GAUGE TRANSFORMATION PROPERTIES OF VECTOR AND TENSOR POTENTIALS REVISITED: A GROUP QUANTIZATION APPROACH *

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Abstract

The possibility of non-trivial representations of the gauge group on wavefunctionals of a gauge invariant quantum field theory leads to a generation of mass for intermediate vector and tensor bosons. The mass parameters m show up as central charges in the algebra of constraints, which then become of second-class nature. The gauge group coordinates acquire dynamics outside the null-mass shell and provide the longitudinal field degrees of freedom that massless bosons need to form massive bosons. This is a *non-Higgs* mechanism that could provide new clues for the best understanding of the symmetry breaking mechanism in unified field theories. A unified quantization of massless and massive non-Abelian Yang-Mills, linear Gravity and Abelian two-form gauge field theories are fully developed from this new approach, where a cohomological origin of mass is pointed out.

Keywords: algebraic and geometric quantization, group cohomology, constraints.

1 Introduction

In this paper we discuss a new approach to quantum gauge theories, from a group-theoretic perspective, in which mass enters the theory in a *natural* way. More precisely, the presence of mass will manifest through non-trivial responses $U\Psi = D^{(m)}_{\tilde{T}}(U)\Psi$ of the wavefunctional Ψ under the action of gauge transformations $U \in \tilde{T}$, where we denote by $D^{(m)}_{\tilde{T}}$ a specific

*Work partially supported by the DGICYT.

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representation of the gauge group $\tilde{T}$ with index m. The standard case $D^{(m)}_{\tilde{T}}(U) = 1\,, \forall U \in \tilde{T}$ corresponds to the well-known 'Gauss law' condition, which also reads $X_a \Psi = 0$ for infinitesimal gauge transformations $U \sim 1 + \varphi^a X_a$. The case of Abelian representations $D^{(\vartheta)}_{\tilde{T}}(U_n) = e^{in\vartheta}$ of $\tilde{T}$, where n denotes the winding number of U_n, leads to the well-known ϑ-vacuum phenomena. We shall see that more general (non-Abelian) representations $D^{(m)}_{\tilde{T}}$ of the gauge group $\tilde{T}$ entail *non-equivalent quantizations* (in the sense of, e.g. [1, 2]) and a *generation of mass*.

This non-trivial response of Ψ under gauge transformations U causes a *deformation* of the corresponding Lie-algebra commutators and leads to the appearance of central terms proportional to mass parameters (eventually parametrizing the *non-equivalent quantizations*) in the algebra of constraints, which then become a mixture of first- and second-class constraints. As a result, extra (internal) field degrees of freedom emerge out of second-class constraints and are transferred to the gauge potentials to conform massive bosons (without Higgs fields!).

Thus, the 'classical' case $D^{(m)}_{\tilde{T}} = 1$ is not in general preserved in passing to the quantum theory. Upon quantization, first-class constraints (connected with a gauge invariance of the classical system) might become second-class, a metamorphosis which is familiar when quantizing *anomalous* gauge theories. Quantum "anomalies" change the picture of physical states being singlets under the constraint algebra. Anomalous (unexpected) situations generally go with the standard viewpoint of quantizing classical systems; however, these breakdowns, which sometimes are inescapable obstacles for canonical quantization, could be reinterpreted as normal (even essential) situations in a wider setting. Dealing with constraints directly in the quantum arena, this transmutation in the nature of constraints should be naturally allowed, as it provides new richness to the quantum theory.

A formalism which intends to place the familiar correspondence (canonical) rules of quantum mechanics, $q \to \hat{q},\ p \to \hat{p} = -i\hbar\frac{\partial}{\partial q}$, within a rigorous frame is Geometric Quantization (GQ) [3]. The basic idea in this formalism is that the quantum theory should be an irreducible representation of the Poisson algebra of observables of the classical phase space. However, it is well known that this program cannot be fully executed because technical obstructions arise, mainly due to ordering problems (see, for example, [4, 5]). Some of these limitations can be avoided by replacing the phase-space manifold by a group, which is the spirit of the Group Approach to Quantization (GAQ) program [6]. Needless to say, the requirement of an underlying group structure represent some drawback, although less, in practice, than it might seem. After all, any consistent (non-perturbative) quantization is nothing other than a unitary irreducible representation of a suitable (Lie, Poisson) algebra. Also, constrained quantization (see below and Refs. [7, 8]) increases the range of applicability of the formalism. Nonetheless, we should remark that the GAQ formalism, which is at heart an operator description of a quantum system, is not meant to quantize a classical system (a phase space), but rather a *quantizing group* $\tilde{G}$ is the primary object. Even more, in some cases (anomalous groups [9, 10, 11], for example), it is unclear how to associate a definite classical phase space with the quantum theory obtained, thus weakening the notion itself of classical limit. Furthermore, this *cohomological* mechanism of mass-generation makes perfect sense from the GAQ framework and we are going to use its concepts and tools to work out the quantization of vector and tensor potentials.

Quantizing on a group requires the revision of some standard concepts, such as *gauge transformations*, in order to deal properly with them. The meaning of gauge transformations in Quantum Mechanics is not well understood at present (see, for example, [12]); thus, a reexamination of it is timely.

In a previous article [13], a revision of the traditional concept of gauge transformation for the electromagnetic vector potential,

$$\varphi(x) \to \varphi(x) + \varphi'(x), \quad \mathcal{A}_\mu(x) \to \mathcal{A}_\mu(x) - \partial_\mu \varphi'(x)\,, \tag{1}$$

was necessary to arrange this transformation inside a group law; that is, to adapt this operation to an action of a group on itself: the group law of the (infinite-dimensional) *electromagnetic quantizing group* $\tilde{G}$. The proposed Lie group $\tilde{G}$ had a principal bundle structure $\tilde{G} \to \tilde{G}/\tilde{T}$ and was parameterized, roughly speaking, by the coordinates $A_\mu(\vec{x},t)$ of the Abelian subgroup G_A of Lie-algebra valued vector potentials, the coordinates $v = (y_\mu, \Lambda_{\mu\nu})$ (space-time translations and Lorentz transformations) of the Poincaré group P and the coordinates $\varphi(x)$ of the local group $T \equiv \mathrm{Map}(\Re^4, U(1))$, which took part of the structure group $\tilde{T} \sim T \times U(1)$ and generalized the standard $U(1)$-phase invariance, $\Psi \sim e^{i\alpha}\Psi$, in Quantum Mechanics. In this way, the extra $\tilde{T}$-equivariance conditions on wave functions [complex valued functions $\Psi(\tilde{g})$ on $\tilde{G}$], i.e. $\Psi(\tilde{g}_t * \tilde{g}) \sim \Psi(\tilde{g})$, $\forall \tilde{g} \in \tilde{G}, \tilde{g}_t \in \tilde{T}$, provided the traditional constraints of the theory (we denote by $*$ the composition law of $\tilde{G}$).

The above-mentioned revision was motivated by the fact that the transformation (1) is not compatible with a quantizing group $\tilde{G}$. In fact, the general property $g*e = e*g = g$ for a composition law $g'' = g' * g$ of a group G (e denotes the identity element), precludes the existence of linear terms, in the group law $g''^j = g''^j(g'^k, g^l)$ of a given parameter g^j of G, other than g'^j and g^j; that is, near the identity we have $g''^j = g'^j + g^j + O(2)$ —in canonical coordinates. Therefore, the group law for the field parameter $\mathcal{A}_\mu$ cannot have linear terms in φ. The natural way to address this situation is just to choose $A_\mu \equiv \mathcal{A}_\mu - \partial_\mu\varphi$, which stays unchanged under gauge transformations, and change the phase $\zeta = e^{i\alpha}$ of the quantum-mechanical wave functional $\Psi(A)$ accordingly, as follows:

$$\begin{gathered}\varphi(x) \to \varphi(x) + \varphi'(x), \quad A_\mu(x) \to A_\mu(x), \\ \zeta \to \zeta \exp\left\{-\frac{i}{2c\hbar^2}\int_\Sigma d\sigma_\mu(x)\eta^{\rho\sigma}\partial_\rho\varphi'(x)\overleftrightarrow{\partial^\mu}A_\sigma(x)\right\}\,,\end{gathered} \tag{2}$$

where $\eta^{\rho\sigma}$ denotes the Minkowski metric, Σ denotes a spatial hypersurface and $\hbar$ is the Plank constant, which is required to kill dimensions of $\partial_\rho\varphi'\overleftrightarrow{\partial^\mu}A^\rho \equiv \partial_\rho\varphi'\partial^\mu A^\rho - A^\rho\partial^\mu\partial_\rho\varphi'$ and gives a *quantum* character to the transformation (2) versus the *classical* character of (1) [from now on, we shall use natural unities $\hbar = 1 = c$]. The piece $\partial_\rho\varphi'\overleftrightarrow{\partial^\mu}A^\rho$ in (2) takes part of a *symplectic* current

$$J^\mu(g'|g)(x) \equiv \frac{1}{2}\eta^{\rho\sigma}[(vA')_\rho(x) - \partial_\rho(v\varphi')(x)]\overleftrightarrow{\partial^\mu}[A_\sigma(x) - \partial_\sigma\varphi(x)]\,, \tag{3}$$

[we are denoting $g \equiv (A, \varphi, v)$ and $(vA')_\rho(x) \equiv \frac{\partial v^\alpha(x)}{\partial x^\rho} A'_\alpha(v(x))$, $(v\varphi')(x) \equiv \varphi'(v(x))$, with $v^\alpha(x) = \Lambda^\alpha_\beta x^\beta + y^\alpha$ the general action of the restricted Poincaré group P on Minkowski space-time] which is conserved, $\partial_\mu J^\mu = 0$, if A_ν and φ satisfy the field equations $(\partial_\mu \partial^\mu + m^2)A_\nu = 0$ and $(\partial_\mu \partial^\mu + m^2)\varphi = 0$ (m is a parameter with mass dimension), so that the integral in (2) does not depend on the chosen spacelike hypersurface Σ. The integral $\xi(g'|g) \equiv \int_\Sigma d\sigma_\mu(x) J^\mu(g'|g)(x)$ is a two-cocycle $\xi : G \times G \to \Re$ [G denotes the semi-direct product $(G_A \times T) \times_v P$], which fulfills the well-known properties:

$$\begin{array}{c} \xi(g_2|g_1) + \xi(g_2 * g_1|g_3) = \xi(g_2|g_1 * g_3) + \xi(g_1|g_3) \quad, \forall g_i \in G\,, \\ \xi(g|e) = 0 = \xi(e|g)\,, \quad \forall g \in G\,, \end{array} \tag{4}$$

and is the basic ingredient to construct the centrally extended group law $\tilde{g}'' = \tilde{g}' * \tilde{g}$, more explicitly

$$\tilde{g}'' \equiv (g''; \zeta'') = (g' * g; \zeta'\zeta e^{i\xi(g'|g)})\,, \quad g, g', g'' \in G;\ \zeta, \zeta', \zeta'' \in U(1)\,, \tag{5}$$

of the electromagnetic quantizing group $\tilde{G}$ (see below and Ref. [13] for more details).

