 1+1 dimensions can be seen as a homogeneous space of the group $G = SO(1,2)$ ($SO(3,2)$ in 3+1 dimensions). It is then a particular case of curved space on which a group quantization method is especially suited. We must remark that, in restricting to 1+1 dimensions, we find an apparent ambiguity as $SO(1,2) \sim SO(2,1)$. However, the distinction between de Sitter and Anti-de Sitter spaces in 1+1 dimensions is realized by means of two different central extensions, associated with non-equivalent co-adjoint orbits (see [6] for two different pseudo-extensions of $SO(1,2)$; see also [20] for a general discussion on pseudo-extensions and co-adjoint orbits). Furthermore, and except for discrete transformations, which are not relevant for our purpose, G can be replaced by its two-covering

$$SU(1,1) = \left\{U = \begin{pmatrix} z_1 & z_2 \\ z_2^* & z_1^* \end{pmatrix}, z_i, z_i^* \in C/\det(U) = |z_1|^2 - |z_2|^2 = 1\right\} , \qquad (36)$$

which is more directly related to the representation (and central extension) we are going to handle. The group $SU(1,1)$ has the topological structure of a trivial fibre bundle with fibre $U(1)$ and base the hyperboloid (or its projection on the plane: the open unit disk D_1). A system of coordinates adapted to this fibration is the following:

$$\eta \equiv \frac{z_1}{|z_1|}, \quad \alpha \equiv \frac{z_2}{z_1}, \quad \alpha^* \equiv \frac{z_2^*}{z_1^*}, \qquad \eta \in U(1), \quad \alpha, \alpha^* \in D_1 \ , \tag{37}$$

where $y_0 \equiv -i \log \eta$ will play the role of a (dimensionless) time coordinate and α, α^* is a couple of complex-conjugated (Fock-like) variables [see later on in Eq. (107) for a relationship between these variables, the space-time position x^μ and the covariant momenta p^μ, corresponding to the more usual "configuration-space" image].

The group law $U'' = U'U$ in η, α, α^* coordinates,

$$\begin{aligned} \eta'' &= \frac{z_1''}{|z_1''|} = \frac{\eta'\eta + \eta'\eta^*\alpha'\alpha^*}{\sqrt{(1+\eta^{*2}\alpha'\alpha^*)(1+\eta^2\alpha\alpha^{*\prime})}} \, , \\ \alpha'' &= \frac{z_2''}{z_1''} = \frac{\alpha\eta^2 + \alpha'}{\eta^2 + \alpha'\alpha^*} \, , \\ \alpha^{*\prime\prime} &= \frac{z_2^{\prime\prime *}}{z_1^{\prime\prime *}} = \frac{\alpha^*\eta^{-2} + \alpha^{*\prime}}{\eta^{-2} + \alpha^{*\prime}\alpha} \, , \end{aligned} \tag{38}$$

can be centrally-extended by $U(1)$ through a cocycle (in fact, coboundary) generated by the function $\delta(g) = -2iNy_0$ on the basic group G, in the following form:

$$\zeta'' = \zeta'\zeta e^{i\xi(g',g)} = \zeta'\zeta \left(\eta''\eta'^{-1}\eta^{-1}\right)^{-2N} \quad ; \quad \zeta, \zeta' \in U(1) \, , \tag{39}$$

where the parameter N appears to be quantized for globallity conditions [compactness of the variable ζ and η; $2N$ is the "winding number" of applications from $U(1) \subset SU(1,1) \to U(1)$], the possible values for N being $N = \frac{n}{2}$, $n \in Z$, and characterizes each irreducible representation of $\tilde{G}$. The integrallity condition for $2N$ disappears on the universal covering CAdS of AdS space-time.

The explicit expression of the left- and right-invariant vector fields of the extended group $\tilde{G}$ is:

$$\begin{aligned} \tilde{X}^L_\zeta &= \tilde{X}^R_\zeta = \zeta\frac{\partial}{\partial\zeta} \\ \tilde{X}^L_\eta &= \eta\frac{\partial}{\partial\eta} - 2\alpha\frac{\partial}{\partial\alpha} + 2\alpha^*\frac{\partial}{\partial\alpha^*} \\ \tilde{X}^L_\alpha &= -\frac{1}{2}\eta\alpha^*\frac{\partial}{\partial\eta} + \frac{\partial}{\partial\alpha} - \alpha^{*2}\frac{\partial}{\partial\alpha^*} + N\alpha^*\zeta\frac{\partial}{\partial\zeta} \\ \tilde{X}^L_{\alpha^*} &= \frac{1}{2}\eta\alpha\frac{\partial}{\partial\eta} - \alpha^2\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\alpha^*} - N\alpha\zeta\frac{\partial}{\partial\zeta} \\ \tilde{X}^R_\eta &= \eta\frac{\partial}{\partial\eta} \end{aligned} \tag{40}$$

$$\begin{aligned}
\tilde{X}^R_\alpha &= \frac{1}{2}\eta^{-1}\alpha^*\frac{\partial}{\partial\eta} + \eta^{-2}(1-\alpha\alpha^*)\frac{\partial}{\partial\alpha} - N\eta^{-2}\alpha^*\zeta\frac{\partial}{\partial\zeta} \\
\tilde{X}^R_{\alpha^*} &= -\frac{1}{2}\eta^{3}\alpha\frac{\partial}{\partial\eta} + \eta^{2}(1-\alpha\alpha^*)\frac{\partial}{\partial\alpha^*} + N\eta^{2}\alpha\zeta\frac{\partial}{\partial\zeta} \,.
\end{aligned}$$

Both sets close the Lie algebra:

$$\begin{aligned}
\left[\tilde{X}^L_\eta, \tilde{X}^L_\alpha\right] &= 2\tilde{X}^L_\alpha \\
\left[\tilde{X}^L_\eta, \tilde{X}^L_{\alpha^*}\right] &= -2\tilde{X}^L_{\alpha^*} \\
\left[\tilde{X}^L_\alpha, \tilde{X}^L_{\alpha^*}\right] &= \tilde{X}^L_\eta - 2N\tilde{X}^L_\zeta \\
\left[\tilde{X}^L_\zeta, \mathrm{all}\right] &= 0 \,,
\end{aligned} \tag{41}$$

and the corresponding right counterparts change a sign in the structure constants. The Casimir operator for this Lie algebra is (except for a central term ambiguity):

$$\hat{C} = (\tilde{X}^R_\eta - 2N\tilde{X}^R_\zeta)^2 + 2\tilde{X}^R_\alpha\tilde{X}^R_{\alpha^*} + 2\tilde{X}^R_{\alpha^*}\tilde{X}^R_\alpha = -(\tilde{X}^R_{y_0} - 2N\tilde{X}^R_{y_3})^2 + (\tilde{X}^R_{y_1})^2 + (\tilde{X}^R_{y_2})^2 \,, \tag{42}$$

where we have denoted $\alpha \equiv y_1 + iy_2$ and $\zeta \equiv e^{iy_3}$ for future use.

The Quantization 1-form and the Characteristic Module have the following form:

$$\begin{aligned}
\Theta &= \frac{iN}{1-\alpha\alpha^*}\left(4\alpha\alpha^*\eta^{-1}d\eta + \alpha^* d\alpha - \alpha d\alpha^*\right) - i\zeta^{-1}d\zeta \\
\mathcal{G}_\Theta &= <\tilde{X}^L_\eta> \,.
\end{aligned} \tag{43}$$

We realize that a full polarization (first-order and complete) exist for this holomorphic picture and it is made up of

$$\mathcal{P} = <\tilde{X}^L_\eta, \tilde{X}^L_\alpha> \,. \tag{44}$$

The solution of the T-function condition (5), together with the polarization conditions (6), lead to the following wave functions:

$$\begin{aligned}
\psi^{(N)}(\eta,\alpha,\alpha^*,\zeta) &= W_N(\alpha,\alpha^*,\zeta)\phi(s) \\
W_N(\alpha,\alpha^*,\zeta) &= \zeta(1-\alpha\alpha^*)^N \,,
\end{aligned} \tag{45}$$

where W_N plays the role of a vacuum and ϕ is an arbitrary power series

$$\phi(s) = \sum_{n=0}^{\infty} a_n s^n, \tag{46}$$

in the variable

$$s \equiv \eta^{-2}\alpha^* = \frac{z_2^*}{z_1}. \tag{47}$$

The group $\tilde{G}$ acts on this one-dimensional space by left translation ($\tilde{g}'' = \tilde{g}' * \tilde{g}$) as follows:

$$s \to s' = \frac{s + \alpha^{*\prime}}{\eta'^2(1 + s\alpha')} \,. \tag{48}$$

The left-invariant integration volume has the form

$$\mu(\alpha, \alpha^*, \eta, \zeta) = \frac{-(2\pi)^{-3}}{(1-\alpha\alpha^*)^2}\eta^{-1}d\eta \wedge d\mathrm{Re}(\alpha) \wedge d\mathrm{Im}(\alpha) \wedge \zeta^{-1}d\zeta\,. \tag{49}$$

Let us denote $\check{\psi}^{(N)}_n(\eta, \alpha, \alpha^*, \zeta) = W_N(\alpha, \alpha^*, \zeta)s^n$ a basic wave function. The scalar product of two of them is:

$$\langle \check{\psi}^{(N)}_n | \check{\psi}^{(N)}_m \rangle = \frac{n!(2N-2)!}{(2N+n-1)!}\delta_{nm} \equiv C^{(N)}_n \delta_{nm}, \tag{50}$$

which is well-defined (finite) for values $N > \frac{1}{2}$. This condition can be relaxed to $N > 0$ by going to the universal covering group of G. The set

$$B(\mathcal{H}_N(\tilde{G})) = \left\{ \psi^{(N)}_n \equiv \frac{1}{\sqrt{C^{(N)}_n}}\check{\psi}^{(N)}_n \right\} \tag{51}$$

is then orthonormal and complete, i.e, an orthonormal base of $\mathcal{H}_N(\tilde{G})$.

