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Planar rotations for a gyrostat in the three-body problem

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Abstract

We consider the non-canonical Hamiltonian dynamics of a gyrostat in the three body problem. By means of geometric-mechanics methods we study the approximate Poisson dynamics that arises when we develop the potential in series of Legendre and truncate this in an arbitrary order k. Working in the reduced problem, the existence and number of equilibria, that we denominate *planar rotation type* in analogy with classic results on the topic, is considered. Necessary and sufficient conditions for their existence in a approximate dynamics of order k is obtained and we give explicit expressions of this equilibria, useful for the later study of the stability of the same ones. A complete study of the *planar rotation type* equilibria is made in approximate dynamics or order zero and one.

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1 Introduction

In the study of configurations of relative equilibria by differential geometry methods or by more classical ones; we will mention here the papers of Wang et al. [8], about the problem of a rigid body in a central Newtonian field; Maciejewski [3], about the problem of two rigid bodies in mutual Newtonian attraction. These papers have been generalized to the case of a gyrostat by Mondéjar and Vigueras [4] to the case of two gyrostats in mutual Newtonian attraction.

For the problem of three rigid bodies we would like to mention that Vidiakin [7] and Duboshin [1] proved the existence of Euler and Lagrange configurations of equilibria when the bodies possess symmetries; Zhuravlev [9] made a review of the results up to 1990.

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In Vera [5] and a recent paper of Vera and Vigueras [6] we study the non-canonical Hamiltonian dynamics of $n+1$ bodies in Newtonian attraction, where n of them are rigid bodies with spherical distribution of mass or material points and the other one is a triaxial gyrostat.

Let us remember that a gyrostat is a mechanical system S, composed of a rigid body S', and other bodies S'' (deformable or rigid) connected to it, in such a way that their relative motion with respect to its rigid part do not change the distribution of mass of the total system S, (see Leimanis [2] for details).

In this paper, we take $n = 2$ and as a first approach to the qualitative study of this system, we describe the approximate dynamics that arises in a natural way when we take the Legendre development of the potential function and truncate this until an arbitrary order. We give global conditions on the existence of relative equilibria and in analogy with classic results on the topic, we study the existence of relative equilibria that we will denominate of *planar rotation type* in the case in which S_1, S_2 are spherical or punctual bodies and S_0 is a gyrostat. Necessary and sufficient conditions for their existence in a approximate dynamics of order k are obtained and we give explicit expressions of these equilibria, useful for the later study of the stability of the same ones. A complete study of the planar rotation type equilibria is made in approximate dynamics of order zero and one. One should notice that the studied system, has potential interest both in astrodynamics (dealing with spacecrafts) as well as in the understanding of the evolution of planetary systems recently found (and more to appear), where some of the planets may be modeled like a gyrostat rather than a rigid body. In fact, the equilibria reported might well be compared with the ones taken for the 'parking areas' of the space missions (GENESIS, SOHO, DARWIN, etc) around the Eulerian points of the Sun-Earth and the Earth-Moon systems.

To finish this introduction, we describe the structure of the article. The paper is organized in four sections and the bibliography. In these sections we study the equations of motion, Casimir function and integrals of the system, the relative equilibria and the existence of planar rotation type equilibria in an approximate dynamics of order k, in particular in an approximate dynamics of order zero and one.

2 Equations of motion

Following the line of Vera and Vigueras [6] let S_0 be a gyrostat of mass m_0 and S_1, S_2 two spherical rigid bodies of masses m_1 and m_2. We use the following notation.

For $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^3$, $\mathbf{u} \cdot \mathbf{v}$ is the dot product, $\mid \mathbf{u} \mid$ is the Euclidean norm of the vector $\mathbf{u}$ and $\mathbf{u}\times\mathbf{v}$ is the cross product. $\mathbf{I}_{\mathbb{R}^3}$ is the identity matrix and $\mathbf{0}$ is the zero matrix of order three. Let $\mathbf{z} = (\boldsymbol{\Pi}, \boldsymbol{\lambda}, \mathbf{p}_{\boldsymbol{\lambda}}, \boldsymbol{\mu}, \mathbf{p}_{\mu}) \in \mathbb{R}^{15}$ be a generic element of the twice reduced problem obtained using the symmetries of the system, where $\mathbf{\Pi} = \mathbb{I}\mathbf{\Omega} + \mathbf{l}_r$ is the total rotational angular momentum vector of the gyrostat, $\mathbb{I} = diag(A,\, B,\, C)$ are the diagonal tensor of inertia of the gyrostat and $\mathbf{\Omega}$ the angular velocity of S_0 in the body frame, $\mathfrak{J}$, which is attached to its rigid part and whose axes have the direction of the principal axes of inertia of S_0. The vector $\mathbf{l}_r$ is the gyrostatic momentum that we suppose constant and given by $\mathbf{l}_r = (0, 0, l)$. The elements $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, $\mathbf{p}_{\boldsymbol{\lambda}}$ and $\mathbf{p}_{\mu}$ are respectively the barycentric coordinates and the linear momenta expressed in the body frame $\mathfrak{J}$.