It is worth mentioning that the required review of the concepts of gauge transformations and constraint conditions to construct the quantizing group $\tilde{G}$ has led, as a byproduct, to a unified quantization of both the electromagnetic and Proca fields [13], within the same general scheme of quantization based on a group (GAQ) [6]; clearly, a unified scheme of quantization for massless and massive gauge theories is suitable as an alternative to the standard Spontaneous Symmetry Breaking mechanism, which is intended to supply mass whereas preserving renormalizability. On the other hand, the standard (classical) transformation (1) is regained as the trajectories associated with generalized equations of motion generated by vector fields with null Noether invariants (*gauge subalgebra*, see Refs. [13, 14] and below).

This particular revision also applies for the non-Abelian case (Yang-Mills potentials), and for gravity itself, as they are gauge theories. However, for these cases, the situation seems to be a little more subtle and complicated. The goal of this article is to present a (non-perturbative) group approach to quantization of non-Abelian gauge theories and to point out a cohomological origin of mass, as a consequence of a new look to gauge transformations and constraints. Furthermore, although a proper group formulation of a quantum theory of gravity is beyond the scope of this article, the corresponding linearized version of this theory can be a useful model to work out, as a preliminary step towards the more complicated, non-linear case. Even more so, some two-dimensional toy models of quantum gravity, such as *string inspired* or *BF* theories [15], also admit a group formulation similar to the non-Abelian Yang-Mills theories.

The organization of the present paper is the following. In Sec. 2, a quantizing group $\tilde{G}$ for linear gravity and Abelian two-form gauge theories (symmetric and anti-symmetric tensor potentials, respectively) is offered. This simple example contains most of the essential elements of more general cases to which GAQ is applied and we shall make use of it to explain the method; the structure and nature of constraints, the count of field degrees of freedom, the Hilbert space and the physical operators of these theories is presented, for the massless and massive cases, in a unified manner. In Sec. 3, a quantizing group for general, non-Abelian Yang-Mills gauge theories is proposed; the full quantization is then performed inside the GAQ framework, and the connection with other approaches to quantization is given; also, non-trivial representations of the gauge group T are connected with the *ϑ-vacuum* phenomenon and with mass generation for Yang-Mills vector fields; this is a *non-Higgs* mechanism which can provide new clues for the best understanding of the nature of the symmetry-breaking mechanism. Finally, Sec. 4 is devoted to some comments, conclusions and outlooks.

2 Unified quantization of massless and massive tensor potentials

Tensor potentials $\mathcal{A}_{\lambda\nu}(x)$ are primary objects for field theories such as gravity and two-form gauge theories. More specifically, the symmetric and anti-symmetric parts $\mathcal{A}^{(\pm)}_{\lambda\nu}(x) \equiv \frac{1}{2}[\mathcal{A}_{\lambda\nu}(x) \pm \mathcal{A}_{\nu\lambda}(x)]$ of $\mathcal{A}_{\lambda\nu}(x)$ are the corresponding tensor potentials for linearized gravity and Abelian two-form gauge theories, respectively. These theories have a gauge freedom of the form (see, for example, Refs. [16, 17, 18]):

$$\varphi^{(\pm)}_\lambda(x) \to \varphi^{(\pm)\prime}_\lambda(x) + \varphi^{(\pm)}_\lambda(x), \;\; \mathcal{A}^{(\pm)}_{\lambda\nu}(x) \to \mathcal{A}^{(\pm)}_{\lambda\nu}(x) - \frac{1}{2}\left(\partial_\lambda \varphi^{(\pm)\prime}_\nu(x) \pm \partial_\nu \varphi^{(\pm)\prime}_\lambda(x)\right), \quad (6)$$

where $\varphi^{(\pm)}_\lambda$ are vector-valued functions parametrizing the four-dimensional Abelian local group $T^{(\pm)} = \mathrm{Map}(\Re^4, U(1)\times U(1)\times U(1)\times U(1))$. As for the vector potential $\mathcal{A}_\mu$, there is a quantum version of the transformation (6) compatible with a group law, which is explicitly written as:

$$\begin{aligned} &\varphi^{(\pm)}_\lambda(x) \to \varphi^{(\pm)}_\lambda(x) + \varphi^{(\pm)\prime}_\lambda(x), \quad A^{(\pm)}_{\lambda\nu}(x) \to A^{(\pm)}_{\lambda\nu}(x), \\ &\zeta \to \zeta \exp\left\{-\frac{i}{2}\int_\Sigma d\sigma_\mu(x) N^{\lambda\nu\rho\sigma}_{(\pm)} \partial_\lambda \varphi^{(\pm)\prime}_\nu(x) \overleftrightarrow{\partial^\mu} A^{(\pm)}_{\rho\sigma}(x)\right\}, \end{aligned} \quad (7)$$

where we denote $N^{\lambda\nu\rho\sigma}_{(\pm)} \equiv \eta^{\lambda\rho}\eta^{\nu\sigma} \pm \eta^{\lambda\sigma}\eta^{\nu\rho} - \kappa_{(\pm)}\eta^{\lambda\nu}\eta^{\rho\sigma}$ with $\kappa_{(+)} = 1$ and $\kappa_{(-)} = 0$. The symplectic current for this case is (for the moment, let us restrict the theory to the simplified situation where $v = e_P =$ the identity of the Poincaré subgroup P)

$$J^\mu_{(\pm)}(g'|g)(x) \equiv \frac{1}{2} N^{\lambda\nu\rho\sigma}_{(\pm)} [A^{(\pm)\prime}_{\lambda\nu}(x) - \partial_\lambda \varphi^{(\pm)\prime}_\nu(x)] \overleftrightarrow{\partial^\mu} [A^{(\pm)}_{\rho\sigma}(x) - \partial_\rho \varphi^{(\pm)}_\sigma(x)], \quad (8)$$

which is conserved for solutions of the field equations $(\partial_\mu\partial^\mu + m^2)A^{(\pm)}_{\rho\sigma} = 0$ and $(\partial_\mu\partial^\mu + m^2)\varphi^{(\pm)}_\rho = 0$. The integral of this current on an arbitrary spatial hypersurface Σ splits up into three distinguishable and typical (see later on Sec. 3) cocycles

$$\begin{aligned} \xi^{(\pm)}_1(g'|g) &= \frac{1}{2}\int_\Sigma d\sigma_\mu N^{\lambda\nu\rho\sigma}_{(\pm)} A^{(\pm)\prime}_{\lambda\nu} \overleftrightarrow{\partial^\mu} A^{(\pm)}_{\rho\sigma}, \\ \xi^{(\pm)}_2(g'|g) &= -\frac{1}{2}\int_\Sigma d\sigma_\mu N^{\lambda\nu\rho\sigma}_{(\pm)} [\partial_\lambda \varphi^{(\pm)\prime}_\nu \overleftrightarrow{\partial^\mu} A^{(\pm)}_{\rho\sigma} + A^{(\pm)\prime}_{\lambda\nu} \overleftrightarrow{\partial^\mu} \partial_\rho \varphi^{(\pm)}_\sigma], \end{aligned} \quad (9)$$

$$\begin{aligned}
\xi_3^{(\pm)}(g'|g) &= \frac{1}{2}\int_\Sigma d\sigma_\mu N_{(\pm)}^{\lambda\nu\rho\sigma}\partial_\lambda\varphi_\nu^{(\pm)\prime}\overleftrightarrow{\partial^\mu}\partial_\rho\varphi_\sigma^{(\pm)} \\
&= \frac{m^2}{2}\int_\Sigma d\sigma_\mu[\varphi_\lambda^{(\pm)}\overleftrightarrow{\partial^\mu}\varphi^{(\pm)\lambda} - \frac{\kappa_{(\mp)}}{m^2}\partial_\lambda\varphi^{(\pm)\lambda}\overleftrightarrow{\partial^\mu}\partial_\rho\varphi^{(\pm)\rho}]\,.
\end{aligned}$$

The first cocycle ξ_1 is meant to provide *dynamics* to the tensor potential, so that the couple $(A^{(\pm)}, \dot{A}^{(\pm)})$ corresponds to a canonically-conjugate pair of variables; the second cocycle ξ_2, the *mixed* cocycle, provides a non-trivial (non-diagonal) action of constraints on tensor potentials and determines the number of degrees of freedom of the constrained theory [it is also responsible for the transformation appearing in the second line of (7)]; the third cocycle ξ_3, the *mass* cocycle, determines the structure of constraints (first- or second-class) and modifies the dynamical content of the tensor potential coordinates $A^{(\pm)}$ in the massive case $m \neq 0$ by transferring degrees of freedom between the $A^{(\pm)}$ and $\varphi^{(\pm)}$ coordinates, thus conforming the massive field. In this way, the appearance of mass in the theory has a *cohomological origin*. To make more explicit the intrinsic significance of these three quantities $\xi_j\,,\ j = 1, 2, 3$, let us construct the quantum theory of these (anti-)symetric tensor potentials for the massless and massive cases in a unified manner, the physical interpretation of which will be a free theory of massless and massive spin 2 particles (gravitons) for the symmetric case, and a free theory of massless spin 0 pseudo-scalar particles, and massive spin 1 pseudo-vector particles for the anti-symmetric case.

The starting point will be the *Tensor quantizing group* $\tilde{G}^{(\pm)} = \{\tilde{g} = (A^{(\pm)}, \varphi^{(\pm)}, v; \zeta)\}$ with group law $\tilde{g}'' = \tilde{g}' * \tilde{g}$, which can be explicitly written as:

$$\begin{aligned}
v'' &= v' * v\,, \quad v, v', v'' \in P\,, \\
A_{\lambda\nu}^{(\pm)\prime\prime}(x) &= (vA^{(\pm)\prime})_{\lambda\nu}(x) + A_{\lambda\nu}^{(\pm)}(x)\,, \\
\varphi_\rho^{(\pm)\prime\prime}(x) &= (v\varphi^{(\pm)\prime})_\rho(x) + \varphi_\rho^{(\pm)}(x)\,, \\
\zeta'' &= \zeta'\zeta\exp\left\{i\sum_{j=1}^{3}\xi_j^{(\pm)}(A^{(\pm)\prime}, \varphi^{(\pm)\prime}, v'|A^{(\pm)}, \varphi^{(\pm)}, v)\right\}\,,
\end{aligned} \tag{10}$$

where we denote $(vA^{(\pm)})_{\lambda\nu}(x) \equiv \frac{\partial v^\alpha(x)}{\partial x^\lambda}\frac{\partial v^\beta(x)}{\partial x^\nu}A_{\alpha\beta}^{(\pm)}(v(x))$ and so on. The entire group $\tilde{G}$ will be regarded as a principal fibre bundle $\tilde{G} \to \tilde{G}/\tilde{T}$, with structure group $\tilde{T} \to \tilde{T}/U(1)$ [in general, a non-trivial, central extension $\tilde{T}$ of T by $U(1)$], and it will be the driver of the quantization procedure.