The action of the right-invariant vector fields (operators in the theory) on polarized wave functions in (45) has the explicit form:

$$\begin{aligned}
\tilde{X}^R_\eta \psi^{(N)} &= W_N \cdot (-2s\frac{\partial}{\partial s})\phi(s)\,, \\
\tilde{X}^R_\alpha \psi^{(N)} &= W_N \cdot (-s^2\frac{\partial}{\partial s} - 2Ns)\phi(s)\,, \\
\tilde{X}^R_{\alpha^*} \psi^{(N)} &= W_N \cdot (\frac{\partial}{\partial s})\phi(s)\,, \\
\tilde{X}^R_\zeta \psi^{(N)} &= \psi^{(N)},
\end{aligned} \tag{52}$$

providing an action on the ϕ-space once the common factor W_N has been factored out.

The action of the Casimir operator (42) on polarized wave functions is:

$$\hat{C}\psi^{(N)} = 4N(N-1)\psi^{(N)}\ . \tag{53}$$

The finite (left) action (12) of an arbitrary element $\tilde{g}' = \tilde{g}'(\eta', \alpha', \alpha^{*\prime}, \zeta') \in \tilde{G}$ on an arbitrary wave function

$$\psi^{(N)}(\tilde{g}) = \sum_{n=0}^{\infty} a_n \psi^{(N)}_n(\tilde{g})\ , \tag{54}$$

can be given through the matrix elements $\rho^{(N)}_{mn}(\tilde{g}') \equiv \langle \psi^{(N)}_m | \rho(\tilde{g}') | \psi^{(N)}_n \rangle$ of ρ in the base $B(\mathcal{H}_N(\tilde{G}))$. They have the following expression:

$$\begin{aligned}
\rho^{(N)}_{mn}(\tilde{g}) &= \sqrt{\frac{C^{(N)}_m}{C^{(N)}_n}}\zeta^{-1} \sum_{l=\mathrm{Max}(0,n-m)}^{n} \begin{pmatrix} n \\ l \end{pmatrix} \begin{pmatrix} 2N+m+l-1 \\ m-n+l \end{pmatrix} \times \\
& (-1)^l \eta^{2m} \alpha^{*l} \alpha^{m-n+l} (1-\alpha\alpha^*)^N\ .
\end{aligned} \tag{55}$$

Let us show how to second-quantize this first-quantized theory.

3 Quantum Field Theory on a symmetric curved space

3.1 "Second Quantization" on a group

In this subsection we shall develop a general approach to the quantization of linear, complex quantum fields defined on a group manifold $\tilde{G}$ [more precisely, on the quotient $\tilde{G}/(T\cup P)$]. This formalism can be seen as a "second quantization" of a "first-quantized" theory defined by a group $\tilde{G}$ and a Hilbert space $\mathcal{H}(\tilde{G})$ of polarized wave functions.

The construction of the quantizing group $\tilde{G}^{(2)}$ for this complex quantum field is as follows. Given a Hilbert space $\mathcal{H}(\tilde{G})$ and its dual $\mathcal{H}^*(\tilde{G})$, we define the direct sum

$$\begin{aligned}\mathcal{F}(\tilde{G}) &\equiv \mathcal{H}(\tilde{G})\oplus\mathcal{H}^*(\tilde{G})\\ &= \Big\{|f\rangle = |A\rangle + |B^*\rangle;\ |A\rangle\in\mathcal{H}(\tilde{G}),\, |B^*\rangle\in\mathcal{H}^*(\tilde{G})\Big\}\ ,\end{aligned} \tag{56}$$

where we have denoted $|B^*\rangle$ according to $\langle\tilde{g}^*_{\mathcal{P}}|B^*\rangle\equiv\langle B|\tilde{g}_{\mathcal{P}}\rangle = B^*_{\mathcal{P}}(\tilde{g})$. The group $\tilde{G}$ acts on this vectorial space as follows:

$$\rho(\tilde{g}')|f\rangle = \rho(\tilde{g}')|A\rangle + \rho(\tilde{g}')|B^*\rangle\ , \tag{57}$$

where

$$\langle\tilde{g}^*_{\mathcal{P}}|\rho(\tilde{g}')|B^*\rangle\equiv\langle B|\rho^\dagger(\tilde{g}')|\tilde{g}_{\mathcal{P}}\rangle = B^*_{\mathcal{P}}(\tilde{g}'^{-1}*\tilde{g}). \tag{58}$$

We can also define the dual space

$$\begin{aligned}\mathcal{F}^*(\tilde{G}) &\equiv \mathcal{H}^*(\tilde{G})\oplus\mathcal{H}^{**}(\tilde{G})\\ &= \Big\{\langle f| = \langle A| + \langle B^*|\,;\ \ \langle A|\in\mathcal{H}^*(\tilde{G}),\, \langle B^*|\in\mathcal{H}^{**}(\tilde{G})\sim\mathcal{H}(\tilde{G})\Big\}\ ,\end{aligned} \tag{59}$$

where $\tilde{G}$ acts according to the adjoint action

$$\langle f|\rho^\dagger(\tilde{g}') = \langle A|\rho^\dagger(\tilde{g}') + \langle B^*|\rho^\dagger(\tilde{g}') \tag{60}$$

and now

$$\langle B^*|\rho^\dagger(\tilde{g}')|\tilde{g}^*_{\mathcal{P}}\rangle\equiv\langle\tilde{g}_{\mathcal{P}}|\rho(\tilde{g}')|B\rangle. \tag{61}$$

Using the closure relation (11), the product of two arbitrary elements of $\mathcal{F}(\tilde{G})$ is

$$\langle f'|f\rangle = \langle A'|A\rangle + \langle A'|B^*\rangle + \langle B'^*|A\rangle + \langle B'^*|B^*\rangle = \langle A'|A\rangle + \langle B'^*|B^*\rangle\,, \tag{62}$$

since the integrals

$$\int_{\tilde{G}}\mu(\tilde{g})A'_{\mathcal{P}}{}^*(\tilde{g})B^*_{\mathcal{P}}(\tilde{g}) = 0 = \int_{\tilde{G}}\mu(\tilde{g})B'_{\mathcal{P}}(\tilde{g})A_{\mathcal{P}}(\tilde{g}) \tag{63}$$

are zero because of the integration on the central parameter $\zeta\in U(1)$. Thus, the subspaces $\mathcal{H}(\tilde{G})$ and $\mathcal{H}^*(\tilde{G})$ are orthogonal with respect to this scalar product in $\mathcal{F}(\tilde{G})$. A basis for $\mathcal{F}(\tilde{G})$ is provided by the set $\mathcal{B}(\mathcal{F}(\tilde{G})) = \{|n\rangle + |m^*\rangle\}_{n,m\in I}$.

The space $\mathcal{M}(\tilde{G}) \equiv \mathcal{F}(\tilde{G}) \otimes \mathcal{F}^*(\tilde{G})$ can be endowed with a symplectic structure

$$S(f', f) \equiv \frac{-i}{2}(\langle f'|f\rangle - \langle f|f'\rangle)\,, \tag{64}$$

thus defining $\mathcal{M}(\tilde{G})$ as a phase space. This symplectic structure in $\mathcal{M}(\tilde{G})$ is paralleled by the symplectic structure, Ω, in the solution manifold $\mathcal{S}$ of the classical equations (1) in other approaches to QFT in globally hyperbolic curved space (see, for example, [3]), or by the canonical product Ω in Ref. [31].