The twice reduced Hamiltonian of the system, obtained by the action of the group $\mathbf{SE}(3)$, has the following expression

$$\mathcal{H}(\mathbf{z}) = \frac{\mid \mathbf{p}_{\boldsymbol{\lambda}} \mid^2}{2g_1} + \frac{\mid \mathbf{p}_{\boldsymbol{\mu}} \mid^2}{2g_2} + \frac{1}{2}\Pi\mathbb{I}^{-1}\Pi - \mathbf{l}_r \cdot \mathbb{I}^{-1}\Pi + \mathcal{V} \tag{2.1}$$

with

$$M_2 = m_1 + m_2, \qquad M_1 = m_1 + m_2 + m_0, \qquad g_1 = \frac{m_1 m_2}{M_2}, \qquad g_2 = \frac{m_0 M_2}{M_1}$$

being $\mathcal{V}$ the potential function of the system given by the formula

$$\mathcal{V}(\boldsymbol{\lambda}, \boldsymbol{\mu}) = -\left(\frac{G m_1 m_2}{\mid \boldsymbol{\lambda} \mid} + \int_{S_0} \frac{G m_1 dm(\mathbf{Q})}{\mid \mathbf{Q} + \boldsymbol{\mu} + \frac{m_2}{M_2}\boldsymbol{\lambda} \mid} + \int_{S_0} \frac{G m_2 dm(\mathbf{Q})}{\mid \mathbf{Q} + \boldsymbol{\mu} - \frac{m_1}{M_2}\boldsymbol{\lambda} \mid} \right). \tag{2.2}$$

Let $\mathbf{M} = \mathbb{R}^{15}$, and we consider the manifold $(\mathbf{M}, \{\ ,\ \}, \mathcal{H})$, with Poisson brackets $\{\ ,\ \}$ defined by means of the Poisson tensor

$$\mathbf{B}(\mathbf{z}) = \begin{pmatrix} \widehat{\mathbf{\Pi}} & \widehat{\boldsymbol{\lambda}} & \widehat{\mathbf{p}_{\boldsymbol{\lambda}}} & \widehat{\boldsymbol{\mu}} & \widehat{\mathbf{p}_{\mu}} \\ \widehat{\boldsymbol{\lambda}} & \mathbf{0} & \mathbf{I}_{\mathbb{R}^3} & \mathbf{0} & \mathbf{0} \\ \widehat{\mathbf{p}_{\boldsymbol{\lambda}}} & -\mathbf{I}_{\mathbb{R}^3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \widehat{\boldsymbol{\mu}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{\mathbb{R}^3} \\ \widehat{\mathbf{p}_{\mu}} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_{\mathbb{R}^3} & \mathbf{0} \end{pmatrix}. \tag{2.3}$$

In $\mathbf{B}(\mathbf{z})$, $\widehat{\mathbf{v}}$ is considered to be the image of the vector $\mathbf{v} \in \mathbb{R}^3$ by the standard isomorphism between the Lie Algebras $\mathbb{R}^3$ and $\mathfrak{so}(3)$, i.e.

$$\widehat{\mathbf{v}} = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}.$$

The equations of the motion is given by the following expression

$$\frac{d\mathbf{z}}{dt} = \{\mathbf{z}, \mathcal{H}(\mathbf{z})\}(\mathbf{z}) = \mathbf{B}(\mathbf{z})\boldsymbol{\nabla}_{\mathbf{z}}\mathcal{H}(\mathbf{z}) \tag{2.4}$$

where $\boldsymbol{\nabla}_{\mathbf{z}} f$ is the gradient of $f \in C^{\infty}(\mathbf{M})$ with respect to an arbitrary vector $\mathbf{z}$.