The Hilbert space $\mathcal{H}(\tilde{G})$ of the theory will be made of complex $\tilde{T}$-equivariant functions $\Psi(\tilde{g})$ on $\tilde{G}$ (wave functionals), i.e.

$$\Psi : \tilde{G} \to C\,, \quad \text{such that} \quad \Psi(\tilde{g}_t * \tilde{g}) = D_{\tilde{T}}^{(\epsilon)}(\tilde{g}_t)\Psi(\tilde{g})\,, \ \forall \tilde{g}_t \in \tilde{T},\ \forall \tilde{g} \in \tilde{G}\,, \tag{11}$$

where $D_{\tilde{T}}^{(\epsilon)}$ symbolizes a specific representation D of $\tilde{T}$ with ϵ-index (the mass $\epsilon = m$ or, in particular, the ϑ-*angle* [19] of non-Abelian gauge theories; see below). On the other hand, the operators will be the, let us say, right-invariant vector fields $\tilde{X}_{\tilde{g}^j}^R$; that is, the generators of the finite left-action of $\tilde{G}$ on itself $L_{\tilde{g}'}(\tilde{g}) = \tilde{g}' * \tilde{g}$ [the corresponding finite right-action $R_{\tilde{g}}(\tilde{g}') = \tilde{g}' * \tilde{g}$ is generated by the left-invariant vector fields $\tilde{X}_{\tilde{g}^j}^L$].

Let us use a Fourier-like parametrization of $\tilde{G}$ according to the standard decomposition of the field into negative, and positive, frequency parts:

$$A^{(\pm)}_{\lambda\nu}(x) \equiv \int d\Omega_k [a^{(\pm)}_{\lambda\nu}(k)e^{-ikx} + \bar{a}^{(\pm)}_{\lambda\nu}(k)e^{ikx}]\,,$$
$$\varphi^{(\pm)}_{\rho}(x) \equiv \int d\Omega_k [f^{(\pm)}_{\rho}(k)e^{-ikx} + \bar{f}^{(\pm)}_{\rho}(k)e^{ikx}]\,, \tag{12}$$

where $d\Omega_k \equiv \frac{d^3k}{2k^0}$ is the standard integration measure on the positive sheet of the mass hyperboloid $k^2 = m^2$. The non-trivial Lie-algebra commutators of the left-invariant vector fields $\tilde{X}^L_{\tilde{g}^j}$ are easily computed from the group law (10) in the parametrization (12), and they are explicitly (for simplicity, we discard the Lorentz subgroup):

$$\begin{aligned}
\left[\tilde{X}^L_{\bar{a}^{(\pm)}_{\lambda\nu}(k)}, \tilde{X}^L_{a^{(\pm)}_{\rho\sigma}(k')}\right] &= iN^{\lambda\nu\rho\sigma}_{(\pm)}\Delta_{kk'}\Xi\,, \\
\left[\tilde{X}^L_{a^{(\pm)}_{\lambda\nu}(k)}, \tilde{X}^L_{\bar{f}^{(\pm)}_{\sigma}(k')}\right] &= k_\rho N^{\lambda\nu\rho\sigma}_{(\pm)}\Delta_{kk'}\Xi\,, \quad \left[\tilde{X}^L_{\bar{a}^{(\pm)}_{\lambda\nu}(k)}, \tilde{X}^L_{f^{(\pm)}_{\sigma}(k')}\right] = k_\rho N^{\lambda\nu\rho\sigma}_{(\pm)}\Delta_{kk'}\Xi\,, \\
\left[\tilde{X}^L_{\bar{f}^{(\pm)}_{\rho}(k)}, \tilde{X}^L_{f^{(\pm)}_{\sigma}(k')}\right] &= ik^2(\eta^{\rho\sigma} - \kappa_{(\mp)}\frac{k^\rho k^\sigma}{k^2})\Delta_{kk'}\Xi\,, \\
\left[\tilde{X}^L_{y^\mu}, \tilde{X}^L_{a^{(\pm)}_{\lambda\nu}(k)}\right] &= ik_\mu \tilde{X}^L_{a^{(\pm)}_{\lambda\nu}(k)}\,, \quad \left[\tilde{X}^L_{y^\mu}, \tilde{X}^L_{\bar{a}^{(\pm)}_{\lambda\nu}(k)}\right] = -ik_\mu \tilde{X}^L_{\bar{a}^{(\pm)}_{\lambda\nu}(k)}\,, \\
\left[\tilde{X}^L_{y^\mu}, \tilde{X}^L_{f^{(\pm)}_{\rho}(k)}\right] &= ik_\mu \tilde{X}^L_{f^{(\pm)}_{\rho}(k)}\,, \quad \left[\tilde{X}^L_{y^\mu}, \tilde{X}^L_{\bar{f}^{(\pm)}_{\rho}(k)}\right] = -ik_\mu \tilde{X}^L_{\bar{f}^{(\pm)}_{\rho}(k)}\,,
\end{aligned} \tag{13}$$

where $\Delta_{kk'} = 2k^0\delta^3(k - k')$ is the generalized delta function on the positive sheet of the mass hyperboloid, and we denote by $\Xi \equiv i\tilde{X}^L_\zeta = i\tilde{X}^R_\zeta$ the central generator, in order to distinguish it from the rest, in view of its "central" (important) role in the quantization procedure; it behaves as i times the identity operator, $\Xi\Psi(\tilde{g}) = i\Psi(\tilde{g})$, when the $U(1)$ part of the $\tilde{T}$-equivariance conditions (11), $D^{(\epsilon)}_{\tilde{T}}(\zeta) = \zeta$ (always faithful, except in the classical limit $U(1) \to \Re$ [6]), is imposed.

The representation $L_{\tilde{g}'}\Psi(\tilde{g}) = \Psi(\tilde{g}' * \tilde{g})$ of $\tilde{G}$ on wave functions Ψ proves to be reducible. The reduction can be achieved by means of those right-conditions $R_{\tilde{g}}\Psi(\tilde{g}') = \Psi(\tilde{g}' * \tilde{g}) \equiv \Psi(\tilde{g}')$ (which commute with the left-action L) compatible with the above $U(1)$-equivariant condition $\Xi\Psi = i\Psi$, i.e., by means of the so-called *polarization conditions* $\tilde{X}^L\Psi = 0$ [6]. Roughly speaking, a polarization corresponds to a maximal left-subalgebra $\mathcal{G}_p$ of the Lie algebra $\tilde{\mathcal{G}}^L$ of $\tilde{G}$ which exclude the central generator Ξ; or, in other words, a maximal, *horizontal* left-subalgebra $\mathcal{G}_p$. The horizontality property for a subalgebra $\mathcal{G}_H =< \tilde{X}^L_{\tilde{g}^j} >$ can be formally stated as $\Theta(\tilde{X}^L_{\tilde{g}^j}) = 0,\ \forall \tilde{X}^L_{\tilde{g}^j} \in \mathcal{G}_H$, where we denote by Θ the "vertical" component $\tilde{\theta}^{L(\zeta)}$ [dual to $\Xi = i\tilde{X}^L_\zeta$, i.e., $\tilde{\theta}^{L(\zeta)}(\tilde{X}^L_{\tilde{g}^j}) = \delta^\zeta_{\tilde{g}^j}$] of the standard canonical (Lie-valued) left-invariant 1-form $\tilde{\theta}^L$ of $\tilde{G}$. It can be easily calculated from the general expression:

$$\Theta = \frac{\partial}{\partial g^j}\xi(g'|g)\bigg|_{g'=g^{-1}} dg^j - i\zeta^{-1}d\zeta\,. \tag{14}$$

For cases such as the centrally-extended Galilei group [6], the so called *quantization 1-form* Θ reduces to the Poincaré-Cartan form of Classical Mechanics, $\Theta_{PC} = pdq - \frac{p^2}{2m}dt$, except for the (typically quantum) phase term $-i\zeta^{-1}d\zeta$. In the same way as Θ_{PC}, the quantization 1-form Θ gives the generalized *classical* equations of motion of the system. The trajectories for (14) are given by the integral curves of the characteristic module $\mathrm{Ker}\Theta \cap \mathrm{Ker}d\Theta = \{\text{vector fields } \tilde{X} \,/\, \Theta(\tilde{X}) = 0,\, d\Theta(\tilde{X}) = 0\}$ of the *presymplectic* (in general, it has a non-trivial kernel) two-form $d\Theta$ on the group $\tilde{G}$. The characteristic module of Θ is generated by the *characteristic subalgebra* $\mathcal{G}_c$, which includes non-symplectic (non-dynamical) generators; that is, horizontal left-invariant vector fields which, under commutation, do not give rise to central terms proportional to Ξ (i.e., they do not have any conjugated counterpart). In fact, $d\Theta$ at the identity of the group regains the Lie-algebra cocycle Σ:

$$\Sigma \equiv d\Theta|_e : \mathcal{G}^L \times \mathcal{G}^L \to \Re\,, \;\; \Sigma(X^L_a, X^L_b) = d\Theta(X^L_a, X^L_b)|_e = \Theta(\big[\tilde{X}^L_a, \tilde{X}^L_b\big])|_e \tag{15}$$

A glance at the commutators (13) tells us the content of the characteristic subalgebra for $\tilde{G}$:

$$\mathcal{G}^{(\pm)}_c =< \tilde{X}^L_{y^\mu},\, (\tilde{X}^L_{\Lambda^\mu_\nu}),\, \tilde{X}^L_{h^{(\pm)}_\rho(k)},\, \tilde{X}^L_{\bar{h}^{(\pm)}_\sigma(k)} > \;\forall k\,; \tag{16}$$

That is, $\mathcal{G}_c$ contains the whole Poincaré subalgebra (when the Lorentz transformations Λ are kept) and two particular combinations

$$\mathcal{G}^{(\pm)}_{\rm gauge} =< \tilde{X}^L_{h^{(\pm)}_\rho(k)} \equiv \tilde{X}^L_{f^{(\pm)}_\rho(k)} + ik_\lambda \tilde{X}^L_{a^{(\pm)}_{\lambda\rho}(k)}\,,\; \tilde{X}^L_{\bar{h}^{(\pm)}_\sigma(k)} \equiv \tilde{X}^L_{\bar{f}^{(\pm)}_\sigma(k)} - ik_\lambda \tilde{X}^L_{\bar{a}^{(\pm)}_{\lambda\sigma}(k)} >\,, \tag{17}$$

which define the *gauge subalgebra* $\mathcal{G}_{\rm gauge}$ of the theory. Let us justify this name for $\mathcal{G}_{\rm gauge}$. As already commented, the trajectories associated with vector fields $\tilde{X}^L \in \mathcal{G}_c$ represent the generalized *classical* equations of motion (they generalize the standard classical equations of motion because they contain also the evolution of the phase parameter ζ); for example, the flow of $\tilde{X}^L_{y^\mu}$ represents the space-time evolution $a^{(\pm)}_{\lambda\nu}(k) \to e^{-iky} a^{(\pm)}_{\lambda\nu}(k)$, whereas the flow of (17),

$$f^{(\pm)}_\lambda(k) \to f^{(\pm)}_\lambda(k) + h^{(\pm)}_\lambda(k)\,, \quad a^{(\pm)}_{\lambda\nu}(k) \to a^{(\pm)}_{\lambda\nu}(k) + \frac{i}{2}[k_\lambda h^{(\pm)}_\nu(k) \pm k_\nu h^{(\pm)}_\lambda(k)]\,, \tag{18}$$

(and the complex-conjugated counterpart) recovers the classical gauge transformations (6) in Fourier coordinates. The invariant quantities under the above-mentioned classical equations of motion are $F_{\tilde{g}^j} \equiv i_{\tilde{X}^R_{\tilde{g}^j}}\Theta = \Theta(\tilde{X}^R_{\tilde{g}^j})$, which define the (generalized) Noether invariants of the theory (total energy, momentum, initial conditions, etc.). It can be seen that the Noether invariants associated with gauge vector-fields are identically zero $F_{h^{(\pm)}_\lambda(k)} = 0 = F_{\bar{h}^{(\pm)}_\lambda(k)}$. Even more, in general, the gauge subalgebra $\mathcal{G}_{\rm gauge}$ constitutes a horizontal *ideal* of the whole Lie algebra $\tilde{\mathcal{G}}$ of $\tilde{G}$ (a *normal* horizontal subgroup N for finite transformations) for which the right-invariant vector fields can be written as a linear combination of the corresponding left-invariant vector fields. In fact, the coefficients of this linear combination provide a representation of the complement of $\mathcal{G}_{\rm gauge}$ in $\mathcal{G}_c$ (in this case, the Poincaré subalgebra). All these properties characterize a gauge subgroup of $\tilde{G}$.

