Algebraic characterization of constraints and generation of mass in gauge theories

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The possibility of non-trivial representations of the gauge group on wavefunctionals of a gauge invariant quantum field theory leads to a generation of mass for intermediate vector and tensor bosons. The mass parameters m show up as central charges in the algebra of constraints, which then become of second-class nature. The gauge group coordinates acquire dynamics outside the null-mass shell and provide the longitudinal field degrees of freedom that massless bosons need to form massive bosons.

1. INTRODUCTION

In this paper we discuss a new approach to quantum gauge theories, from a group-theoretic perspective, in which mass enters the theory in a *natural* way. More precisely, the presence of mass will manifest through non-trivial responses

$$U\Psi = D_{\tilde{T}}^{(m)}(U)\Psi, \tag{1}$$

of the wavefunctional Ψ under the action of gauge transformations $U \in \tilde{T}$, where we denote by $D_{\tilde{T}}^{(m)}$ a specific representation of the gauge group $\tilde{T}$ with index m. The standard case $D_{\tilde{T}}^{(m)}(U) = 1$, $\forall U \in \tilde{T}$ corresponds to the well-known 'Gauss law' condition, which also reads $\Phi_a\Psi = 0$ for infinitesimal gauge transformations $U \sim 1 + \phi^a\Phi_a$. The case of Abelian representations $D_{\tilde{T}}^{(\vartheta)}(U_n) = e^{in\vartheta}$ of $\tilde{T}$, where n denotes the winding number of U_n, leads to the well-known ϑ-vacuum phenomena. We shall see that more general (non-Abelian) representations $D_{\tilde{T}}^{(m)}$ of the gauge group $\tilde{T}$ entail *non-equivalent quantizations* (in the sense of, e.g. [1,2]) and a *generation of mass*.

This non-trivial response of Ψ under gauge transformations U causes a *deformation* of the corresponding Lie-algebra commutators and leads

to the appearance of central terms proportional to mass parameters (eventually parametrizing the *non-equivalent quantizations*) in the algebra of constraints, which then become a mixture of first- and second-class constraints. As a result, extra (internal) field degrees of freedom emerge out of second-class constraints and are transferred to the gauge potentials to conform massive bosons (without Higgs fields!).

Thus, the 'classical' case $D_{\tilde{T}}^{(m)} = 1$ is not in general preserved in passing to the quantum theory. Upon quantization, first-class constraints (connected with a gauge invariance of the classical system) might become second-class, a metamorphosis which is familiar when quantizing *anomalous* gauge theories. Quantum "anomalies" change the picture of physical states being singlets under the constraint algebra. Anomalous (unexpected) situations generally go with the standard viewpoint of quantizing classical systems and the avoidance of them is evident when quantizing, for example, Yang-Mills theory with chiral fermions, where a cancellation of gauge anomalies is apparently needed; however, these breakdowns, which sometimes are inescapable obstacles for canonical quantization, could be reinterpreted as normal (even essential) situations in a wider setting. Dealing with constraints directly in the quantum arena, this transmutation in the

nature of constraints should be naturally allowed, as it provides new richness to the quantum theory.

This *cohomological* mechanism of mass generation makes perfect sense from a Group Approach to Quantization (GAQ [3]) framework, which is, at heart, an operator description of a quantum system. Thus, our essential ingredient to define a quantum system will be a given underlying symmetry algebra $\tilde{\mathcal{G}}$ rather than an action functional S, which is the standard starting point in the usual ("classically-oriented") formulation of QFT.

In order to set the context, let us describe a simple, but illustrative, example of an abstract quantizing algebra $\tilde{\mathcal{G}}$ which eventually applies to a diversity of physical systems.