Developing $\{\mathbf{z}, \mathcal{H}(\mathbf{z})\}$, we obtain the following group of vectorial equations of the motion

$$\frac{d\mathbf{\Pi}}{dt} = \mathbf{\Pi}\times\mathbf{\Omega} + \boldsymbol{\lambda}\times\nabla_{\boldsymbol{\lambda}}\mathcal{V} + \boldsymbol{\mu}\times\nabla_{\boldsymbol{\mu}}\mathcal{V},$$

$$\frac{d\boldsymbol{\lambda}}{dt} = \frac{\mathbf{p}_{\boldsymbol{\lambda}}}{g_1} + \boldsymbol{\lambda}\times\mathbf{\Omega}, \qquad \frac{d\mathbf{p}_{\boldsymbol{\lambda}}}{dt} = \mathbf{p}_{\boldsymbol{\lambda}}\times\mathbf{\Omega} - \nabla_{\boldsymbol{\lambda}}\mathcal{V}, \tag{2.5}$$

$$\frac{d\boldsymbol{\mu}}{dt} = \frac{\mathbf{p}_{\boldsymbol{\mu}}}{g_2} + \boldsymbol{\mu}\times\mathbf{\Omega}, \qquad \frac{d\mathbf{p}_{\boldsymbol{\mu}}}{dt} = \mathbf{p}_{\boldsymbol{\mu}}\times\mathbf{\Omega} - \nabla_{\boldsymbol{\mu}}\mathcal{V}.$$

Important elements of $\mathbf{B}(\mathbf{z})$ are the associate Casimir functions. We consider the total angular momentum $\mathbf{L}$ given by

$$\mathbf{L} = \mathbf{\Pi} + \boldsymbol{\lambda}\times\mathbf{p}_{\boldsymbol{\lambda}} + \boldsymbol{\mu}\times\mathbf{p}_{\boldsymbol{\mu}}. \tag{2.6}$$

Then the following result is verified (see Vera and Vigueras [6] for details).

Proposition 1. *If φ is a real smooth function no constant, then $\varphi(\frac{|\mathbf{L}|^2}{2})$ is a Casimir function of the Poisson tensor $\mathbf{B}(\mathbf{z})$. Moreover $Ker\mathbf{B}(\mathbf{z}) =< \nabla_{\mathbf{z}}\varphi >$. Also, we have $\frac{d\mathbf{L}}{dt} = \mathbf{0}$, that is to say the total angular momentum vector remains constant.*

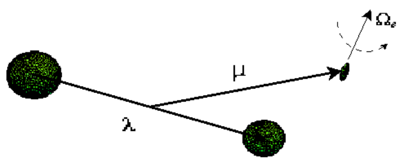

Figure 2.1: Gyrostat in the three body problem

2.1 Approximate Poisson dynamics

To simplify the problem we assume that the gyrostat S_0 is symmetrical around the third axis of inertia OZ and with respect to the plane OXY being OX, OY, OZ the coordinated axes of the body frame $\mathfrak{J}$. If the mutual distances are bigger than the individual dimensions of the bodies, then we can develop the potential in fast convergent series. Under these hypotheses, we will be able to carry out a study of equilibria in different approximate dynamics.

Applying the Legendre development of the potential, we have

$$\mathcal{V}(\boldsymbol{\lambda}, \boldsymbol{\mu}) = -\left(\frac{Gm_1m_2}{|\boldsymbol{\lambda}|} + \sum_{i=0}^{\infty}\frac{Gm_1A_{2i}}{|\boldsymbol{\mu}+\frac{m_2}{M_2}\boldsymbol{\lambda}|^{2i+1}} + \sum_{i=0}^{\infty}\frac{Gm_2A_{2i}}{|\boldsymbol{\mu}-\frac{m_1}{M_2}\boldsymbol{\lambda}|^{2i+1}}\right)$$

where $A_0 = m_0$, $A_2 = (C-A)/2$ and A_{2i} are certain coefficients related with the geometry of the gyrostat, see Vera and Vigueras [6] for details.

Definition 2. We call approximate potential of order k, to the following expression

$$\mathcal{V}_k(\boldsymbol{\lambda},\boldsymbol{\mu}) = -\left(\frac{Gm_1m_2}{\mid \boldsymbol{\lambda} \mid} + \sum_{i=0}^{k} \frac{Gm_1A_{2i}}{\mid \boldsymbol{\mu}+\frac{m_2}{M_2}\boldsymbol{\lambda} \mid^{2i+1}} + \sum_{i=0}^{k} \frac{Gm_2A_{2i}}{\mid \boldsymbol{\mu}-\frac{m_1}{M_2}\boldsymbol{\lambda} \mid^{2i+1}}\right).$$

It is easy to demonstrate the following lemmas.