Note the subtle distinction between *gauge symmetries* and *constraints* inside the GAQ framework. Constraints take part of the structure group $\tilde{T}$ and have a non-trivial action (11) on wave functions. The Lie algebra of $\tilde{T}$,

$$\tilde{\mathcal{T}}^{(\pm)} = < \tilde{X}^L_{f^{(\pm)}_\rho(k)}\,,\ \tilde{X}^L_{\bar{f}^{(\pm)}_\sigma(k)}\,,\ \Xi >\,, \tag{19}$$

is not a horizontal ideal but, rather, $\tilde{T}$ is itself a quantizing group [more precisely, a central extension of T by $U(1)$] with its own quantization 1-form $\Theta_{\tilde{T}}$. *First-class constraints* will be defined as the characteristic subalgebra $\mathcal{T}_c$ of $\tilde{\mathcal{T}}$ with respect to $\Theta_{\tilde{T}}$. *Second-class constaints* are defined as the complement of $\mathcal{T}_c$ in $\tilde{\mathcal{T}}$, and can be arranged into couples of conjugated generators. The constrained Hilbert space $\mathcal{H}_{\text{phys.}}$ will be made of complex wave functionals (11) which are annihilated

$$\tilde{X}^R_{\tilde{g}_t}\Psi_{\text{phys.}} = 0, \quad \forall \tilde{X}^R_{\tilde{g}_t} \in \mathcal{T}_p \tag{20}$$

by a polarization subalgebra $\mathcal{T}_p \subset \tilde{\mathcal{T}}$ of right-invariant vector fields, which contains $\mathcal{T}_c$ and 'half' of second-class constraints (the, let us say, 'positive modes'). The algebra $\mathcal{T}_p$ is then the maximal right-subalgebra of $\tilde{\mathcal{T}}$ that can be consistently imposed to be zero on wavefunctionals as constraint equations. The condition (20) selects those wave functionals $\Psi_{\text{phys.}}$ which transform as 'highest weight vectors' under $\tilde{T}$.

Since constraint conditions (20) are imposed as infinitesimal right-restictions (finite left-restrictions), it is obvious that not all the operators $\tilde{X}^R$ will preserve the constraints; we shall call $\tilde{\mathcal{G}}_{\text{good}} \subset \tilde{\mathcal{G}}^R$ the subalgebra of (*good*∼physical) operators which do so. These have to be found inside the *normalizer* of the constraints; for example, a sufficient condition for $\tilde{\mathcal{G}}_{\text{good}}$ to preserve the constraints is:

$$\left[\tilde{\mathcal{G}}_{\text{good}}, \mathcal{T}_p\right] \subset \mathcal{T}_p\,. \tag{21}$$

From this characterization, we see that first-class constraints $\mathcal{T}_c$ become a horizontal ideal (a gauge subalgebra) of $\tilde{\mathcal{G}}_{\text{good}}$, which now defines the constrained theory. Even more, it can be proved that gauge generators (17) are trivial (zero) on polarized wave functions (see later).

With all this information at hand, let us go back to the reduction process of the representation (11). An operative method to obtain a *full* polarization subalgebra $\mathcal{G}_p$ is to complete the characteristic subalgebra $\mathcal{G}_c$ to a maximal, horizontal left-subalgebra. Roughly speaking, $\mathcal{G}_p$ will include non-symplectic generators in $\mathcal{G}_c$ and half of the symplectic ones (either "positions" or "momenta"). The above-mentioned polarization conditions

$$\tilde{X}^L_{\tilde{g}_p}\Psi = 0,\ \forall \tilde{X}^L_{\tilde{g}_p} \in \mathcal{G}_p \tag{22}$$

[or its finite counterparts $R_{\tilde{g}_p}\Psi(\tilde{g}) = \Psi(\tilde{g})$] represent the generalized *quantum* equations of motion; for example, $\tilde{X}^L_{y^0}\Psi = 0$ represents the Schrödinger equation. In this way, the concept of polarization here generalizes the analogous one in Geometric Quantization [3], where no characteristic module exists (since all variables are symplectic). The content of

$\mathcal{G}_p$ will depend on the value of k^2, as also happens for $\mathcal{T}_p$. From now on we shall distinguish between the cases $k^2 = 0$ and $k^2 = m^2 \neq 0$, and between symmetric and antis-ymmetric tensor potentials, placing each one in separate subsections. Let us see how to obtain the corresponding constrained Hilbert space and the action of the physical operators on wave functions.

2.1 $\tilde{G}(k^2 = 0)$: Massless tensor fields

Firstly, we shall consider the massless case. A polarization subalgebra for the symmetric and anti-symmetric cases is:

$$\mathcal{G}_p^{(\pm)} =< \mathcal{G}_c\,,\ \tilde{X}^L_{a^{(\pm)}_{\lambda\nu}(k)} > \ \forall k\,. \tag{23}$$

The corresponding $U(1)$-equivariant, polarized wave functions (22) have the following general form:

$$\begin{aligned}
\Psi^{(\pm)}\left(y,(\Lambda),a^{(\pm)},\bar{a}^{(\pm)},f^{(\pm)},\bar{f}^{(\pm)},\zeta\right) &= W^{(\pm)}\cdot\Phi^{(\pm)}\left(\bar{c}^{(\pm)}_{\lambda\nu}(k)e^{-iky}\right)\,,\\
W^{(\pm)} &= \zeta\exp\left\{\frac{1}{2}\int d\Omega_k N^{\lambda\nu\rho\sigma}_{(\pm)}\bar{c}^{(\pm)}_{\lambda\nu}(k)c^{(\pm)}_{\rho\sigma}(k)\right\}\,, \qquad (24)\\
\bar{c}^{(\pm)}_{\lambda\nu} &\equiv \bar{a}^{(\pm)}_{\lambda\nu} + \frac{i}{2}(k_\lambda\bar{f}^{(\pm)}_\nu \pm k_\nu\bar{f}^{(\pm)}_\lambda)
\end{aligned}$$

where Φ is an arbitrary power series on its arguments. For example, the zero-order wave function (the *vacuum*), and the one-particle states of momentum k are $|0\rangle \equiv W$ and $\hat{a}^{(\pm)\dagger}_{\lambda\nu}(k)|0\rangle \equiv \frac{1}{4}N^{(\pm)}_{\lambda\nu\rho\sigma}\tilde{X}^R_{a^{(\pm)}_{\rho\sigma}(k)}|0\rangle = W^{(\pm)}\cdot[\bar{c}^{(\pm)}_{\lambda\nu}(k)e^{-iky}]$, respectively. The last equality defines the creation operators of the theory whereas the corresponding annihilation operators are $\hat{a}^{(\pm)}_{\lambda\nu}(k) \equiv \frac{1}{4}N^{(\pm)}_{\lambda\nu\rho\sigma}\tilde{X}^R_{\bar{a}^{(\pm)}_{\rho\sigma}(k)}$ and its action on polarized wave functions (24) is $\hat{a}^{(\pm)}_{\lambda\nu}(k)\Psi^{(\pm)} = W^{(\pm)}\frac{\delta\Phi^{(\pm)}}{\delta\bar{c}^{(\pm)}_{\lambda\nu}(k)}$. One can easily check that the action of the gauge operators [the right version of (17)] $\tilde{X}^R_{h^{(\pm)}_\rho(k)}$ and $\tilde{X}^R_{\bar{h}^{(\pm)}_\rho(k)}$ on polarized wave functions (24) is trivial (zero), since they close a *horizontal ideal* of $\tilde{\mathcal{G}}$.

The representation $L_{\tilde{g}'}\Psi(\tilde{g}) = \Psi(\tilde{g}' * \tilde{g})$ of $\tilde{G}$ on polarized wave functions (24) is irreducible and unitary with respect to the natural scalar product,

$$\begin{aligned}
\langle\Psi|\Psi'\rangle &= \int_{\tilde{G}}\mu(\tilde{g})\bar{\Psi}(\tilde{g})\Psi'(\tilde{g})\,, \qquad (25)\\
\mu^{(\pm)}(\tilde{g}) &= \tilde{\theta}^{L(1)}\wedge \stackrel{\dim(\tilde{G}^{(\pm)})}{\cdots} \wedge\tilde{\theta}^{L(n)} = \mu_P\wedge N^{\lambda\nu\rho\sigma}_{(\pm)}\prod_k d\,\mathrm{Re}[c^{(\pm)}_{\lambda\nu}(k)]\wedge d\,\mathrm{Im}[c^{(\pm)}_{\rho\sigma}(k)]\,,
\end{aligned}$$

where μ_P means the standard left-invariant measure of the Poincaré subgroup [exterior product $\wedge$ of the components $\tilde{\theta}^{L(j)}$ of the standard (Lie-valued) left-invariant 1-form $\tilde{\theta}^L$ of the corresponding group]. The finiteness of this scalar product is ensured when restricting to constrained wave functions (11). Before imposing the rest of $\tilde{T}$-equivariant conditions [the $U(1)$ part has already been imposed], we have to look carefully at the particular fibration of the structure group $\tilde{T} \to \tilde{T}/U(1)$ by $U(1)$ in order to separate first- from second-class constraints. A look at the right-version of the third line in (13) tells us that all constraints are first-class for the massless, symmetric case, whereas the massless, anti-symmetric case possesses a couple of second-class constraints:

$$\left[\check{k}_\rho \tilde{X}^R_{\bar{f}^{(-)}_\rho(k)}, \check{k}'_\sigma \tilde{X}^R_{f^{(-)}_\sigma(k')}\right] = 4i(k^0)^4 \Delta_{kk'} \Xi \,, \tag{26}$$

where $\check{k}_\rho \equiv k^\rho$. Thus, first-class constraints for the massless anti-symmetric case are $\mathcal{T}^{(-)}_c =< \epsilon^\mu_\rho(k)\tilde{X}^R_{\bar{f}^{(-)}_\rho(k)}, \, \epsilon^\mu_\rho(k)\tilde{X}^R_{f^{(-)}_\rho(k)} >$, $\mu = 0, 1, 2,$ where $\epsilon^\mu_\rho(k)$ is a tetrad which diagonalizes the matrix $P^{\rho\sigma} = k^\rho k^\sigma$; in particular, we choose $\epsilon^3_\rho(k) \equiv \check{k}_\rho$ and $\epsilon^0_\rho(k) \equiv k_\rho$.