2. A SIMPLE ABSTRACT QUANTIZING ALGEBRA

Our particular algebra under study will be the following:

$$
\begin{aligned}
[X_j, P_k] &= i\delta_{jk}I \, , \\
[\Phi_a, \Phi_b] &= if_{ab}^c\Phi_c + im_{ab}I \, , \quad (2) \\
[X_j, \Phi_a] &= i\check{f}_{ja}^k X_k, \quad [P_j, \Phi_a] = i\check{f}_{ja}^k P_k,
\end{aligned}
$$

where X_j and P_k represent standard "position" and "momentum" operators, respectively, corresponding to the *extended phase space* $\mathcal{F}$ of the preconstrained (free-like) theory; The operators Φ_a represent the constraints which, for the moment, are supposed to close a Lie subalgebra $\tilde{\mathcal{T}}$ with structure constants f_{ab}^c and central charges m_{ab}. We also consider a diagonal action of constraints Φ on X and P with structure constants $\check{f}_{ja}^k$ (non-diagonal actions mixing X and P lead to interesting "anomalous" situations which we shall not discuss here [4]). By I we simply denote the identity operator, that is, the generator of the typical phase invariance $\Psi \sim e^{i\beta}\Psi$ of Quantum Mechanics. At this stage, it is worth mentioning that we could have introduced dynamics in our model by adding a Hamiltonian operator H to $\tilde{\mathcal{G}}$. However, we have preferred not to include it because, although we could make compatible the dynamics H and the constraints Φ, the price could result in an unpleasant enlarging of

$\tilde{\mathcal{G}}$, which would make the quantization procedure much more involved. Anyway, for us, the *true* dynamics (that which preserves the constraints) will eventually arise as part of the set of *good* operators (observables) of the theory (see below).

Note that a flexibility in the class of the constraints has being allowed by introducing arbitrary central charges m_{ab} in (2). Thus, the operators Φ_a represent a mixed set of first- and second-class constraints. Let us denote by $\mathcal{T}^{(1)} = \{\Phi_n^{(1)}\}$ the subalgebra of first-class constraints, that is, the ones which do not give rise to central terms proportional to m_{ab} at the right hand side of the commutators (2). The rest of constraints (second-class) will be arranged by conjugated pairs $(\Phi_{+\alpha}^{(2)}, \Phi_{-\alpha}^{(2)})$, so that $m_{+\alpha,-\alpha} \neq 0$.

The simplest ('classical') case is when $m_{ab} = 0$, $\forall a, b$, that is, when all constraints are first class $\mathcal{T}^{(1)} = \mathcal{T} = \tilde{\mathcal{T}}/u(1)$ and wave functions are singlets under $\mathcal{T}$. However, the 'quantum' case $m_{ab} \neq 0$ entails non-equivalent quantizations with important physical consequences. This possibility indicates a non-trivial response (1) of the wave function Ψ under $\tilde{\mathcal{T}}$. That is, Ψ acquires a non-trivial dependence on extra degrees of freedom $\phi_{-\alpha}^{(2)}$ ('negative modes' attached to pairs of second-class constraints), in addition to the usual configuration space variables x_j (attached to X_j).

Let us formally outline the actual construction of the unitary irreducible representations of the group $\tilde{G}$ with Lie-algebra (2). Wave functions Ψ are defined as complex functions on $\tilde{G}$, $\Psi : \tilde{G} \to C$, so that the (let us say) left-action

$$
L_{\tilde{g}'}\Psi(\tilde{g}) \equiv \Psi(\tilde{g}' * \tilde{g}), \quad \tilde{g}', \tilde{g} \in \tilde{G} \quad (3)
$$

defines a reducible (in general) representation of $\tilde{G}$. The reduction is achieved by means of that maximal set of right restrictions on wave functions

$$
R_{\tilde{g}_p}\Psi = \Psi, \quad \forall \tilde{g}_p \in G_p, \quad (4)
$$

(which commute with the left action) compatible with the natural condition $I\Psi = \Psi$. The right restrictions (4) generalize the notion of *polarization conditions* of Geometric Quantization and give rise to a certain representation space depending on the choice of the subgroup $G_p \subset \tilde{G}$.