Lemma 3. *Given the approximate potential of order k, we have*

$$\boldsymbol{\nabla}_{\boldsymbol{\lambda}}\mathcal{V}_k = \frac{Gm_1m_2\boldsymbol{\lambda}}{\mid \boldsymbol{\lambda} \mid^3} + \frac{Gm_1m_2}{M_2}\sum_{i=0}^{k}\frac{(\boldsymbol{\mu}+\frac{m_2}{M_2}\boldsymbol{\lambda})(2i+1)A_{2i}}{\mid \boldsymbol{\mu}+\frac{m_2}{M_2}\boldsymbol{\lambda} \mid^{2i+3}} - \frac{Gm_1m_2}{M_2}\sum_{i=0}^{k}\frac{(\boldsymbol{\mu}-\frac{m_1}{M_2}\boldsymbol{\lambda})(2i+1)A_{2i}}{\mid \boldsymbol{\mu}-\frac{m_1}{M_2}\boldsymbol{\lambda} \mid^{2i+3}},$$

$$\boldsymbol{\nabla}_{\boldsymbol{\mu}}\mathcal{V}_k = Gm_1\sum_{i=0}^{k}\frac{(\boldsymbol{\mu}+\frac{m_2}{M_2}\boldsymbol{\lambda})(2i+1)A_{2i}}{\mid \boldsymbol{\mu}+\frac{m_2}{M_2}\boldsymbol{\lambda} \mid^{2i+3}} + Gm_2\sum_{i=0}^{k}\frac{(\boldsymbol{\mu}-\frac{m_1}{M_2}\boldsymbol{\lambda})(2i+1)A_{2i}}{\mid \boldsymbol{\mu}-\frac{m_1}{M_2}\boldsymbol{\lambda} \mid^{2i+3}}. \tag{2.7}$$

The following identities are verified

$$\boldsymbol{\nabla}_{\boldsymbol{\lambda}}\mathcal{V}_k = \widetilde{A}_{11}\boldsymbol{\lambda} + \widetilde{A}_{12}\boldsymbol{\mu}, \qquad \boldsymbol{\nabla}_{\boldsymbol{\mu}}\mathcal{V}_k = \widetilde{A}_{21}\boldsymbol{\lambda} + \widetilde{A}_{22}\boldsymbol{\mu} \tag{2.8}$$

being

$$\begin{aligned}
\widetilde{A}_{11}(\boldsymbol{\lambda},\boldsymbol{\mu}) &= \frac{Gm_1m_2}{\mid \boldsymbol{\lambda} \mid^3} + \frac{Gm_1m_2^2}{M_2^2}\left(\sum_{i=0}^{k}\frac{\beta_i}{\mid \boldsymbol{\mu}+\frac{m_2}{M_2}\boldsymbol{\lambda} \mid^{2i+3}}\right) + \frac{Gm_1^2m_2}{M_2^2}\left(\sum_{i=0}^{k}\frac{\beta_i}{\mid \boldsymbol{\mu}-\frac{m_1}{M_2}\boldsymbol{\lambda} \mid^{2i+3}}\right),\\
\widetilde{A}_{12}(\boldsymbol{\lambda},\boldsymbol{\mu}) &= \frac{Gm_1m_2}{M_2}\left(\sum_{i=0}^{k}\frac{\beta_i}{\mid \boldsymbol{\mu}+\frac{m_2}{M_2}\boldsymbol{\lambda} \mid^{2i+3}} - \sum_{i=0}^{k}\frac{\beta_i}{\mid \boldsymbol{\mu}-\frac{m_1}{M_2}\boldsymbol{\lambda} \mid^{2i+3}}\right),\\
\widetilde{A}_{22}(\boldsymbol{\lambda},\boldsymbol{\mu}) &= Gm_1\left(\sum_{i=0}^{k}\frac{\beta_i}{\mid \boldsymbol{\mu}+\frac{m_2}{M_2}\boldsymbol{\lambda} \mid^{2i+3}}\right) + Gm_2\left(\sum_{i=0}^{k}\frac{\beta_i}{\mid \boldsymbol{\mu}-\frac{m_1}{M_2}\boldsymbol{\lambda} \mid^{2i+3}}\right),\\
\widetilde{A}_{21}(\boldsymbol{\lambda},\boldsymbol{\mu}) &= \widetilde{A}_{12}(\boldsymbol{\lambda},\boldsymbol{\mu})
\end{aligned} \tag{2.9}$$

with coefficients $\beta_0 = m_0$, $\beta_1 = 3/2(C-A)$, $\beta_i = (2i+1)A_{2i}$ for $i \geqslant 1$.