The constraint equations for the massless, symmetric case are:

$$\begin{array}{lcl} \tilde{X}^R_{\bar{f}^{(+)}_\sigma(k)} \Psi^{(+)}_{\rm phys} = 0 & \Rightarrow & \left(2k_\lambda \frac{\delta}{\delta \bar{c}^{(+)}_{\lambda\sigma}(k)} - k^\sigma \eta_{\lambda\nu} \frac{\delta}{\delta \bar{c}^{(+)}_{\lambda\nu}(k)}\right) \Phi^{(+)}_{\rm phys} = 0 \,, \\ \tilde{X}^R_{f^{(+)}_\sigma(k)} \Psi^{(+)}_{\rm phys} = 0 & \Rightarrow & \left(2k^\lambda \bar{a}^{(+)}_{\lambda\sigma}(k) - k^\sigma \eta^{\lambda\nu} \bar{a}^{(+)}_{\lambda\nu}(k)\right) \Psi^{(+)}_{\rm phys} = 0 \,. \end{array} \tag{27}$$

The first condition in (27) says that, in particular, an arbitrary combination of one-particle states of momentum k, $\varepsilon^{\lambda\nu}(k)\hat{a}^{(+)\dagger}_{\lambda\nu}(k)|0\rangle$, is *physical* (observable) if $2k^\lambda \varepsilon^\sigma_\lambda(k) = k^\sigma \varepsilon^\lambda_\lambda(k)$. This condition also guarantees that physical states have positive (or null) norm, since $-\bar{\varepsilon}_{\lambda\nu} N^{\lambda\nu\rho\sigma}_{(+)} \varepsilon_{\rho\sigma} \geq 0$. The second condition in (27) eliminates *null norm vectors* from the theory, since it establishes that the physical wave functions $\Psi^{(+)}_{\rm phys}$ have support only on the surface $2k^\lambda \bar{a}^{(+)}_{\lambda\sigma}(k) - k^\sigma \eta^{\lambda\nu} \bar{a}^{(+)}_{\lambda\nu}(k) = 0$. In summary, the 8 independent conditions (27) keep *two* field degrees of freedom out of the original 10 field degrees of freedom of the symmetric tensor potential. The good (physical) operators (21) of the theory are:

$$\tilde{\mathcal{G}}^{(+)}_{\rm good} =< \varepsilon^{\lambda\nu}(k)\hat{a}^{(+)\dagger}_{\lambda\nu}(k), \, \bar{\varepsilon}^{\lambda\nu}(k)\hat{a}^{(+)}_{\lambda\nu}(k), \, \tilde{X}^R_{y^\mu}, \, \tilde{X}^R_{\Lambda^{\mu\nu}}, \, \Xi > , \tag{28}$$

where the factors $\varepsilon^{\lambda\nu}(k)$ are restricted by the above-mentioned conditions [note that $\mathcal{T}^{(+)} = \tilde{\mathcal{T}}^{(+)}/U(1)$ becomes a horizontal ideal (gauge subalgebra) of $\tilde{\mathcal{G}}^{(+)}_{\rm good}$]. The transformation properties of physical one-particle states under the Poincaré group P declare, in particular, that the symmetric tensor field carries helicity ± 2, as corresponds to a massless spin 2 particle (graviton).

For the massless anti-symmetric case, only a polarization subalgebra $\mathcal{T}^{(-)}_p$ of $\tilde{\mathcal{T}}^{(-)}$ can be consistently imposed as constraint equations, due to the non-trivial fibration of $\tilde{T}^{(-)}$ by $U(1)$. These constraint equations are:

$$\begin{array}{lcll} \epsilon^\mu_\sigma(k)\tilde{X}^R_{\bar{f}^{(-)}_\sigma(k)} \Psi^{(-)}_{\rm phys} = 0 & \Rightarrow & k_\lambda \epsilon^\mu_\sigma(k) \frac{\delta}{\delta \bar{c}^{(-)}_{\lambda\sigma}(k)} \Phi^{(-)}_{\rm phys} = 0 \,, & \mu = 0, 1, 2, 3 \\ \epsilon^\mu_\sigma(k)\tilde{X}^R_{f^{(-)}_\sigma(k)} \Psi^{(-)}_{\rm phys} = 0 & \Rightarrow & k^\lambda \epsilon^{\mu\nu}(k) \bar{a}^{(-)}_{\lambda\nu}(k) \Psi^{(-)}_{\rm phys} = 0 \,, & \mu = 0, 1, 2. \end{array} \tag{29}$$

Only 3 of the four conditions in the first line of (29) are independent, since the combination $\epsilon^0_\sigma \tilde{X}^R_{\bar{f}^{(-)}_\sigma}$ coincides with the gauge operator $k_\sigma \tilde{X}^R_{\bar{h}^{(-)}_\sigma}$, which is identically zero on polarized wave functions (24). The second-class character of the constraints (26) precludes the imposition of the combination $\mu = 3$ in the second line of (29), so that we have only 2 additional independent conditions which, together with the 3 previous ones, keep *one* field degree of freedom out of the 6 original ones corresponding to an anti-symmetric tensor potential. As for the symmetric case, the behaviour of the physical states $\Psi^{(-)}_{\rm phys}$ under the action of the Poincaré group declares that the constrained theory corresponds to a pseudo-scalar particle. The computation of the good operators follows the same steps as for the symmetric case.

2.2 $\tilde{G}(k^2 \neq 0)$: Massive tensor fields

As mentioned above, a remarkable characteristic of the quantizing group law (10) is that it accomplishes the quantization of both the massless and massive cases, according to the value of the central extension parameter m in the third cocycle of (9), in a unified way. The modification of $\xi_3^{(\pm)}$ in the $m \neq 0$ case causes a transfer of degrees of freedom between the $A^{(\pm)}$ and $\varphi^{(\pm)}$ coordinates, so that it is possible to decouple the tensor potential by means of a transformation which diagonalizes the cocycle $\xi^{(\pm)}$. In fact, the combinations:

$$\begin{array}{rcl} \tilde{X}^L_{\bar{b}^{(\pm)}_{\lambda\nu}(k)} & \equiv & \tilde{X}^L_{\bar{a}^{(\pm)}_{\lambda\nu}(k)} + \frac{i}{k^2}\left(k^\lambda \tilde{X}^R_{\bar{f}^{(\pm)}_\nu(k)} \pm k^\nu \tilde{X}^R_{\bar{f}^{(\pm)}_\lambda(k)} - i\eta^{\lambda\nu}k_\alpha k_\beta \tilde{X}^R_{\bar{a}^{(\pm)}_{\alpha\beta}(k)}\right)\,, \\ \tilde{X}^L_{b^{(\pm)}_{\lambda\nu}(k)} & \equiv & \tilde{X}^L_{a^{(\pm)}_{\lambda\nu}(k)} - \frac{i}{k^2}\left(k^\lambda \tilde{X}^R_{f^{(\pm)}_\nu(k)} \pm k^\nu \tilde{X}^R_{f^{(\pm)}_\lambda(k)} + i\eta^{\lambda\nu}k_\alpha k_\beta \tilde{X}^R_{a^{(\pm)}_{\alpha\beta}(k)}\right)\,, \end{array} \tag{30}$$

commute with the constraints (19) and close the Lie subalgebra

$$\left[\tilde{X}^L_{\bar{b}^{(\pm)}_{\lambda\nu}(k)}, \tilde{X}^L_{b^{(\pm)}_{\rho\sigma}(k')}\right] = iM^{\lambda\nu\rho\sigma}_{(\pm)}(k)\Delta_{kk'}\Xi\,, \tag{31}$$

where $M^{\lambda\nu\rho\sigma}_{(\pm)} \equiv M^{\lambda\rho}M^{\nu\sigma} \pm M^{\lambda\sigma}M^{\nu\rho} - 2\kappa_{(\pm)}M^{\lambda\nu}M^{\rho\sigma}$ and $M^{\lambda\rho}(k) \equiv \eta^{\lambda\rho} - \frac{k^\lambda k^\rho}{k^2}$. The different cohomological structure of the quantizing group $\tilde{G}$ for the present massive case, with regard to the abovementioned massless case, leads to a different polarization subalgebra and a new structure for constraints. The polarization subalgebra is made of the following left generators:

$$\mathcal{G}^{(\pm)}_p =< \mathcal{G}_c\,,\ \tilde{X}^L_{b^{(\pm)}_{\lambda\nu}(k)}\,,\ \tilde{X}^L_{\bar{f}^{(\pm)}_\rho(k)} > \ \forall k\,. \tag{32}$$

The integration of the polarization conditions (22) on $U(1)$-equivariant wave functions leads to

$$\begin{array}{rcl} \Psi^{(\pm)} & = & \zeta \exp\left\{\frac{1}{2}\int d\Omega_k \left(M^{\lambda\nu\rho\sigma}_{(\pm)}\bar{b}^{(\pm)}_{\lambda\nu}b^{(\pm)}_{\rho\sigma} + k^2 M^{\lambda\rho}_{(\pm)}\bar{\chi}^{(\pm)}_\lambda\chi^{(\pm)}_\rho\right)\right\} \\ & & \cdot\Phi^{(\pm)}\left([\bar{\omega}^{\alpha\beta}_{ij}(k)^{(\pm)}\bar{b}^{(\pm)}_{\alpha\beta}(k)]e^{-iky},\, [\varpi^\sigma_l(k)^{(\pm)}\chi^{(\pm)}_\sigma(k)]e^{iky}\right)\,, \\ a^{(\pm)}_{\lambda\nu}(k) & \equiv & b^{(\pm)}_{\lambda\nu}(k) + \eta^{\alpha\beta}\frac{k_\lambda k_\nu}{k^2}b^{(\pm)}_{\alpha\beta}(k),\ \ f^{(\pm)}_\lambda(k) \equiv \chi^{(\pm)}_\lambda(k) - 2i\frac{k_\nu}{k^2}b^{(\pm)}_{\lambda\nu}(k)\,, \end{array} \tag{33}$$

where $M^{\lambda\rho}_{(\pm)}(k) \equiv \eta^{\lambda\rho} - \kappa_{(\mp)}\frac{k^\lambda k^\rho}{k^2}$; $\omega^{\alpha\beta}_{\lambda\nu}(k)^{(\pm)}$ and $\varpi^\sigma_\lambda(k)^{(\pm)}$ are matrices which diagonalize $M^{\lambda\nu\rho\sigma}_{(\pm)}(k)$ and $M^{\lambda\rho}_{(\pm)}(k)$, respectively, and the indices $i, j = 1, 2, 3$ and $l = 0(+), 1, 2, 3$ label the corresponding eigenvectors with non-zero eigenvalue ($l = 0(+)$ means "only for the symmetric case"). From the third line of (33), we realize the already mentioned transfer of degrees of freedom between the $a^{(\pm)}$ and $f^{(\pm)}$ coordinates for the massive case, leading to a new set of variables, $b^{(\pm)}$ and $\chi^{(\pm)}$, which correspond to a massive (anti-)symmetric tensor field and some sort of vector field with *negative* energy.