This phase space can be naturally embedded into a quantizing group

$$\tilde{G}^{(2)} \equiv \left\{ \tilde{g}^{(2)} = (g^{(2)}; \varsigma) = (\tilde{g}, |f\rangle, \langle f|; \varsigma) \right\}\,, \tag{65}$$

which is a (true) central extension by $U(1)$, with parameter ς, of the semidirect product $G^{(2)} \equiv \tilde{G} \otimes_\rho \mathcal{M}(\tilde{G})$ of the basic group $\tilde{G}$ and the phase space $\mathcal{M}(\tilde{G})$. The group law of $\tilde{G}^{(2)}$ is formally:

$$\begin{aligned} \tilde{g}'' &= \tilde{g}' * \tilde{g} \\ |f''\rangle &= |f'\rangle + \rho(\tilde{g}')|f\rangle \\ \langle f''| &= \langle f'| + \langle f|\rho^\dagger(\tilde{g}') \\ \varsigma'' &= \varsigma'\varsigma e^{i\xi^{(2)}(g^{(2)\prime}, g^{(2)})}\,, \end{aligned} \tag{66}$$

where $\xi^{(2)}(g^{(2)\prime}, g^{(2)})$ is a cocycle defined as

$$\xi^{(2)}(g^{(2)\prime}, g^{(2)}) \equiv \kappa S(f', \rho(\tilde{g}')f) \tag{67}$$

and κ is intended to kill any possible dimension of S. The group law (66) generalizes in a natural way the semi-direct action of the time evolution uniparametric group by the Heisenberg-Weyl group modeled on the solution manifold of a linear field, which has the structure of an infinte-dimensional symplectic vector space. Although the representation $\rho(\tilde{g})$ of the space-time symmetry group $\tilde{G}$ on the fields f could be more general (even more, $\tilde{G}^{(2)}$ might admit other structure than the semi-direct product), we shall restrict ourselves in this paper to the left-finite action (12) coming from a first quantization on the group $\tilde{G}$, thus leading to a "second quantization on a group $\tilde{G}$" paralleling the standard concept in QFT of "second quantization". The unitarity of ρ ensures the hermiticity of the space-time symmetry operators (in particular, the Hamiltonian), something which is guaranteed when ρ comes from a first quantization on $\tilde{G}$.

A system of coordinates for $\tilde{G}^{(2)}$ corresponds to a choice of representation associated with a given polarization $\mathcal{P}$

$$\begin{aligned} f^{(+)}_{\mathcal{P}}(\tilde{g}) &\equiv \langle \tilde{g}_{\mathcal{P}}|f\rangle\,, \quad f^{(-)}_{\mathcal{P}}(\tilde{g}) \equiv \langle \tilde{g}^*_{\mathcal{P}}|f\rangle\,, \\ f^{*(+)}_{\mathcal{P}}(\tilde{g}) &\equiv \langle f|\tilde{g}^*_{\mathcal{P}}\rangle\,, \quad f^{*(-)}_{\mathcal{P}}(\tilde{g}) \equiv \langle f|\tilde{g}_{\mathcal{P}}\rangle\,. \end{aligned} \tag{68}$$

This splitting of f is the group generalization of the more standard decomposition of a field in positive and negative frequency parts. If we make use of the closure relation

$1 = \int_{\tilde{G}} \mu(\tilde{g})\{|\tilde{g}_{\mathcal{P}}\rangle\langle\tilde{g}_{\mathcal{P}}| + |\tilde{g}^*_{\mathcal{P}}\rangle\langle\tilde{g}^*_{\mathcal{P}}|\}$ for $\mathcal{F}(\tilde{G})$, the explicit expression of the cocycle (67) in this coordinate system (for simplicity, we discard the semidirect action of $\tilde{G}$),

$$\begin{aligned} \xi^{(2)}(g^{(2)\prime}, g^{(2)}) &= \frac{-i\kappa}{2} \iint_{\tilde{G}} \mu(\tilde{g}')\mu(\tilde{g}) \left\{ f'^{*(-)}_{\mathcal{P}}(\tilde{g}')\Delta^{(+)}_{\mathcal{P}}(\tilde{g}',\tilde{g}) f^{(+)}_{\mathcal{P}}(\tilde{g}) \right. \\ &- f^{*(-)}_{\mathcal{P}}(\tilde{g}')\Delta^{(+)}_{\mathcal{P}}(\tilde{g}',\tilde{g}) f'^{(+)}_{\mathcal{P}}(\tilde{g}) + f'^{*(+)}_{\mathcal{P}}(\tilde{g}')\Delta^{(-)}_{\mathcal{P}}(\tilde{g}',\tilde{g}) f^{(-)}_{\mathcal{P}}(\tilde{g}) \\ &- \left. f^{*(+)}_{\mathcal{P}}(\tilde{g}')\Delta^{(-)}_{\mathcal{P}}(\tilde{g}',\tilde{g}) f'^{(-)}_{\mathcal{P}}(\tilde{g}) \right\}, \end{aligned} \tag{69}$$

where

$$\begin{aligned} \Delta^{(+)}_{\mathcal{P}}(\tilde{g}',\tilde{g}) &\equiv \langle\tilde{g}'_{\mathcal{P}}|\tilde{g}_{\mathcal{P}}\rangle = \sum_{n\in I} \psi_{\mathcal{P},n}(\tilde{g}')\psi^*_{\mathcal{P},n}(\tilde{g}), \\ \Delta^{(-)}_{\mathcal{P}}(\tilde{g}',\tilde{g}) &\equiv \langle\tilde{g}'^*_{\mathcal{P}}|\tilde{g}^*_{\mathcal{P}}\rangle = \Delta^{(+)}_{\mathcal{P}}(\tilde{g},\tilde{g}'), \end{aligned} \tag{70}$$

shows that the vector fields associated with the co-ordinates in (68) are canonically conjugated

$$\begin{aligned} \left[\tilde{X}^L_{f^{*(-)}_{\mathcal{P}}(\tilde{g}')}, \tilde{X}^L_{f^{(+)}_{\mathcal{P}}(\tilde{g})}\right] &= \kappa\Delta^{(+)}_{\mathcal{P}}(\tilde{g}',\tilde{g})\tilde{X}^L_{\varsigma}, \\ \left[\tilde{X}^L_{f^{*(+)}_{\mathcal{P}}(\tilde{g}')}, \tilde{X}^L_{f^{(-)}_{\mathcal{P}}(\tilde{g})}\right] &= \kappa\Delta^{(-)}_{\mathcal{P}}(\tilde{g}',\tilde{g})\tilde{X}^L_{\varsigma}. \end{aligned} \tag{71}$$

Here, the functions $\Delta^{(\pm)}_{\mathcal{P}}(\tilde{g}',\tilde{g})$ play the role of *propagators* (central matrices of the cocycle). The propagators in two different parametrizations of $\tilde{G}^{(2)}$, corresponding to two different polarization subalgebras $\mathcal{P}_1$ and $\mathcal{P}_2$ of $\tilde{\mathcal{G}}^L$ (or $U\tilde{\mathcal{G}}^L$), are related through polarization-changing operators (15) as follows:

$$\begin{aligned} \Delta^{(\pm)}_{\mathcal{P}_2}(\tilde{h}',\tilde{h}) &= \iint_{\tilde{G}} \mu(\tilde{g}')\mu(\tilde{g})\Delta^{(\pm)}_{\mathcal{P}_2\mathcal{P}_1}(\tilde{h}',\tilde{g}')\Delta^{(\pm)}_{\mathcal{P}_1}(\tilde{g}',\tilde{g})\Delta^{(\pm)}_{\mathcal{P}_1\mathcal{P}_2}(\tilde{g},\tilde{h}), \\ \Delta^{(+)}_{\mathcal{P}_i\mathcal{P}_j}(\tilde{h},\tilde{g}) &\equiv \Delta_{\mathcal{P}_i\mathcal{P}_j}(\tilde{h},\tilde{g}), \quad \Delta^{(-)}_{\mathcal{P}_i\mathcal{P}_j}(\tilde{h},\tilde{g}) \equiv \Delta_{\mathcal{P}_i\mathcal{P}_j}(\tilde{g},\tilde{h}). \end{aligned} \tag{72}$$

A particular case of the last expression is the transformation which relate the propagator $\langle p'|p\rangle$ in momentum-space to the propagator $\langle x'|x\rangle$ in configuration-space.

At this point, it is important to stress that a globally hyperbolic structure $(\Re \times \Sigma)$ in the space $Q = G/P$ is not a prerequisite for having a well-defined quantum field theory constructed from the group $\tilde{G}^{(2)}$. The necessity of a spatial hypersurface $\Sigma \subset Q$ —on which Cauchy data are freely specified in the standard approach— to quantize a field on Q is now inessential because we can perform the quantization program in an alternative representation space (momentum p-space, Bargmann-Fock α^*-space, etc) for which the possible undesirable global properties of Q might be absent. Even for the quantization in configuration space, it should be stressed that the existence of Σ (i.e., the necessity of a well-posed *classical* initial value formulation) is not the problem that really matters in GAQ but, rather, the existence of an appropriate polarization intended to reduce unitarily the group representation, as already commented. Note that we are computing inner products by integrating on the whole group $\tilde{G}$, not just on Σ. There may be some cases where we could factorize out the integration on the time parameter in $\int_{\tilde{G}}\mu(\tilde{g})$, together with the extra (momentum) variables (by following a regularization process in the case in which the co-ordinates are non-compact), keeping an integration $\int_\Sigma d\sigma(x)$ on $\Sigma\subset Q$ only; however, this is not the general case, and an integration on the time parameter (even on the whole group $\tilde{G}$) could be necessary to keep hermiticity in the symmetry generators [remember the comments after (32) and see later on in Sec. 4].