For the algebra (2), a polarization subgroup can be $G_p^{(P)} = F_P \times_s T_p$, that is, the semi-direct product of the Abelian group of translations F_P generated by $\mathcal{F}_P \equiv \{P_k\}$ (half of the symplectic generators in $\mathcal{F}$) by a polarization subalgebra $\mathcal{T}_p = \{\Phi_n^{(1)}, \Phi_{+\alpha}^{(2)}\}$ of $\tilde{T}$ consisting of first-class constraints and half of second-class constraints (namely, the 'positive modes'). The polarization conditions (4) lead to the configuration-space representation made of wave functions $\Psi(x_j, \phi_{-\alpha}^{(2)})$ depending arbitrarily on the group coordinates on $\tilde{G}/G_p$ only. Thus, as mentioned above, wave functions transform non-trivially under the left-action $L_\phi \Psi(\tilde{g}) = D_{\tilde{T}}^{(m)}(\phi)\Psi(\tilde{g})$ of $\tilde{T}$ according to a given representation $D_{\tilde{T}}^{(m)}$ like in (1). The physical Hilbert space is made of those wave functions $\Psi_{\text{ph.}}$ that transform as 'highest-weight vectors' under $\tilde{T}$, that is, they stay invariant under the left-action of first-class constraints and (let us say) negative second-class modes:

$$L_{\phi_n^{(1)}}\Psi_{\text{ph.}} = \Psi_{\text{ph.}}\,,\ n = 1, ..., \dim(T^{(1)}),$$
$$L_{\phi_{-\alpha}^{(2)}}\Psi_{\text{ph.}} = \Psi_{\text{ph.}}\,,\ \alpha = 1, ..., \dim(T/T^{(1)})/2\,, \ (5)$$

which close the subgroup $T_p \subset \tilde{T}$.

The counting of *true degrees of freedom* is as follows: polarized-constrained wave functions (5) depend arbitrarily on $d = \dim(\tilde{G}) - \dim(G_p) - \dim(T_p) - 1$ reduced-space coordinates (we are subtracting the phase coordinate $e^{i\beta}$). The algebra of observables of the theory, $\tilde{\mathcal{G}}_{\text{good}} \subset \mathcal{U}(\tilde{\mathcal{G}})$ (the enveloping algebra), has to be found inside the *normalizer* of constraints, that is:

$$\left[\tilde{\mathcal{G}}_{\text{good}}, \mathcal{T}_p\right] \subset \mathcal{T}_p\,. \tag{6}$$

From this characterization, the subalgebra of first-class constraints $\mathcal{T}^{(1)}$ become a horizontal ideal (a *gauge* subalgebra [5]) of $\tilde{\mathcal{G}}_{\text{good}}$. The Hamiltonian operator has to be found inside $\tilde{\mathcal{G}}_{\text{good}}$ by using extra physical arguments.

In what follows, the quantization of massless and massive non-Abelian Yang-Mills, linear Gravity and Abelian two-form gauge field theories are developed from this new approach, where a cohomological origin of mass is pointed out.

3. UNIFIED QUANTIZATION OF MASSLESS AND MASSIVE VECTOR AND TENSOR BOSONS

Let us start with the simplest case of the electromagnetic field. Let us use a Fourier parametrization

$$A_\mu(x) \equiv \int \frac{d^3k}{2k^0}[a_\mu(k)e^{-ikx} + a_\mu^\dagger(k)e^{ikx}]\,,$$
$$\Phi(x) \equiv \int \frac{d^3k}{2k^0}[\varphi(k)e^{-ikx} + \varphi^\dagger(k)e^{ikx}]\,, \quad (7)$$

for the vector potential $A_\mu(x)$ and the constraints $\Phi(x)$ (the generators of local $U(1)(x)$ gauge transformations). The Lie algebra $\tilde{\mathcal{G}}$ of the quantizing electromagnetic group $\tilde{G}$ has the following form [6]