For the present massive case, all constraints are second-class for the symmetric case, since they close an electromagnetic-like subalgebra [see third line of Eq. (13)], whereas, for the anti-symmetric case, constraints close a Proca-like subalgebra which leads to three couples of second-class constraints, $\{\bar{\varpi}_\lambda^{j(-)}\tilde{X}^R_{\bar{f}_\lambda^{(-)}},\, \varpi_\lambda^{j(-)}\tilde{X}^R_{f_\lambda^{(-)}}\}$ $j = 1, 2, 3$, and a couple of gauge vector fields $\{k_\lambda \tilde{X}^R_{\bar{f}_\lambda^{(-)}},\, k_\lambda \tilde{X}^R_{f_\lambda^{(-)}}\}$. The constraint equations (related to a polarization subalgebra $\mathcal{T}_p$ of $\tilde{\mathcal{T}}$) eliminate the $\chi^{(\pm)}$ dependence of wave functions in (33) and keep $6 = 10 - 4$ field degrees of freedom for the symmetric case (massive spin 2 particle + massive scalar field=trace of the symmetric tensor), and $3 = 6 - 3$ field degrees of freedom for the anti-symmetric case (massive pseudo-vector particle).

3 Group approach to quantization of non-Abelian Yang-Mills theories

Once we know how GAQ works on Abelian gauge field theories, let us tackle the non-Abelian case, the underlying structure of which is similar to the previous case. However, new richness and subtle distinctions are introduced because of the non-Abelian character of the constraint subgroup $T = \mathrm{Map}(\Re^4, \mathbf{T}) = \{U(x) = e^{\varphi_a(x)T^a}\}$, where $\mathbf{T}$ is some non-Abelian, compact Lie-group with Lie-algebra commutation relations $[T^a, T^b] = C_c^{ab} T^c$.

As for the Abelian case, the traditional gauge transformation properties,

$$U(x) \to U'(x)U(x)\,, \quad \mathcal{A}_\nu(x) \to U'(x)\mathcal{A}_\nu(x)U'(x)^{-1} + U'(x)\partial_\nu U'(x)^{-1}, \tag{34}$$

for Lie-algebra valued vector potentials $\mathcal{A}^\nu(x) = r_a^b \mathcal{A}_b^\nu(x) T^a$ (r_a^b denotes a coupling-constant matrix) have to be revised in order to adapt it to an action of a group on itself. The solution for this is to consider $A_\nu \equiv \mathcal{A}_\nu - U\partial_\nu U^{-1}$, which transforms homogeneously under the adjoint action of T, whereas the non-tensorial part $U'(x)\partial_\nu U'(x)^{-1}$ modifies the phase ζ of the wavefunctional Ψ according to:

$$\begin{aligned} U(x) \to U'(x)U(x)\,, \quad A_\nu(x) \to U'(x)A_\nu(x)U'(x)^{-1}\,, \\ \zeta \to \zeta \exp\left\{\frac{i}{r^2}\int_\Sigma d\sigma_\mu(x)\,\mathrm{tr}\left[U'(x)^{-1}\partial_\nu U'(x)\overleftrightarrow{\partial^\mu} A^\nu(x)\right]\right\}\,, \end{aligned} \tag{35}$$

where we are restricting ourselves, for the sake of simplicity, to special unitary groups $\mathbf{T}$, so that the structure constants C_c^{ab} are totally anti-symmetric, and the anti-hermitian generators T^a can be chosen such that the Killing-Cartan metric is just $\mathrm{tr}(T^a T^b) = -\frac{1}{2}\delta^{ab}$. For simple groups, the coupling-constant matrix r_a^b reduces to a multiple of the identity $r_a^b = r\delta_a^b$, and we have $A_a^\mu = -\frac{2}{r}\mathrm{tr}(T^a A^\mu)$. As above, the argument of the exponential in (35) can be considered to be a piece of a two-cocycle $\xi : G \times G \rightarrow \Re$ constructed through a conserved current, $\xi(g'|g) = \int_\Sigma d\sigma_\mu(x) J^\mu(g'|g)(x), \;\; g', g \in G$, so that it does not depend on the chosen spacelike hypersurface Σ. Let us also discard the Poincaré subgroup P in the group G; that is, we shall consider, roughly speaking, $G = G_A \times_s T$, i.e. the semidirect product of the Abelian group G_A of Lie-valued vector potentials and the group T. The reason for so doing is something more than a matter of symplicitly. In fact, we could make the kinematics (P) and the constraints (T) compatible at the price of introducing an infinite number of extra generators in the enveloping algebra of G_A, in a way that makes the quantization procedure quite unwieldy. We could also introduce a free-like (Abelian) kinematics without the need for extra generators at the price of appropriately adjusting the constraint set and the kinematics into a stable system, that is, by introducing secondary constraints. Nevertheless, unlike other standard approaches to quantum mechanics, GAQ still holds, even in the absence of a well-defined (space-)time evolution, an interesting and desirable property concerning the quantization of gravity (see, for example, [20]). The *true* dynamics [which preserves the constraints (11)] will eventually arise as part of the set of *good* operators of the theory (see below).

We shall adopt a non-covariant approach and choose a $t =$ constant Σ-hypersurface to write the cocycle ξ. Also, we shall make partial use of the gauge freedom to set the temporal component $A^0 = 0$, so that the electric field is simply $\vec{E}_a = -\partial_0 \vec{A}_a$ [from now on, and for the sake of simplicity, we shall put any three-vector $\vec{A}$ as A, and we will understand $AE = \sum_{j=1}^3 A^j E^j$, in the hope that no confusion will arise]. In this case, there is still a residual gauge invariance $T = \mathrm{Map}(\Re^3, \mathbf{T})$ (see [21]).

Taking all of this into account, the explicit group law $\tilde{g}'' = \tilde{g}' * \tilde{g}$ [with $\tilde{g} = (g; \zeta) = (A, E, U; \zeta)$] for the proposed infinite-dimensional *Yang-Mills quantizing group* $\tilde{G}$ is:

$$
\begin{aligned}
U''(x) &= U'(x)U(x)\,, \\
A''(x) &= A'(x) + U'(x)A(x)U'(x)^{-1}\,, \\
E''(x) &= E'(x) + U'(x)E(x)U'(x)^{-1}\,, \\
\zeta'' &= \zeta'\zeta \exp\left\{ -\frac{i}{r^2}\sum_{j=1}^{2} \xi_j(A', E', U'|A, E, U) \right\}\,; \\
\xi_1(g'|g) &\equiv \int d^3x \, \mathrm{tr}\left[\begin{pmatrix} A' & E' \end{pmatrix} S \begin{pmatrix} U'AU'^{-1} \\ U'EU'^{-1} \end{pmatrix} \right] \\
\xi_2(g'|g) &\equiv \int d^3x \, \mathrm{tr}\left[\begin{pmatrix} \nabla U'U'^{-1} & E' \end{pmatrix} S \begin{pmatrix} U'\nabla U U^{-1}U'^{-1} \\ U'EU'^{-1} \end{pmatrix} \right],
\end{aligned}
\tag{36}
$$

where $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is a symplectic matrix. As above, the first cocycle ξ_1 is meant to provide *dynamics* to the vector potential, so that the couple (A, E) corresponds to a canonically-conjugate pair of coordinates, and ξ_2 is the non-covariant analogue of (35). It is noteworthy that, unlike the Abelian case $T = \mathrm{Map}(\Re^4, U(1))$, the simple (and non-Abelian) character of T for the present case precludes a non-trivial central extension

$\tilde{T}$ of T by $U(1)$ given by the cocycle ξ_3 [see (9)]. However, mass will enter the non-Abelian Yang-Mills theories through *pseudo-cocycles* [in fact, coboundaries $\xi_\lambda(g'|g) = \eta_\lambda(g' * g) - \eta_\lambda(g') - \eta_\lambda(g)$, where $\eta_\lambda(g)$ is the generating function (of the coboundary)] with 'mass' parameters $\lambda_a = m_a^3$, which define trivial extensions as such, but provide new commutation relations in the Lie algebra of $\tilde{G}$ and alter the number of degrees of freedom of the theory (see below). To make more explicit the intrinsic significance of the two quantities $\xi_j\,,\ j = 1, 2$, let us calculate the non-trivial Lie-algebra commutators of the right-invariant vector fields from the group law (36). They are explicitly:

$$\begin{aligned}
\left[\tilde{X}^R_{A^j_a(x)}, \tilde{X}^R_{E^k_b(y)}\right] &= -\delta^{ab}\delta_{jk}\delta(x-y)\Xi\,, \\
\left[\tilde{X}^R_{E_a(x)}, \tilde{X}^R_{\varphi_b(y)}\right] &= -C^{ab}_c\delta(x-y)\tilde{X}^R_{E_c(x)} + \frac{1}{r}\delta^{ab}\nabla_x\delta(x-y)\Xi\,, \\
\left[\tilde{X}^R_{A_a(x)}, \tilde{X}^R_{\varphi_b(y)}\right] &= -C^{ab}_c\delta(x-y)\tilde{X}^R_{A_c(x)} \\
\left[\tilde{X}^R_{\varphi_a(x)}, \tilde{X}^R_{\varphi_b(y)}\right] &= -C^{ab}_c\delta(x-y)\tilde{X}^R_{\varphi_c(x)}\,,
\end{aligned} \tag{37}$$

which agree with those of Ref. [21] when the identification $\hat{E}_a \equiv i\tilde{X}^R_{A_a}\,,\ \hat{A}_a \equiv i\tilde{X}^R_{E_a}\,,\ \hat{G}_a \equiv i\tilde{X}^R_{\varphi_a}$ is made [note that $\tilde{X}^R_{A_a} \sim \frac{\delta}{\delta A_a}$ and $\tilde{X}^R_{E_a} \sim \frac{\delta}{\delta E_a}$ near the identity element $\tilde{g} = e$ of $\tilde{G}$, which motivates this particular identification].

Let us construct the Hilbert space of the theory. As already mentioned, the representation $L_{\tilde{g}'}\Psi(\tilde{g}) = \Psi(\tilde{g}' * \tilde{g})$ of $\tilde{G}$ on $\tilde{T}$-equivariant wave functions (11) proves to be reducible. The reduction is achieved by means of polarization conditions (22) which contain finite right-transformations generated by left-invariant vector fields $\tilde{X}^L$ devoid of dynamical content (that is, without a canonically conjugated counterpart), and half of the left-invariant vector fields related to dynamical coordinates (either "positions" or "momenta"). The left-invariant vector fields without a canonically conjugated counterpart are the combinations

$$\mathcal{G}_c = <\tilde{X}^L_{\theta_a} \equiv \tilde{X}^L_{\varphi_a} - \frac{1}{r}\nabla\cdot\tilde{X}^L_{A_a}>\,, \tag{38}$$

which close a Lie subalgebra isomorphic to the *higher-order* gauge subalgebra (it generates a horizontal ideal of the right-enveloping algebra $\mathcal{U}(\tilde{\mathcal{G}}^R)$ and proves to be zero on polarized wave functionals)

$$\mathcal{G}_{\rm gauge} = <\tilde{X}^R_{\phi_a(x)} = \tilde{X}^R_{\varphi_a(x)} + \frac{1}{r}\nabla\cdot\tilde{X}^R_{A_a(x)} + iC^{ab}_c\tilde{X}^R_{A_b(x)}\cdot\tilde{X}^R_{E_c(x)}>\ . \tag{39}$$

As for the Abelian case, the flow of (39) recovers the time-independent (residual) transformation (34).