As regard the general problem of the "back-reaction" effects of the quantum field on the curved space-time [39], it might also be worthmentioning the *Hadamard* condition on states of the first-quantized theory as a necessary condition for a state to have a finite expected value for the stress-energy tensor [39, 40]. For Hadamard states, two-point functions (the analog of our $\Delta^{(+)}_{\mathcal{P}}+\Delta^{(-)}_{\mathcal{P}}$) satisfy a certain asymptotic condition [41]. Nevertheless, the proper discussion of this general problem in a group framework requires the stress-energy tensor to be incorporated as an infinite set of extra generators (related to a gauge-like algebra) in the group $\tilde{G}^{(2)}$ of the second quantized theory. This enlargement of $\tilde{G}^{(2)}$ is beyond the scope of this article; however, a treatment of this kind has been performed in connection with 2d-Gravity [42].

In applying the GAQ formalism to $\tilde{G}^{(2)}$, it is appropriate to use a "Fourier-like" parametrization asociated with the basis $\mathcal{B}(\mathcal{F}(\tilde{G}))=\{|n\rangle+|m^*\rangle\}_{n,m\in I}$ made of Hamiltonian eigenstates, alternative to the field-like parametrization above [see (68)]¶. If we denote by

$$\begin{array}{ll} a_n\equiv\langle n|f\rangle\,, & b_m\equiv\langle m^*|f\rangle\,,\\ a^*_n\equiv\langle f|n\rangle\,, & b^*_m\equiv\langle f|m^*\rangle\,, \end{array} \tag{73}$$

the Fourier coefficients of the "particle" and the "antiparticle", a polarization subalgebra $\mathcal{P}^{(2)}$ for $\tilde{G}^{(2)}$ can always be given by

$$\mathcal{P}^{(2)}=<\tilde{X}^L_{a_n},\tilde{X}^L_{b_m};\ \mathcal{G}_{\Theta^{(2)}}\simeq\tilde{\mathcal{G}}^L\ >,\quad\forall n,m\in I, \tag{74}$$

that is, the corresponding left-invariant vector fields $\tilde{X}^L_{a_n},\tilde{X}^L_{b_n}$ and the whole Lie algebra $\tilde{\mathcal{G}}^L$ of $\tilde{G}$, which is the characteristic subalgebra $\mathcal{G}_{\Theta^{(2)}}$ of the second-quantized theory (see next subsection). The operators of the theory are the right-invariant vector fields of $\tilde{G}^{(2)}$; in particular, the basic operators are: the annihilation operators of particles and anti-particles, $\hat{a}_n\equiv\tilde{X}^R_{a^*_n}$, $\hat{b}_m\equiv\tilde{X}^R_{b^*_m}$, and the corresponding creation operators $\hat{a}^\dagger_n\equiv-\frac{1}{\kappa}\tilde{X}^R_{a_n}$, $\hat{b}^\dagger_m\equiv-\frac{1}{\kappa}\tilde{X}^R_{b_m}$. The operators corresponding to the subgroup $\tilde{G}$ [the second-quantized version $\tilde{X}^{R(2)}_{\tilde{g}^j}$ of the first-quantized operators $\tilde{X}^R_{\tilde{g}^j}$ in (4)] are written in terms of the basic ones (it is worthmentioning that they appear, in a natural way, *normally ordered*), since they are in the characteristic subalgebra $\mathcal{G}_{\Theta^{(2)}}$ of the second-quantized theory.

The group $\tilde{G}$ plays a key role in characterizing vacuum states in the curved space Q, in the same way as the Poincaré group plays a central role in relativistic quantum theories defined on Minkowski space. In general, standard QFT in curved space suffers from the lack

¶Of course, a "manifestly covariant" parametrization $f(x)=\langle x|f\rangle$ in configuration-space relative to boost-like eigenstates $|\tilde{g}_{\mathcal{P}}=x\rangle$ is also possible [31], although the characterization of the Polarization subalgebra (see bellow) is much more involved as it makes use of the manifestly covariant propagator $\Delta(x,x')=\langle x|x'\rangle$ (see e.g. Eq. (90)), versus the more manageable $\Delta(n,n')=\langle n|n'\rangle=\delta_{n,n'}$.

of a preferred definition of particles. The infinite-dimensional character of the symplectic solution manifold of a field system is responsible for the existence of an infinite number of unitarily inequivalent irreducible representations of the Heisenberg-Weyl (H-W) relations and there is no criterion to select a preferred vacuum of the corresponding quantum field (see, for example, [2, 3, 45, 46]). This situation is not present in the finite-dimensional case, according to the Stone-von Neuman theorem [43, 44]. In our language, the origin of this fact is related to the infinite, arbitrary, non-equivalent directions that the wave functions $\Psi(a, a^*, b, b^*)$ can be polarized for the infinite-dimensional H-W subgroup $\tilde{G}^{(2)}/\tilde{G}$ of $\tilde{G}^{(2)}$ itself. In practice, this means that arbitrary choices of annihilation operators, through the canonical transformation (Bogolyubov transformation)

$$\hat{a}'_l = \sum_{n\in I} \alpha_{ln}\hat{a}_n + \beta_{ln}\hat{a}^\dagger_n\,, \quad \hat{a}'^\dagger_l = \sum_{n\in I} \bar{\beta}_{ln}\hat{a}_n + \bar{\alpha}_{ln}\hat{a}^\dagger_n \tag{75}$$

(and similarly for the antiparticle) lead to non-identical (non-unitarily equivalent) vacua charactericed by $\hat{a}_n|0\rangle = 0$ and $\hat{a}'_l|0'\rangle = 0$,$\forall n, l \in I$, when β_{ln} is not a Hilbert-Schmidt operator (see [46]). The situation is rather different, however, when we can embed the curved space Q into a given group $\tilde{G}$. In fact, the existence of a characteristic module —generated by $\mathcal{G}_{\Theta^{(2)}} \sim \tilde{\mathcal{G}}^L$— in the polarization subalgebra $\mathcal{P}^{(2)}$ strongly restricts the possible transformations (75) to automorphysms of $\mathcal{P}^{(2)}$, that is, the transformation (75) has now to preserve the maximality and horizontality properties characterizing a full polarization subalgebra like (74). In other words, the transformation (75) has to be one-to-one (in order for $\mathcal{P}^{(2)}$ to be maximal), and to fulfill:

$$\left[\tilde{X}^{L(2)}_{\tilde{g}^j}, \tilde{X}^L_{a'_l}\right] = \rho'^j_{lk}\tilde{X}^L_{a'_k}\,,\ \ \forall \tilde{g} \in \tilde{G},\ \forall l \in I\,, \tag{76}$$

in order to close a horizontal subalgebra, where the transformed structure constants ρ'^j_{lk} must be related to the original $\rho^j_{mn} \equiv \partial\rho_{mn}(\tilde{g})/\partial\tilde{g}^j\big|_{\tilde{g}=e}$ by

$$\rho'^j_{lk} = \alpha_{lm}\rho^j_{mn}\alpha^{-1}_{nk} = \beta_{lm}\rho^{j*}_{mn}\beta^{-1}_{nk}\,. \tag{77}$$

The last equation imposes strong restrictions to the coeficients α, β, which also have to fulfill the typical relations for an infinite-dimensional symplectic transformation (see e.g. [2, 3, 45, 46]), leaving a limited number of possible polarization subalgebras $\mathcal{P}^{(2)}$. For each $\mathcal{P}^{(2)}$, the corresponding vacuum state will be characterized as being annihilated by the right version of the polarization subalgebra dual to $\mathcal{P}^{(2)}$. For example, in the case of (74), the vacumm will be invariant under the action of $\tilde{G} \subset \tilde{G}^{(2)}$ and annihilated by the right-invariant vector fields $\tilde{X}^R_{a^*_n}, \tilde{X}^R_{b^*_n}$ (the standard annihilation operators $\hat{a}_n$, $\hat{b}_n$, respectively). In more physical terms, the vacuum states are those which "look the same" to any freely falling observer, anywhere in the curved space $Q = G/P$.