$$\begin{aligned}
\left[a_\mu(k), a_\nu^\dagger(k')\right] &= \eta_{\mu\nu}\Delta_{kk'}I\,, \\
\left[\varphi(k), \varphi^\dagger(k')\right] &= k^2\Delta_{kk'}I\,, \\
\left[a_\mu^\dagger(k), \varphi(k')\right] &= -ik_\mu\Delta_{kk'}I\,, \\
\left[a_\mu(k), \varphi^\dagger(k')\right] &= -ik_\mu\Delta_{kk'}I\,, \tag{8}
\end{aligned}$$

where $\Delta_{kk'} = 2k^0\delta^3(k - k')$ is the generalized delta function on the positive sheet of the mass hyperboloid and $k^2 = m^2$ is the squared mass. Constraints are first-class for $k^2 = 0$ and constraint equations $\varphi\Psi = 0 = \varphi^\dagger\Psi$ keep 2 field degrees of freedom out of the original 4, as corresponds to a photon. For $k^2 \neq 0$, constraints are second-class and the restrictions $\varphi\Psi = 0$ keep 3 field degrees of freedom out of the original 4, as corresponds to a Proca field.

For symmetric and anti-symmetric tensor potentials $A_{\mu\nu}^{(\pm)}$, the algebra is the following [7]:

$$\begin{aligned}
\left[a_{\lambda\nu}^{(\pm)}(k), a_{\rho\sigma}^{\dagger(\pm)}(k')\right] &= N_{\lambda\nu\rho\sigma}^{(\pm)}\Delta_{kk'}I\,, \\
\left[\varphi_\rho^{(\pm)}(k), \varphi_\sigma^{\dagger(\pm)}(k')\right] &= k^2M_{\rho\sigma}^{(\pm)}(k)\Delta_{kk'}I\,, \\
\left[a_{\lambda\nu}^{\dagger(\pm)}(k), \varphi_\sigma^{(\pm)}(k')\right] &= -ik^\rho N_{\lambda\nu\rho\sigma}^{(\pm)}\Delta_{kk'}I\,, \\
\left[a_{\lambda\nu}^{(\pm)}(k), \varphi_\sigma^{\dagger(\pm)}(k')\right] &= -ik^\rho N_{\lambda\nu\rho\sigma}^{(\pm)}\Delta_{kk'}I\,, \quad (9)
\end{aligned}$$

where $M_{\rho\sigma}^{(\pm)}(k) \equiv \eta_{\rho\sigma} - \kappa_{(\mp)}\frac{k_\rho k_\sigma}{k^2}$ and $N_{\lambda\nu\rho\sigma}^{(\pm)} \equiv \eta_{\lambda\rho}\eta_{\nu\sigma} \pm \eta_{\lambda\sigma}\eta_{\nu\rho} - \kappa_{(\pm)}\eta_{\lambda\nu}\eta_{\rho\sigma}$, with $\kappa_{(+)} = 1$ and $\kappa_{(-)} = 0$. For the massless $k^2 = 0$ case, all constraints are first-class for the symmetric

case, whereas the massless, anti-symmetric case possesses a couple of second-class constraints:

$$\left[\check{k}^\rho\varphi_\rho^{(-)}(k),\check{k}'^\sigma\varphi_\sigma^{(-)\dagger}(k')\right]=4(k^0)^4\Delta_{kk'}I\,,\quad(10)$$

where $\check{k}^\rho\equiv k_\rho$. Thus, first-class constraints for the massless anti-symmetric case are $\mathcal{T}_{(-)}^{(1)}=\{\epsilon_\mu^\rho\varphi_\rho^{(-)},\,\epsilon_\mu^\rho\varphi_\rho^{(-)\dagger}\}$, $\mu=0,1,2$, where ϵ_μ^ρ is a tetrad which diagonalizes the matrix $P_{\rho\sigma}=k_\rho k_\sigma$; in particular, we choose $\epsilon_3^\rho\equiv\check{k}^\rho$ and $\epsilon_0^\rho\equiv k^\rho$. There are $2=10-8$ true degrees of freedom for the symmetric case (a massless graviton) and $1=6-5$ for the anti-symmetric case (a pseudo-scalar particle).