The characteristic subalgebra $\mathcal{G}_c$ can be completed to a full polarization subalgebra $\mathcal{G}_p$ in two different ways:

$$\mathcal{G}^{(A)}_p \equiv <\mathcal{G}_c,\ \tilde{X}^L_{A_b}\ \forall b>,\quad \mathcal{G}^{(E)}_p \equiv <\mathcal{G}_c,\ \tilde{X}^L_{E_b}\ \forall b>, \tag{40}$$

each one giving rise to a different representation space: a) the electric field representation and b) the magnetic field representation, respectively. The polarized, $U(1)$-equivariant functions are:

$$\begin{aligned}\tilde{X}^L\Psi_A=0,\ \forall\tilde{X}^L\in\mathcal{G}_p^{(A)} &\Rightarrow \Psi_A(A,E,U;\zeta)=\zeta e^{-\frac{i}{r^2}\int d^3x\,\mathrm{tr}\left[AE-U\nabla U^{-1}E\right]}\Phi_A(E)\,,\\ \tilde{X}^L\Psi_E=0,\ \forall\tilde{X}^L\in\mathcal{G}_p^{(E)} &\Rightarrow \Psi_E(A,E,U;\zeta)=\zeta e^{\frac{i}{r^2}\int d^3x\,\mathrm{tr}\left[AE-U\nabla U^{-1}E\right]}\Phi_E(\mathcal{A})\,,\end{aligned}\qquad(41)$$

where Φ_A and Φ_E are arbitrary functionals of E and $\mathcal{A}\equiv A+\nabla UU^{-1}$, respectively. The trivial fibration of $\tilde{T}=T\times U(1)$ for the massless case allows us to impose the whole $T_p=T$ group as constraint conditions $L_{\tilde{g}'_t}\Psi(\tilde{g})=\Psi$, $\forall\tilde{g}'_t=(0,0,U';1)\in T$ on wavefunctionals. For each representation space, the constraint conditions read:

$$\begin{aligned}L_{\tilde{g}'_t}\Psi_A(\tilde{g})=\Psi_A(\tilde{g}) &\Rightarrow \Phi_A(E)=e^{-2\frac{i}{r^2}\int d^3x\,\mathrm{tr}\left[U'^{-1}\nabla U'E\right]}\Phi_A(U'EU'^{-1})\,,\\ L_{\tilde{g}'_t}\Psi_E(\tilde{g})=\Psi_E(\tilde{g}) &\Rightarrow \Phi_E(\mathcal{A})=\Phi_E(U'\mathcal{A}U'^{-1}+\nabla U'U'^{-1})\,,\end{aligned}\qquad(42)$$

which are the finite counterpart of the infinitesimal, quantum implementation of the Gauss law $\tilde{X}^R_{\varphi_a(x)}\Psi=-i\hat{G}_a(x)\Psi=0$. The polarized and constrained wavefunctionals (41,42) define the constrained Hilbert space $\mathcal{H}_{\rm phys}(\tilde{G})$ of the theory, and the infinitesimal form $\tilde{X}^R_{\tilde{g}}\Psi(\tilde{g})$ of the finite left-action $L_{\tilde{g}'}\Psi(\tilde{g})$ of $\tilde{G}$ on $\mathcal{H}(\tilde{G})$ provides the action of the operators $\hat{A}_a,\hat{E}_a,\hat{G}_a$ on wave functions (see [22] for more details).

The good operators $\tilde{\mathcal{G}}_{\rm good}$ for this case have to be found inside the right-enveloping algebra $\mathcal{U}(\tilde{\mathcal{G}}^R)$ of polynomials of the basic operators $\hat{A}_a(x)$, $\hat{E}_b(x)$, as forming part of the *normalizer* of T (see Eq. (21)). In particular, some good operators are:

$$\tilde{\mathcal{G}}_{\rm good}=<\mathrm{tr}[\hat{E}^j(x)\hat{B}^k(x)],\ \mathrm{tr}[\hat{E}^j(x)\hat{E}^k(x)],\ \mathrm{tr}[\hat{B}^j(x)\hat{B}^k(x)],\ \Xi>\,,\qquad(43)$$

where $\hat{B}_a\equiv\nabla\wedge\hat{A}_a-\frac{1}{2}rC_c^{ab}\hat{A}_b\wedge\hat{A}_c$ (the magnetic field) can be interpreted as a "correction" to $\hat{A}_a$ that, unlike $\hat{A}_a$, transforms homogeneously under the adjoint action of T [see 2nd line of (37)]. The components $\hat{\Theta}^{\mu\nu}(x)$ of the standard canonical energy-momentum tensor for Yang-Mills theories are linear combinations of operators in (43); for example, $\hat{\Theta}^{00}(x)=-\mathrm{tr}[E^2(x)+B^2(x)]$ is the Hamiltonian density. In this way, Poincaré invariance is retrieved in the constrained theory.

Let us mention, for the sake of completeness, that the actual use of good operators is not restricted to first- and second-order operators. Higher-order operators can constitute a useful tool in finding the whole constrained Hilbert space $\mathcal{H}_{\rm phys}(\tilde{G})$. In fact, it can be obtained from a constrained (physical) state $\Phi^{(0)}$, i.e. $\hat{G}_a\Phi^{(0)}=0$, on which the energy-momentum tensor has null expectation value $\langle\Phi^{(0)}|\hat{\Theta}^{\mu\nu}|\Phi^{(0)}\rangle=0$, by taking the orbit of the rest of good operators passing through this "vacuum". This has been indeed a rather standard technique (the Verma module approach) in theories where null vector states are present in the original Hilbert space [23, 24, 11]. From an other point of view, with regard to confinement, exponentials of the form $\varepsilon_{\Sigma_2}\equiv\mathrm{tr}\left[\exp(\epsilon_{jkl}\int_{\Sigma_2}d\sigma^{jk}\hat{E}^l)\right]$ and $\beta_{\Sigma_2}\equiv\mathrm{tr}\left[\exp(\epsilon_{jkl}\int_{\Sigma_2}d\sigma^{jk}\hat{B}^l)\right]$, where Σ_2 is a two-dimensional surface in three-dimensional space, are good operators related to Wilson loops.

As a previous step before examining the massive case, let us show how new physics can enter the theory by considering non-trivial representations $D^{(\epsilon)}_{\tilde{T}}$ of $\tilde{T}$ or, in an equivalent way, by introducing certain extra coboundaries in the group law (36). Indeed, more general representations for the constraint subgroup T, namely the one-dimensional representation $D^{(\epsilon)}_{\tilde{T}}(U) = e^{i\epsilon_U}$, can be considered if we impose additional boundary conditions like $U(x) \stackrel{x\to\infty}{\longrightarrow} \pm I$; this means that we compactify the space $\Re^3 \to S^3$, so that the group T fall into disjoint homotopy classes $\{U_l\,,\,\epsilon_{U_l} = l\vartheta\}$ labelled by integers $l \in Z = \Pi_3(\mathbf{T})$ (the third homotopy group). The index ϑ (the *ϑ-angle* [19]) parametrizes *non-equivalent quantizations*, as the Bloch momentum ϵ does for particles in periodic potentials, where the wave function acquires a phase $\psi(q+2\pi) = e^{i\epsilon}\psi(q)$ after a translation of, let us say, 2π. The phenomenon of non-equivalent quantizations can also be reproduced by keeping the constraint condition $D^{(\epsilon)}_{\tilde{T}}(U) = 1$, as in (42), at the expense of introducing a new (pseudo) cocycle ξ_ϑ which is added to the previous cocycle $\xi = \xi_1 + \xi_2$ in (36). The generating function of ξ_ϑ is $\eta_\vartheta(g) = \vartheta \int d^3x\, \mathcal{C}^0(x)$, where $\mathcal{C}^0$ is the temporal component of the *Chern-Simons secondary characteristic class*

$$\mathcal{C}^\mu = -\frac{1}{16\pi^2}\epsilon^{\mu\alpha\beta\gamma}\mathrm{tr}(\mathcal{F}_{\alpha\beta}\mathcal{A}_\gamma - \frac{2}{3}\mathcal{A}_\alpha\mathcal{A}_\beta\mathcal{A}_\gamma)\,, \tag{44}$$

which is the vector whose divergence equals the Pontryagin density $\mathcal{P} = \partial_\mu\mathcal{C}^\mu = -\frac{1}{16\pi^2}\mathrm{tr}$ $({}^*\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu})$ (see [21], for instance). Like some total derivatives (namely, the Pontryagin density), which do not modify the classical equations of motion when added to the Lagrangian but have a non-trivial effect in the quantum theory, the coboundaries ξ_ϑ give rise to non-equivalent quantizations parametrized by ϑ when the topology of the space is affected by the imposition of certain boundary conditions ("compactification of the space"), even though they are trivial cocycles of the "unconstrained" theory. The phenomenon of non-equivalent quantizations can also be understood sometimes as a *Aharonov-Bohm-like effect* (an effect experienced by the quantum particle but not by the classical particle) and $\delta\eta(g) = \frac{\delta\eta(g)}{\delta g^j}\delta g^j$ can be seen as an *induced gauge connection* (see [8] for the example of a superconducting ring threaded by a magnetic flux) which modify the momenta according to minimal coupling.

There exist other kind of coboundaries generated by functions $\eta(g)$ with non-trivial gradient $\delta\eta(g)|_{g=e} \neq 0$ at the identity $g = e$, which provide a contribution to the connection form of the theory (14) and the structure constants of the original Lie algebra (37). We shall call these *pseudo-cocycles*, since they give rise to *pseudo-cohomology classes* related with coadjoint orbits of semisimple groups [10]. Whereas coboundaries generated by global functions on the original (infinite-dimensional) group G having trivial gradient at the identity, namely ξ_ϑ, contribute the quantization with global (topological) effects, pseudo-cocycles can give dynamics to some non-dynamical operators and provide new couples of conjugated field operators, thus substantially modifying the theory. Let us see how, in fact, the possibility of *non-Abelian* representations of $\tilde{T}$ is equivalent to the introduction of new pseudo-cocycles in the centrally extended group law (36).

3.1 Cohomological origin of mass and alternatives to the Higgs mechanism

Non-trivial transformations of the wavefunctional Ψ under the action of T can also be reproduced by considering the pseudo-cocycle

$$\xi_\lambda(g'|g) \equiv -2\int d^3x\,\mathrm{tr}[\lambda\,(\log(U'U) - \log U' - \log U)]\,, \tag{45}$$

which is generated by $\eta_\lambda(g) = -2\int d^3x\,\mathrm{tr}[\lambda \log U]$, where $\lambda = \lambda_a T^a$ is a matrix carrying some parameters λ_a (with mass-cubed dimension) which actually characterize the representation of $\tilde{G}$. However, unlike ξ_ϑ, this pseudo-cocycle (whose generating function η_λ has a non-trivial gradient at the identity $g = e$) alters the Lie-algebra commutators of T and leads to the appearance of new central terms at the right-hand side of the last line of Eq. (37). More explicitly:

$$\left[\tilde{X}^R_{\varphi_a(x)}, \tilde{X}^R_{\varphi_b(y)}\right] = -C^{ab}_c\delta(x-y)\tilde{X}^R_{\varphi_c(x)} - C^{ab}_c\frac{\lambda^c}{r^2}\delta(x-y)\Xi\,. \tag{46}$$

The appearance of new central terms proportional to the parameter λ^c at the right-hand side of (46) restricts the number of vector fields in the characteristic subalgebra (38), which now consists of

$$\mathcal{G}_c = <\tilde{X}^L_{\theta_a}\ /\ C^{ab}_c\lambda^c \neq 0\ \forall b> \tag{47}$$

(that is, the subalgebra of non-dynamical generators), with respect to the case $\lambda^c = 0$, where $\mathcal{G}_c$ is isomorphic to $\mathcal{T}$. Therefore, the pseudo-cocycle ξ_λ provide new degrees of freedom to the theory; that is, new pairs of generators $(\tilde{X}^R_{\varphi_a}, \tilde{X}^R_{\varphi_b})$, with $C^{ab}_c\lambda^c \neq 0$, become conjugated and, therefore, new basic field operators enter the theory. Let us see how these new degrees of freedom are transfered to the vector potentials to conform *massive* vector bosons with mass cubed $m_a^3 = \lambda_a$.