Moreover, it is also always possible to choose particular states which behave as vacua with respect to a given subgroup $\tilde{G}_K \subset \tilde{G}$; that is, those states which are invariant under $\tilde{G}_K$ only. For example, $\tilde{G}_K$ could be the uniparametric subgroup of time evolution (see e.g. [47] for a discussion of vacuum states in de Sitter space). Extremely interesting are the physical phenomena related to the choice of Weyl (Poincaré + dilatation) invariant pseudo-vacua (zero-mode coherent states) in a conformally invariant QFT. Vacuum radiation in relativistic acceletated frames and Fulling-Unruh effect [1, 5] are discussed in this framework in Ref. [6].

¿From a formal point of view, the choice of particular pseudo-vacua would correspond to a breakdown of the symmetry and could be understood as a constrained version of the original theory. Indeed, let us comment on the influence of the constraints in the first-quantized theory at the second quantization level. Associated with a constrained wave function satisfying (17), there is a corresponding constrained quantum field subjected to the condition:

$$\mathrm{ad}_{\tilde{X}^{R(2)}_{\tilde{\tau}}}\left(\tilde{X}^R_{|f\rangle}\right) \equiv \left[\tilde{X}^{R(2)}_{\tilde{\tau}}, \tilde{X}^R_{|f\rangle}\right] = dD^{(\epsilon)}_T(\tilde{\tau})\tilde{X}^R_{|f\rangle}\,, \tag{78}$$

where $\tilde{X}^{R(2)}_{\tilde{\tau}}$ stands for the "second-quantized version" of $\tilde{X}^R_{\tilde{\tau}}$. It is straightforward to generalize the last condition to higher-order constraints:

$$\begin{aligned} \tilde{X}^R_1\tilde{X}^R_2...\tilde{X}^R_j|\psi\rangle &= \epsilon|\psi\rangle \rightarrow \\ \mathrm{ad}_{\tilde{X}^{R(2)}_1}\left(\mathrm{ad}_{\tilde{X}^{R(2)}_2}\left(...\mathrm{ad}_{\tilde{X}^{R(2)}_j}\left(\tilde{X}^R_{|f\rangle}\right)...\right)\right) &= \epsilon\tilde{X}^R_{|f\rangle}\,. \end{aligned} \tag{79}$$

The selection of a given Hilbert subspace $\mathcal{H}^{(\epsilon)}(\tilde{G}) \subset \mathcal{H}(\tilde{G})$ made of wave functions obeying a higher-order constraint $K\psi = \epsilon\psi$, where $K = \tilde{X}^R_1\tilde{X}^R_2...\tilde{X}^R_j$ is some Casimir operator of $\tilde{G}_K \subset \tilde{G}$, manifests itself, at second quantization level, as a new (broken) QFT. The vacuum for the new observables of this broken theory (the good operators in (79)) does not have to coincide with the vacuum of the original theory, and the action of the rest of the operators (the bad operators) could make this new vacuum radiate (see Ref. [6]).

In general, constraints lead to gauge symmetries in the constrained theory and, also, the property for a subgroup $N \subset \tilde{G}$ of being gauge is inheritable at the second-quantization level.

To conclude this subsection, it is important to note that the representation of $\tilde{G}$ on $\mathcal{M}(\tilde{G})$ is reducible, but it is irreducible under $\tilde{G}$ together with the *charge conjugation* operation $a_n \leftrightarrow b_n$, which could be implemented on $\tilde{G}^{(2)}$. For simplicity, we have preferred to discard this transformation; however, a treatment including it, would be relevant as a revision of the CPT symmetry in quantum field theory. The Noether invariant associated with $\tilde{X}^{R(2)}_\zeta$ is nothing other than the *total electric charge* (the total number of particles in the case of a real field $b_n \equiv a_n$) and its central character, inside the "dynamical" group $\tilde{G}$ of the first-quantized theory, now ensures its conservation under the action of the subgroup $\tilde{G} \subset \tilde{G}^{(2)}$. To account for non-Abelian charges (iso-spin, color, etc), a non-Abelian structure group $T \subset \tilde{G}$ is required.

3.2 Example of the AdS space

Let us apply the GAQ formalism to the centrally extended group $\tilde{G}^{(2)}$ given through the group law in (66) for the case of $G = SO(2,1)$ and the "holomorphic polarization" used in Subsec. 2.3. For simplicity, we shall consider the case of a real field and we shall use the "Fourier" parameterization (73) in terms of the coefficients a_n rather than the "field" parameterization (68) in terms of $f_{\mathcal{P}}^{(\pm)}(\tilde{g})$. The explicit group law is:

$$\begin{aligned}
\tilde{g}'' &= \tilde{g}' * \tilde{g} \\
a''_m &= a'_m + \sum_{n=0}^{\infty} \rho^{(N)}_{mn}(\tilde{g}')a_n \\
a^{*}_m{}'' &= a^{*}_m{}' + \sum_{n=0}^{\infty} \rho^{(N)^*}_{mn}(\tilde{g}')a^*_n \\
\varsigma'' &= \varsigma'\varsigma \exp\frac{\kappa}{2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a^{*}_m{}'\rho^{(N)}_{mn}(\tilde{g}')a_n - a'_m\rho^{(N)^*}_{mn}(\tilde{g}')a^*_n)\,.
\end{aligned} \tag{80}$$

Let us denote $\partial_m \equiv \frac{\partial}{\partial a_m}$, $\partial^*_m \equiv \frac{\partial}{\partial a^*_m}$ and use the parameterization $\tilde{g}(\eta,\alpha,\alpha^*,\zeta) = \tilde{g}(y_\nu), \nu = 0,1,2,3$ after (37) and (42), in order to deal with hermitian operators $\tilde{X}^{R(2)}_{y_\nu}$. With this notation, the left- and right-invariant vector fields are:

$$\begin{aligned}
\tilde{X}^L_\varsigma &= \tilde{X}^R_\varsigma = \varsigma\frac{\partial}{\partial\varsigma} \\
\tilde{X}^L_{a_n} &= \sum_{m=0}^{\infty}\rho^{(N)}_{mn}(y)\partial_m + \frac{\kappa}{2}\sum_{m=0}^{\infty}\rho^{(N)}_{mn}(y)a^*_m\tilde{X}^L_\varsigma \\
\tilde{X}^L_{a^*_n} &= \sum_{m=0}^{\infty}\rho^{(N)^*}_{mn}(y)\partial^*_m - \frac{\kappa}{2}\sum_{m=0}^{\infty}\rho^{(N)^*}_{mn}(y)a_m\tilde{X}^L_\varsigma \\
\tilde{X}^{L(2)}_{y_\nu} &= \tilde{X}^L_{y_\nu} \\
\\
\tilde{X}^R_{a_n} &= \partial_n - \frac{\kappa}{2}a^*_n\tilde{X}^L_\varsigma \\
\tilde{X}^R_{a^*_n} &= \partial^*_n + \frac{\kappa}{2}a_n\tilde{X}^L_\varsigma \\
\tilde{X}^{R(2)}_{y_0} &= \tilde{X}^R_{y_0} + 2i\sum_{m=0}^{\infty}m(a_m\partial_m - a^*_m\partial^*_m) \\
\tilde{X}^{R(2)}_{y_1} &= \tilde{X}^R_{y_1} + \sum_{m=0}^{\infty}\sqrt{(m+1)(2N+m)}(a_m\partial_{m+1} - a_{m+1}\partial_m + a^*_m\partial^*_{m+1} - a^*_{m+1}\partial^*_m) \\
\tilde{X}^{R(2)}_{y_2} &= \tilde{X}^R_{y_2} + i\sum_{m=0}^{\infty}\sqrt{(m+1)(2N+m)}(a_m\partial_{m+1} + a_{m+1}\partial_m - a^*_m\partial^*_{m+1} - a^*_{m+1}\partial^*_m) \\
\tilde{X}^{R(2)}_{y_3} &= \tilde{X}^R_{y_3} - i\sum_{m=0}^{\infty}(a_m\partial_m - a^*_m\partial^*_m)\,.
\end{aligned} \tag{81}$$