For $k^2\neq0$, all constraints are second-class for the symmetric case, whereas, for the anti-symmetric case, constraints close a Proca-like subalgebra which leads to three pairs of second-class constraints, and a pair of gauge vector fields $(k^\lambda\varphi_\lambda^{(-)},\,k^\lambda\varphi_\lambda^{(-)\dagger})$. The constraint equations keep $6=10-4$ field degrees of freedom for the symmetric case (massive spin 2 particle + massive scalar field —the trace of the symmetric tensor), and $3=6-3$ field degrees of freedom for the anti-symmetric case (massive pseudo-vector particle).

For non-Abelian $SU(N)$ Yang-Mills theories in the Weyl gauge $A^0=0$ there is still a residual gauge invariance $T=\mathrm{Map}(\Re^3,SU(N))$. The basic commutators between the non-Abelian vector potentials $A_a^j(x)$, $j=1,2,3;a=1,...,N^2-1$, the electric field $E_a^j(x)$ and the (Gauss law) constraints $\Phi_a(x)$ are [8]

$$\begin{aligned}\left[A_a^j(x),E_b^k(y)\right]&=i\delta_{ab}\delta^{jk}\delta(x-y)I\,,\\\left[\Phi_a(x),\Phi_b(y)\right]&=-if_{ab}^c\delta(x-y)\Phi_c(x)\\&\quad-if_{ab}^c\frac{\lambda_c}{r^2}\delta(x-y)I\,,\\\left[A_a^j(x),\Phi_b(y)\right]&=-if_{ab}^c\delta(x-y)A_c^j(x)\\&\quad-\frac{i}{r}\delta_{ab}\partial_x^j\delta(x-y)I\,,\\\left[E_a^j(x),\Phi_b(y)\right]&=-if_{ab}^c\delta(x-y)E_c^j(x)\,,\quad(11)\end{aligned}$$

where r is the coupling constant and $\lambda_{ab}=f_{ab}^c\lambda_c$ is a mass matrix $(\lambda\sim m^3)$. Let us denote by $c\equiv\dim(T^{(1)})$ and $\tau\equiv N^2-1$ the dimensions of the rigid subgroup of first-class constraints and

$SU(N)$ respectively. Unpolarized wave functions $\Psi(A_a^j,E_a^j,\phi_a)$ depend on $n=2\times3\tau+\tau$ field coordinates in $d=3$ dimensions; polarization equations introduce $p=c+\frac{n-c}{2}$ independent restrictions on wave functions, corresponding to c non-dynamical coordinates in $T^{(1)}$ and half of the dynamical ones; finally, constraints (5) impose $q=c+\frac{\tau-c}{2}$ additional restrictions which leave $f=n-p-q=2c+3(\tau-c)$ field degrees of freedom (in $d=3$). These fields correspond to c massless vector bosons (2 polarizations) attached to $T^{(1)}$ and $\tau-c$ massive vector bosons. In particular, for the massless case, we have $c=\tau$, since constraints are *first-class* (that is, we can impose $q=\tau$ restrictions) and constrained wave functions have support on $f_{m=0}=3\tau-\tau=2\tau\leq f_{m\neq0}$ arbitrary fields corresponding to τ massless vector bosons. The subalgebra $\mathcal{T}^{(1)}$ corresponds to the unbroken gauge symmetry of the constrained theory. There are distinct symmetry-breaking patterns $\tilde{\mathcal{T}}\to\mathcal{T}^{(1)}$ according to the different choices of mass-matrices $\lambda_{ab}=f_{ab}^c\lambda_c$ in (11).

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