In order to count the number of degrees of freedom for a given structure subgroup $\tilde{T}$ and a given "mass" matrix λ, let us denote by $\tau = \dim(\mathbf{T})$ and $c = \dim(\mathbf{G}_c)$ the dimensions of the rigid subgroup of T and the characteristic subgroup G_c, respectively. In general, for an arbitrary mass matrix λ, we have $c \leq \tau$. Unpolarized, $U(1)$-equivariant functions $\Psi(A^j_a, E^j_a, \varphi_a)$ depend on $n = 2\times 3\tau + \tau$ field coordinates in $d = 3$ spatial dimensions; polarization equations introduce $p = c + \frac{n-c}{2}$ independent restrictions on wave funtions, corresponding to c non-dynamical coordinates in G_c and half of the dynamical ones; finally, constraints provide $q = c + \frac{\tau - c}{2}$ additional restrictions which leave $f = n - p - q = 3\tau - c$ field degrees of freedom (in $d = 3$). Indeed, for the massive case, constraints are *second-class* and we can only impose a polarization subalgebra $\mathcal{T}_p \subset \tilde{\mathcal{T}}$, which contains a characteristic subalgebra $\mathcal{T}_c = <\tilde{X}^R_{\varphi_a}\ /\ C^{ab}_c\lambda^c = 0\ \forall b> \subset \tilde{\mathcal{T}}$ (which is isomorphic to $\mathcal{G}_c$) and half of the rest of generators in $\tilde{\mathcal{T}}$ (excluding Ξ); that is, $q = c + \frac{\tau-c}{2} \leq \tau$ independent constraints which lead to constrained wave functions having support on $f_{m\neq 0} = 2c + 3(\tau - c) \geq f_{m=0}$ arbitrary fields corresponding to c massless vector bosons attached to $\mathcal{T}_c$ and $\tau - c$ massive vector bosons. In particular, for the massless case we have $\mathcal{T}_c = \mathcal{T}$, i.e. $c = \tau$, since constraints are *first-class* (that is, we can impose $q = \tau$

restrictions) and constrained wave functions have support on $f_{m=0} = 3\tau - \tau = 2\tau \leq f_{m\neq 0}$ arbitrary fields corresponding to τ massless vector bosons. The subalgebra $\mathcal{T}_c$ corresponds to the unbroken gauge symmetry of the constrained theory and proves to be an *ideal* of $\tilde{\mathcal{G}}_{\rm good}$ [remember the characterization of *good* operators before Eq. (43)].

Let us work out a couple of examples. Cartan (maximal Abelian) subalgebras of T will be privileged as candidates for the unbroken electromagnetic gauge symmetry. Thus, let us use the Cartan basis $< H_i, E_{\pm\alpha} >$ instead of $< T^a >$, and denote $\{\varphi_i, \varphi_{\pm\alpha}\}$ the coordinates of T attached to this basis (i.e. $\varphi_{\pm\alpha}$ are complex field coordinates attached to each root $\pm\alpha$ and φ_i are real field coordinates attached to the maximal torus of $\mathbf{T}$). For $T = \mathrm{Map}(\Re^3, SU(2))$ and $\lambda = \lambda_1 H_1$, the characteristic, polarization and constraint subalgebras (leading to the electric field representation) are:

$$\mathcal{G}_c =< \tilde{X}^L_{\theta_1} >, \;\; \mathcal{G}_p^{(A)} =< \tilde{X}^L_{\theta_1}, \tilde{X}^L_{\theta_{+1}}, \tilde{X}^L_A >, \;\; \mathcal{T}_p =< \tilde{X}^R_{\varphi_1}, \tilde{X}^R_{\varphi_{-1}} > . \tag{48}$$

which corresponds to a self-interacting theory of a massless vector boson A_1 [with unbroken gauge subgroup $T_c = \mathrm{Map}(\Re^3, U(1)) \subset \mathrm{Map}(\Re^3, SU(2))$] and two charged vector bosons $A_{\pm 1}$ with mass cubed $m_1^3 = \lambda_1$. For $T = \mathrm{Map}(\Re^3, SU(3))$ and $\lambda = \lambda_2 H_2$, we have

$$\begin{array}{c} \mathcal{G}_c =< \tilde{X}^L_{\theta_{1,2}}, \tilde{X}^L_{\theta_{\pm 1}} >, \;\; \mathcal{G}_p^{(A)} =< \tilde{X}^L_{\theta_{1,2}}, \tilde{X}^L_{\theta_{\pm 1}}, \tilde{X}^L_{\theta_{+2,+3}}, \tilde{X}^L_A >, \\ \mathcal{T}_p =< \tilde{X}^R_{\varphi_{1,2}}, \tilde{X}^R_{\varphi_{\pm 1}}, \tilde{X}^R_{\varphi_{-2,-3}} > \end{array} . \tag{49}$$

Thus, the constrained theory corresponds to a self-interacting theory of two massless vector bosons $A_{1,2}$, two massless charged vector bosons $A_{\pm 1}$ [the unbroken gauge subgroup is now $T_c = \mathrm{Map}(\Re^3, SU(2) \times U(1))$] and four charged vector bosons $A_{\pm 2, \pm 3}$ with mass cubed $m_2^3 = \lambda_2$. For $SU(N)$ we have several symmetry breaking patterns related to the different choices of mass matrix $\lambda = \sum_{i=1}^{N-1} \lambda_i H_i$.

Summarizing, new basic operators $\hat{G}_{\pm\alpha} \equiv \tilde{X}^R_{\varphi_{\pm\alpha}}$, with $C_i^{\alpha -\alpha}\lambda^i \neq 0$, and new *non-trivial* good operators $\hat{C}_i = \{\text{Casimir operators of } \tilde{T}\}$ (i runs the range of $\mathbf{T}$) enter the theory, in contrast to the massless case. For example, for $T = \mathrm{Map}(\Re^3, SU(2))$, the Casimir operator is

$$\hat{C}(x) = (\hat{G}_1(x) + \frac{\lambda_1}{r^2})^2 + 2(\hat{G}_{+1}(x)\hat{G}_{-1}(x) + \hat{G}_{-1}(x)\hat{G}_{+1}(x)) . \tag{50}$$

Also, the Hamiltonian density $\hat{\Theta}^{00}(x) = -\mathrm{tr}\,[E^2(x) + B^2(x)]$ for $m = 0$ is affected in the massive case $m \neq 0$ by the presence of extra terms proportional to these non-trivial Casimir operators (which are zero on constrained wave functionals in the massless case), as follows:

$$\hat{\Theta}^{00}_{m\neq 0}(x) = \hat{\Theta}^{00}_{m=0}(x) + \sum_i \frac{r^2}{m_i^2} \hat{C}_i(x) . \tag{51}$$

Thus, the Schödinger equation $\int d^3x \hat{\Theta}^{00}_{m\neq 0}(x)\Phi = \mathcal{E}\Phi$ is also modified by the presence of extra terms.

4 Conclusions and outlooks

We have seen how the appearance of (quantum) central terms in the Lie-algebra of symmetry of gauge theories provides new degrees of freedom which are transferred to the potentials to conform massive bosons. Thus, the appearance of mass seems to have a *cohomological origin*, beyond any introduction of extra (Higgs) particles. Nevertheless, the introduction of mass through the pseudo-cocycle ξ_λ is equivalent to the choice of a vacuum in which some generators of the unbroken gauge symmetry T_c have a non-zero expectation value proportional to the mass parameter (see [22]). This fact reminds us of the Higgs mechanism in non-Abelian gauge theories, where the Higgs fields point to the direction of the non-null vacuum expectation values. However, the spirit of this standard approach to supply mass and the one explained in this paper are radically different, even though they have some characteristics in common. In fact, we are not making use of extra scalar fields in the theory to provide mass to the vector bosons, but it is the gauge group itself which acquires dynamics for the massive case and transfers degrees of freedom to the vector potentials.

It is also worth devoting some words with regard to renormalizability for the case of a non-trivial mass matrix $\lambda \neq 0$. Obviously we should refer to finitness simply, since we are dealing with a non-perturbative formulation. But the major virtue of a group-theoretic algorithm is that we automatically arrive at normal-ordered, finite quantities, and this is true irrespective of the "breaking" of the symmetry. We must notice that when we use the term 'unbroken gauge symmetry', in referring to T_c, we mean only that this subgroup of $\tilde{T}$ is devoid of dynamical content; the gauge group of the constrained theory is, in both the massless and massive cases, the group $T = \tilde{T}/U(1)$ although, for the massive case, only a polarization subgroup T_p can be consistently imposed as a constraint. This is also the case of the Virasoro algebra

$$[\hat{L}_n, \hat{L}_m] = (n-m)\hat{L}_{n+m} + \frac{1}{12}(cn^3 - c'n)\delta_{n,-m}\hat{1}\,, \tag{52}$$

in String Theory, where the appearance of central terms does not spoil gauge invariance but forces us to impose half of the Virasoro operators only (the positive modes $\hat{L}_{n\geq 0}$) as constraints. In general, although diffeomorphisms usually appear as a constriant algebra under which one might expect all the physical states to be singlets, dealing with them in the quantum arena, the possibility of central extensions should be naturally allowed and welcomed, as they provide new richness to the theory.

Pseudo-cocycles similar to ξ_λ do also appear in the representation of Kac-Moody and conformally invariant theories in general, although the pseudo-cocycle parameters are usually hidden in a redefinition of the generators involved in the pseudo-extension (the argument of the Lie-algebra generating function). For the Virasoro algebra, the redefinition of the $\hat{L}_0$ generator produces a non-trivial expectation value in the vacuum $h \equiv (c - c')/24$ [11].

Fermionic matter can also be incorporated into the theory through extra (Dirac fields) coordinates $\psi_l(x)$, $l = 1, \ldots, p$ and an extra two-cocycle

$$\xi_{\text{matter}} = i \int d^3x \, \left(\bar{\psi}' \gamma^0 \rho(U') \psi - \bar{\psi} \rho(U'^{-1}) \gamma^0 \psi' \right) , \tag{53}$$

where $\rho(U)$ is a p-dimensional representation of $\mathbf{T}$ acting on the column vectors ψ, and γ^0 is the time component of the standard Dirac matrices γ^μ (see [22] for more details).

Acknowledegment

M. Calixto thanks the University of Granada for a Postdoctoral grant and the Department of Physics of Swansea for its hospitality.

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