The non-trivial commutators among those vector fields are:

$$\begin{aligned}
\left[\tilde{X}^{L(2)}_{y_0}, \tilde{X}^{L(2)}_{y_1}\right] &= -2\tilde{X}^{L(2)}_{y_2} \\
\left[\tilde{X}^{L(2)}_{y_0}, \tilde{X}^{L(2)}_{y_2}\right] &= 2\tilde{X}^{L(2)}_{y_1} \\
\left[\tilde{X}^{L(2)}_{y_1}, \tilde{X}^{L(2)}_{y_2}\right] &= -2\tilde{X}^{L(2)}_{y_0} + 4N\tilde{X}^{L(2)}_{y_3}
\end{aligned}$$

$$
\begin{aligned}
\left[\tilde{X}^{L(2)}_{y_0}, \tilde{X}^L_{a_n}\right] &= 2in\tilde{X}^L_{a_n} \\
\left[\tilde{X}^{L(2)}_{y_1}, \tilde{X}^L_{a_n}\right] &= -\sqrt{n(2N+n-1)}\tilde{X}^L_{a_{n-1}} + \sqrt{(n+1)(2N+n)}\tilde{X}^L_{a_{n+1}} \\
\left[\tilde{X}^{L(2)}_{y_2}, \tilde{X}^L_{a_n}\right] &= i\sqrt{n(2N+n-1)}\tilde{X}^L_{a_{n-1}} + i\sqrt{(n+1)(2N+n)}\tilde{X}^L_{a_{n+1}} \\
\left[\tilde{X}^{L(2)}_{y_3}, \tilde{X}^L_{a_n}\right] &= -i\tilde{X}^L_{a_n} \\
\left[\tilde{X}^{L(2)}_{y_0}, \tilde{X}^L_{a^*_n}\right] &= -2in\tilde{X}^L_{a^*_n} \\
\left[\tilde{X}^{L(2)}_{y_1}, \tilde{X}^L_{a^*_n}\right] &= -\sqrt{n(2N+n-1)}\tilde{X}^L_{a^*_{n-1}} + \sqrt{(n+1)(2N+n)}\tilde{X}^L_{a^*_{n+1}} \\
\left[\tilde{X}^{L(2)}_{y_2}, \tilde{X}^L_{a^*_n}\right] &= -i\sqrt{n(2N+n-1)}\tilde{X}^L_{a^*_{n-1}} - i\sqrt{(n+1)(2N+n)}\tilde{X}^L_{a^*_{n+1}} \\
\left[\tilde{X}^{L(2)}_{y_3}, \tilde{X}^L_{a^*_n}\right] &= i\tilde{X}^L_{a^*_n} \\
\left[\tilde{X}^L_{a_n}, \tilde{X}^L_{a^*_m}\right] &= -\kappa\delta_{nm}\tilde{X}^L_{\varsigma}.
\end{aligned}
\tag{82}
$$

The quantization 1-form and the characteristic module are:

$$
\begin{aligned}
\Theta^{(2)} &= \frac{i\kappa}{2}\sum_{n=0}^{\infty}(a_n da^*_n - a^*_n da_n) - i\varsigma^{-1}d\varsigma \\
\mathcal{G}_{\Theta^{(2)}} &= <\tilde{X}^{L(2)}_{y_\nu}>, \ \nu = 0,1,2,3.
\end{aligned}
\tag{83}
$$

The Noether invariants of the second-quantized theory are:

$$
\begin{aligned}
F_{a_n} &= i_{\tilde{X}^R_{a_n}}\Theta^{(2)} = -i\kappa a^*_n \\
F_{a^*_n} &= i_{\tilde{X}^R_{a_n}}\Theta^{(2)} = i\kappa a_n \\
F^{(2)}_{y_0} &= i_{\tilde{X}^{R(2)}_{y_0}}\Theta^{(2)} = 2\kappa\sum_{n=0}^{\infty} n a^*_n a_n \\
F^{(2)}_{y_1} &= i_{\tilde{X}^{R(2)}_{y_1}}\Theta^{(2)} = i\kappa\sum_{n=0}^{\infty}\sqrt{(n+1)(2N+n)}(a^*_n a_{n+1} - a^*_{n+1}a_n) \\
F^{(2)}_{y_2} &= i_{\tilde{X}^{R(2)}_{y_2}}\Theta^{(2)} = \kappa\sum_{n=0}^{\infty}\sqrt{(n+1)(2N+n)}(a^*_n a_{n+1} + a^*_{n+1}a_n) \\
F^{(2)}_{y_3} &= i_{\tilde{X}^{R(2)}_{y_3}}\Theta^{(2)} = -\kappa\sum_{n=0}^{\infty} a^*_n a_n \,.
\end{aligned}
\tag{84}
$$

A full polarization subalgebra can be:

$$
\mathcal{P}^{(2)} =< \tilde{X}^{L(2)}_{y_\nu}, \tilde{X}^L_{a_n} >, \ \ \forall n \geq 0, \ \ \nu = 0,1,2,3\,,
\tag{85}
$$

and the polarized $U(1)$-functions have the form:

$$
\Psi[a_n, a^*_n, y_\nu, \varsigma] = \varsigma \exp\left\{-\frac{\kappa}{2}\sum_{n=0}^{\infty} a^*_n a_n\right\}\Phi[a^*] \equiv \Omega\Phi[a^*]\,,
\tag{86}
$$

where Ω is the vacuum of the second quantized theory and Φ is an arbitrary power series in its arguments. As already commented, Ω looks the same to any freely falling observer anywhere in the AdS space. It is annihilated by the right version of the dual of (85) as can be seen from the general action of the right-invariant vector fields (operators in the second-quantized theory) on polarized wave functions in (86). This action has the explicit form:

$$\begin{aligned}
\tilde{X}^R_{a_n}\Psi &= \Omega\cdot(-\kappa a^*_n)\Phi \equiv \Omega\cdot(-\kappa\hat{a}^\dagger_n)\Phi \\
\tilde{X}^R_{a^*_n}\Psi &= \Omega\cdot(\partial^*_n)\Phi \equiv \Omega\cdot(\hat{a}_n)\Phi \\
\tilde{X}^{R(2)}_{y_0}\Psi &= \Omega\cdot\left(-2i\sum_{n=0}^{\infty} n\hat{a}^\dagger_n\hat{a}_n\right)\Phi \equiv -i\Omega\hat{F}^{(2)}_{y_0}\Phi \\
\tilde{X}^{R(2)}_{y_1}\Psi &= \Omega\cdot\left(\sum_{n=0}^{\infty}\sqrt{(n+1)(2N+n)}(\hat{a}^\dagger_n\hat{a}_{n+1}-\hat{a}^\dagger_{n+1}\hat{a}_n)\right)\Phi \equiv -i\Omega\hat{F}^{(2)}_{y_1}\Phi \\
\tilde{X}^{R(2)}_{y_2}\Psi &= \Omega\cdot\left(-i\sum_{n=0}^{\infty}\sqrt{(n+1)(2N+n)}(\hat{a}^\dagger_n\hat{a}_{n+1}+\hat{a}^\dagger_{n+1}\hat{a}_n)\right)\Phi \equiv -i\Omega\hat{F}^{(2)}_{y_2}\Phi \\
\tilde{X}^{R(2)}_{y_3}\Psi &= \Omega\cdot\left(i\sum_{n=0}^{\infty}\hat{a}^\dagger_n\hat{a}_n\right)\Phi \equiv -i\Omega\hat{F}^{(2)}_{y_3}\Phi\,,
\end{aligned} \tag{87}$$

where $\hat{a}_n$ and $\hat{a}^\dagger_n$ are interpreted as annihilation and creation operators, $\hat{F}^{(2)}_{y_0}$ is interpreted as the total energy operator (Hamiltonian), $\hat{F}^{(2)}_{y_3}$ represents the total number of particles (the total electric charge in the complex case), and the remainder corresponds to other conserved quantities of the theory. As already said, all those quantities appear, in a natural way, *normally ordered*. This is one of the advantages of this method of quantization: normal order does not have to be imposed by hand but, rather, it is implicitly inside the formalism.

Let us go back to the expression (83) of the Quantization 1-form. Note that its simple appearance is due to the fact that it is written in terms of the initial condition variables (84) masking the dynamical content of it. Let us perform a change of variables induced by a general action of the group $\tilde{G}$ (80)

$$a_n = \sum_{m=0}^{\infty}\rho^{(N)}_{nm}(y)c_m\,, \tag{88}$$

and express $\Theta^{(2)}$ in terms of the "evolving" variables c_m. The final form of it is as follows:

$$\begin{aligned}
\Theta^{(2)} &= \frac{i\kappa}{2}\sum_{n=0}^{\infty}(c_n dc^*_n - c^*_n dc_n + T^\nu_n(y)dy_\nu) - i\varsigma^{-1}d\varsigma \\
T^\nu_n(y) &\equiv \sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\left(\rho^{(N)}_{nm}(y)\frac{\partial\rho^{(N)*}_{nl}(y)}{\partial y_\nu} - \rho^{(N)*}_{nl}(y)\frac{\partial\rho^{(N)}_{nm}(y)}{\partial y_\nu}\right)c^*_l c_m.
\end{aligned} \tag{89}$$

The quantities $T^\nu_n(y)$ play the role of momenta (for instance, $T^0_n(y)$ is the n-mode energy).

For completeness, we shall give the explicit expression of the propagator (70) in the present holomorphic polarization case. After some calculations, it proves to be:

$$\Delta^{(+)}(\tilde{g}',\tilde{g}) = \sum_{n=0}^{\infty} \psi_n^{(N)}(\tilde{g}')\psi_n^{(N)*}(\tilde{g}) = (2N-1)\frac{\zeta'\zeta^*(1-\alpha'\alpha^{*\prime})^N(1-\alpha\alpha^*)^N}{(1-\alpha^{*\prime}\alpha\eta'^{-2}\eta^2)^{2N}}\,. \quad (90)$$

In the configuration-space image, the corresponding propagator can be calculated by making use of the expression (72) and the polarization-changing operator given in [48] ("relativistic Bargmann transform"). Let us discuss a bit more on the connection between the "holomorphic picture" used through the paper and the more usual "configuration-space picture" used in standard approaches to QFT, within the GAQ framework.

4 Configuration-space image

The Quantum Mechanics on AdS can be also built in the position representation, which is the usual image when one makes use of the intrinsic differentiable structure of this space-time. In order to make contact with this standard approach, let us consider the 3D hyperboloid defining the de Sitter and AdS universes (see e.g. [38, 37])

$$\eta_{\mu\nu}x^\mu x^\nu + \frac{\lambda^2}{k} = \frac{1}{k}\,, \quad (91)$$

where $\eta_{\mu\nu} = (+,-), x^\mu = (x^0, x^1)$, λ is the extra co-ordinate and k stands for the curvature of the space. For $k > 0$, Eq. (91) defines the AdS space-time ($SO(1,2)$ homogeneous space), for $k < 0$, it defines the de Sitter space-time ($SO(2,1)$ homogeneous space), and for $k = 0$, we get the Minkowski space-time.

The line element in the flat 3-D space defined by (91) is

$$c^2(d\tau)^2 = \eta_{\mu\nu}dx^\mu dx^\nu + \frac{1}{k}(d\lambda)^2, \quad (92)$$

where τ stands for the proper time. We solve for λ in (91) and differentiate,

$$\begin{aligned} \lambda &= \sqrt{1 - k\eta_{\mu\nu}x^\mu x^\nu} \\ d\lambda &= -\frac{k\eta_{\mu\nu}x^\mu dx^\nu}{\lambda}, \end{aligned} \quad (93)$$

and, introducing (93) into (92),

$$c^2(d\tau)^2 = \eta_{\mu\nu}dx^\mu dx^\nu + \frac{k}{\lambda^2}x^\rho\eta_{\rho\mu}x^\sigma\eta_{\sigma\nu}dx^\mu dx^\nu\,, \quad (94)$$

we obtain the metric induced by (92) in the tangent space to the hyperboloid (91); this is the de Sitter metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + k\frac{\eta_{\mu\lambda}x^\lambda\eta_{\nu\kappa}x^\kappa}{1 - k\eta_{\rho\sigma}x^\rho x^\sigma}, \quad (95)$$

with inverse

$$g^{\mu\nu} = \eta^{\mu\nu} - kx^\mu x^\nu . \tag{96}$$

Let us define the contravariant momenta for a particle of mass m, $p^\mu = m\frac{dx^\mu}{dt}$. Then, Eq. (94) may be regarded as a constraint between the momenta p^μ, i.e. the de Sitter mass shell

$$(p^0)^2 - (p)^2 + \frac{k}{\lambda^2}(p^0 x^0 - px^1)^2 = m^2c^2 . \tag{97}$$

Solving p^0 in terms of p and x^μ, we get

$$p^0 = \frac{kpx^1x^0 + \lambda\sqrt{m^2c^2 + p^2 + m^2c^2kx^{1^2}}}{1 + kx^{1^2}} . \tag{98}$$

From (95) it follows the connection

$$\Gamma^\mu_{\nu\lambda} = kx^\mu g_{\nu\lambda} , \tag{99}$$

so that the geodesics are given by the solutions of the equation

$$\frac{d^2x^\mu}{d\tau^2} = -kx^\mu , \tag{100}$$

which could also be interpreted as the equation of motion of a (general) relativistic oscillator [48].

To give the action of the (anti-)de Sitter group G over space-time, we shall adopt the prescription that AdS-parameters become those of the Poincaré group action under the limit $k \to 0$. By denoting a^0, a^1, p the AdS space-time translation parameters and boost parameters, respectively, as well as Λ the group version of the extra co-ordinate λ, the above mentioned group action proves to be:

$$\begin{pmatrix} x'^0 \\ x'^1 \\ \lambda' \end{pmatrix} = \begin{pmatrix} \frac{p^0}{mc} & \frac{p}{mc} & a^0 \\ \frac{p}{mc} & \frac{p^0}{mc} & a^1 \\ -k\frac{p^0a^0 - pa^1}{\Lambda mc} & kK & \Lambda \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ \lambda \end{pmatrix} , \tag{101}$$

where we have defined

$$K \equiv \frac{p^0a^1 - pa^0}{mc}; \;\; \Lambda \equiv \sqrt{1 - k(a^{0^2} - a^{1^2})} , \tag{102}$$

and p^0 is defined in terms of a^0, a^1, p as in (98). ¿From the action (101) it is straightforward to derive the group law by letting it act twice ($g'' = g' * g$). The explicit form of this group law is:

$$\begin{aligned} a^{0''} &= \frac{p'^0}{mc}a^0 + \frac{P'}{mc}a^1 + \Lambda a^{0'} \\ a^{1''} &= \frac{p'}{mc}a^0 + \frac{P^{0'}}{mc}a^1 + \Lambda a^{1'} \\ p'' &= \frac{p'p^0}{mc} + \frac{P^{0'}p}{mc} - \frac{k}{\Lambda}(p^0a^0 - pa^1)a^{1'} , \end{aligned} \tag{103}$$

where we have defined

$$P^0 \equiv \frac{p^0 + mcka^1 K}{\Lambda}; \;\; P \equiv \frac{p + mcka^0 K}{\Lambda}\,. \tag{104}$$

It is a very convenient practice to chose a compact time parameter in order to work out the quantum dynamics of a system with bound states. If we denote

$$t \equiv \frac{1}{\omega}\arcsin\frac{\omega a^0}{c\beta}; \;\; \beta \equiv \sqrt{1+kx^2}; \;\; \omega \equiv \sqrt{k}c; \; x \equiv a^1 \tag{105}$$

and then consider $g = g(t,x,p)$ a parameterization of G, a pseudo-extension by $U(1)$ (i.e. a cocycle generated by a function $\delta(g)$ on the basic group, usually linear in time) can be chosen to be:

$$\begin{aligned}
\zeta'' &= \zeta'\zeta e^{\frac{i}{\hbar}(\delta(g'')-\delta(g')-\delta(g))}\,, \\
\delta(g) &\equiv -mc^2\left(t + \frac{1}{mc^2}f(x,p)\right)\,, \\
f(x,p) &\equiv -\frac{2mc^2}{\omega}\arctan\left[\frac{mc^2}{\omega px}(\beta-1)(\frac{P_0}{mc}-\beta)\right]\,, \\
P_0 &\equiv \sqrt{mc^2+p^2+m^2\omega^2x^2}\,.
\end{aligned} \tag{106}$$

The new coordinates x,p,t are related to the old coordinates α,α^*,η as follows:

$$\begin{aligned}
\sqrt{\frac{m\omega}{2\hbar}}x + \frac{i}{\sqrt{2m\hbar\omega}}p &\equiv \sqrt{2N}\frac{\alpha}{1-\alpha\alpha^*}\,, \\
\exp -i\frac{\omega}{2mc^2}\delta &\equiv \eta = \exp iy_0\,,
\end{aligned} \tag{107}$$

where the extension parameter N proves to be $N = \frac{mc^2}{\hbar\omega}$, and suffers from the same requirements as before [see the comment after (39)]. The function $f(x,p)$ generalizes the simple expression $f(x,p) \sim xp$, for the flat and non-relativistic case, to the curved case.

The commutation relations among left-invariant vector fields associated with the coordinates t,x,p,ζ of $\tilde{G}$ are:

$$\begin{aligned}
\left[\tilde{X}^L_t,\, \tilde{X}^L_x\right] &= -mkc^2\tilde{X}^L_p \\
\left[\tilde{X}^L_t,\, \tilde{X}^L_p\right] &= \frac{1}{m}\tilde{X}^L_x \\
\left[\tilde{X}^L_x,\, \tilde{X}^L_p\right] &= \frac{1}{mc^2}\tilde{X}^L_t - i\frac{1}{\hbar}\tilde{X}^L_\zeta \\
\left[\tilde{X}^L_\zeta,\, \text{all}\right] &= 0\,,
\end{aligned} \tag{108}$$

and the quantization 1-form and its characteristic module are:

$$\begin{aligned}
\Theta &= \frac{p\,(P_0^2+m^2\omega^2x^2)}{\beta^2 P_0\,(P_0+mc)}dx - \frac{m^2c^2x}{P_0\,(P_0+mc)}dp - c\,(P_0-mc)\,dt + \hbar\frac{d\zeta}{i\zeta} \\
\mathcal{G}_\Theta &= <\tilde{X}^L_t>\,.
\end{aligned} \tag{109}$$

The Noether invariants are immediately calculated as $i_{\tilde{X}^R}\Theta$:

$$\begin{aligned} i_{\tilde{X}^R_t}\Theta &= -c(P_0 - mc) \equiv E \\ i_{\tilde{X}^R_x}\Theta &= \frac{p}{\beta}\cos\omega t + \frac{P_0}{mc\beta}m\omega x \sin\omega t \equiv P \\ i_{\tilde{X}^R_p}\Theta &= -\frac{P_0}{mc\beta}x\cos\omega t + \frac{p}{m\omega\beta}\sin\omega t \equiv -K\,, \end{aligned} \tag{110}$$

which fulfill the classical mass-shell restriction

$$E^2 - m^2c^2\omega^2K^2 - c^2P^2 = m^2c^4. \tag{111}$$

Let us now consider the set $\mathcal{B}(\tilde{G})$ of complex valued $U(1)$-functions on $\tilde{G}$. As we have already mentioned, the representation of $\tilde{G}$ on $\mathcal{B}(\tilde{G})$ is reducible. The reduction can be achieved by means of polarization conditions (6). We seek an explicit representation of $\tilde{G}$ on wave functions defined on the AdS space-time, this implying that the generator $\tilde{X}^L_p$ (generator corresponding to the subgroup P) has to be included in the polarization. The commutation relations (108) force us to "deform" the characteristic module $\tilde{X}^L_t$ to a "higher order" $\tilde{X}^{HO}_t$ operator, which we can choose to be basically the left version of the Casimir operator plus an arbitrary central term; more explicitly:

$$\mathcal{P}^{HO} =< \tilde{X}^{HO}_t \equiv (\tilde{X}^L_t)^2 - c^2(\tilde{X}^L_x)^2 - \frac{2imc^2}{\hbar}\tilde{X}^L_t - \frac{mc^2\omega}{\hbar}\tilde{X}^L_\zeta,\, \tilde{X}^L_p >\,, \tag{112}$$

The $U(1)$-function condition (5) together with the polarization conditions lead to wave functions which fulfill:

$$\begin{aligned} \tilde{X}^L_\zeta\psi = \psi &\quad\rightarrow\quad \psi = \zeta\phi(x,p,t) \\ \tilde{X}^L_p\psi = 0 &\quad\rightarrow\quad \psi = \zeta e^{\frac{i}{\hbar}(f+mc^2)}\varphi(x,t) \\ \tilde{X}^{HO}_t\psi = 0 &\quad\rightarrow\quad \left(\Box_x + \frac{m^2c^2}{\hbar^2} + \nu R\right)\varphi = 0\,, \end{aligned} \tag{113}$$

where $R \equiv -2k$ can be viewed as the scalar curvature (see [2] for instance), $\nu \equiv \frac{N}{2}$ and

$$\Box_x \equiv \frac{1}{c^2\beta^2}\frac{\partial^2}{\partial t^2} - \frac{2\omega^2 x}{c^2}\frac{\partial}{\partial x} - \beta^2\frac{\partial^2}{\partial x^2}\,, \tag{114}$$

is the D' Alambertian operator on AdS space-time. Clearly, for $k \rightarrow 0$, equation (113) goes to the usual Klein-Gordon equation for a free particle moving on Minkowski space-time.

The general equation (113) can be solved by power-series expansion, leading to functions

$$\varphi_n \equiv e^{-iE^{(N)}_n\omega t}\beta^{-E^{(N)}_n}H^{(N)}_n\,, \tag{115}$$

where $E_n^{(N)} = N + n$, and $H_n^{(N)}(\chi)$ are polynomials in the variable $\chi \equiv \sqrt{\frac{m\omega}{\hbar}}x$, which can be written via the Rodrigues formula ($\lambda = 1$ in Ref. [48]) as

$$H_n^{(N)}(\chi) = (-1)^n \left(1 + \frac{\chi^2}{N}\right)^{N+n} \frac{d^n}{d\chi^n}\left[\left(1 + \frac{\chi^2}{N}\right)^{-N}\right] . \tag{116}$$

There is a polarization-changing operator $\langle x|\alpha\rangle$ (called "relativistic Bargmann transform" and explicitly calculated in [48]) relating wave functions (115) in configuration space representation to those given in (51) as corresponding to a holomorphic (Fock-like) representation. The polynomials $H_n^{(N)}$ are related to the Gegenbauer [49] and Jacobi [50] polynomials. The wave functions (115) reproduce, for the 1+1D case, those found in [51] (see also the 2nd paper in Ref. [52]) for the massive case, provided that we perform the change of variables $x \to \rho \equiv \arctan(\frac{\omega}{c}x)$, $t \to \tau \equiv \frac{\pi}{2} - \omega t$. In fact, except for normalization constants and time dependence, our even wave functions can be written as

$$\begin{aligned} \varphi_{2n} &\sim \beta^{-N}\beta^{-2n}H_{2n}^{(N)}(\chi) \\ &\sim \beta^{-N}P_n^{(n-\frac{1}{2},-\frac{1}{2})}\left(\frac{\chi^2 - N}{\chi^2 + N}\right) \sim (\cos\rho)^N P_n^{(-\frac{1}{2},N-\frac{1}{2})}(\cos 2\rho)\,, \end{aligned} \tag{117}$$

which correspond to the 1+1D version of the wave functions in Ref. [51]. The restriction to even wave functions is simply because we are comparing 1+1D wave functions with the radial part of the 3+1D ones. Note also that, in 1+1D, the range of $M = N$ would be $M = 1, 2, 3, \ldots$ instead of $M = 3, 4, 5 \ldots$ [N is half-integer when working in the two-covering of AdS group $SO(1, 2)$].

The invariant integration volume in $\tilde{G}$ now adopts the following form:

$$\mu(\tilde{g}) = -i\frac{dp \wedge dx \wedge dt \wedge d\zeta}{\zeta P_0}\,, \tag{118}$$

respect to which the group $\tilde{G}$ is unitarily represented in the present configuration-space. Thus, quantization in a highly symmetric non-globally hyperbolic space-time Q, such as AdS, can proceed on the basis of symmetry by translating the problem of existence of a well-defined, deterministic classical evolution in the standard (canonical) quantization scheme, to a problem of unitarily reducing the representations of $\tilde{G} \supset Q$ in the GAQ framework. In general, unitarity implies self-adjointness of the generators of the group and, in particular, of the time generator, leading to an inherently conserved Noether invariant (the energy) in GAQ. Both properties are not always ensured in the canonical quantization on non-globally hyperbolic spaces, unless additional boundary conditions are imposed [51, 23].

As mentioned above, the reduction $\int_{\tilde{G}} \mu(\tilde{g}) \to \int_\Sigma d\sigma(x)$ to a minimal canonical representation of $\tilde{G}$ on $\Sigma \subset Q$ could lead, in general, to a loss of hermiticity of some part of the operators in $\tilde{\mathcal{G}}^R$. Indeed, we must stress the different structure of the time evolution in Minkowski space in Subsec. 2.2 (or, in general, globally hyperbolic space-times) as compared with the AdS case. The time parameter cannot be factorized out in a natural way.

The appearance of the partial weights β^{-n} in the wave functions in configuration space is traced back to the presence of a time derivative term in the quantum operators. Another consequence of the structure of time evolution (manifest covariance of our configuration space representation) is the need for the time integration in the scalar product defined through the left-invariant integration volume in (118). In fact, a naive factorization of the time dependence in the wave functions, operators and scalar product leads to a non-unitary representation; the functions in (115) are no longer orthogonal nor the operators $\tilde{X}^R_x, \tilde{X}^R_p$ are hermitian. The x-representation can nevertheless be "unitarized" by changing the integration measure, $dx \to \frac{dx}{\beta^2}$, and by redefining the operators $\tilde{X}^R_x$ and $\tilde{X}^R_p$ accordingly. The redefinition process really parallels the *multipliers method* used in the literature [32, 53, 54] to construct unitary representations of a group G when a natural invariant volume is absent. The measure $\frac{dx}{\beta^2}$ is left invariant under the $U(1)$ subgroup of $SO(1,2)$, i.e. the time evolution, so that the energy operator is not affected by the multipliers. Furthermore, the space-time parts of our wave functions $\varphi(x,t)$ satisfy a Klein-Gordon-like equation (113) with an operator $\Box_x$ (114) associated with the metric $ds^2 = c^2\beta^2 dt^2 - \beta^{-2}dx^2$ and, according to the standard techniques in Quantum Mechanics, one would define the time-invariant scalar product $\int \frac{dx}{\beta^2}\varphi^*\varphi'$ in order to have a conserved probability density current. The new representation obtained by this way is a minimal representation and it really constitutes a well-defined theory of orthogonal functions (polynomials with partial weights; see e.g. [48, 49]).

We should remark that, even though the Bargmann-Fock-like representation (45, 52) can be directly restricted to the α, α^*-dependence without losing unitarity, a time-independent polarization-changing operator is not directly defined.

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