Definition 4. Let be $\mathbf{M} = \mathbb{R}^{15}$ and the manifold $(\mathbf{M}, \{\ ,\ \}, \mathcal{H}_k)$, with Poisson brackets $\{\ ,\ \}$ defined by means of the Poisson tensor (2.3). We call *approximate dynamics of order* k to the differential equations of motion given by the following expression

$$\frac{d\mathbf{z}}{dt} = \{\mathbf{z}, \mathcal{H}_k(\mathbf{z})\}(\mathbf{z}) = \mathbf{B}(\mathbf{z})\boldsymbol{\nabla}_{\mathbf{z}}\mathcal{H}_k(\mathbf{z})$$

being

$$\mathcal{H}_k(\mathbf{z}) = \frac{\mid \mathbf{p}_{\boldsymbol{\lambda}} \mid^2}{2g_1} + \frac{\mid \mathbf{p}_{\boldsymbol{\mu}} \mid^2}{2g_2} + \frac{1}{2}\Pi\mathbb{I}^{-1}\Pi - \mathbf{l}_r \cdot \mathbb{I}^{-1}\Pi + \mathcal{V}_k(\boldsymbol{\lambda},\boldsymbol{\mu}).$$

2.1.1 Integrals of the system

On the other hand, it is easy to verify that

$$\nabla_{\mathbf{z}}(\mid \mathbf{\Pi} \mid^2))\mathbf{B}(\mathbf{z})\nabla_{\mathbf{z}}\mathcal{H}_0(\mathbf{z}) = 0$$

and similarly when the gyrostat is of revolution

$$\nabla_{\mathbf{z}}(\boldsymbol{\pi}_3)\mathbf{B}(\mathbf{z})\nabla_{\mathbf{z}}\mathcal{H}_k(\mathbf{z}) = 0$$

where $\boldsymbol{\pi}_3$ is the third component of the rotational angular momentum of the gyrostat. It is verified the following result.

Theorem 5. *In the approximate dynamics of order* 0, $\mid \mathbf{\Pi} \mid^2$ *is an integral of motion and also when the gyrostat is of revolution* $\boldsymbol{\pi}_3$ *is another integral of motion.*

2.2 Relative Equilibria

The relative equilibria are the equilibria of the twice reduced problem whose Hamiltonian function is obtained in Vera and Vigueras [6] for the case $n = 2$. If we denote by $\mathbf{z}_e = (\mathbf{\Pi}_e, \boldsymbol{\lambda}^e, \mathbf{p}^e_{\boldsymbol{\lambda}}, \boldsymbol{\mu}^e, \mathbf{p}^e_{\mu})$ a generic relative equilibrium of an approximate dynamics of order k, then this verifies the equations

$$\mathbf{\Pi}_e \times \mathbf{\Omega}_e \; + \; \boldsymbol{\lambda}^e \times (\boldsymbol{\nabla}_{\boldsymbol{\lambda}} \mathcal{V}_k)_e \; + \; \boldsymbol{\mu}^e \times (\boldsymbol{\nabla}_{\boldsymbol{\mu}} \mathcal{V}_k)_e = \mathbf{0},$$

$$\frac{\mathbf{p}^e_{\boldsymbol{\lambda}}}{g_1} + \boldsymbol{\lambda}^e \times \mathbf{\Omega}_e = \mathbf{0}, \quad \mathbf{p}^e_{\boldsymbol{\lambda}} \times \mathbf{\Omega}_e = (\boldsymbol{\nabla}_{\boldsymbol{\lambda}} \mathcal{V}_k)_e, \tag{2.10}$$

$$\frac{\mathbf{p}^e_{\boldsymbol{\mu}}}{g_2} + \boldsymbol{\mu}^e \times \mathbf{\Omega}_e = \mathbf{0}, \quad \mathbf{p}^e_{\boldsymbol{\mu}} \times \mathbf{\Omega}_e = (\boldsymbol{\nabla}_{\boldsymbol{\mu}} \mathcal{V}_k)_e.$$

Also by virtue of the relationships obtained in Vera and Vigueras [6], we have the following result.

Lemma 6. *If* $\mathbf{z}_e = (\mathbf{\Pi}_e, \boldsymbol{\lambda}^e, \mathbf{p}^e_{\boldsymbol{\lambda}}, \boldsymbol{\mu}^e, \mathbf{p}^e_{\mu})$ *is a relative equilibrium of an approximate dynamics of order* k *the following relationships are verified*

$$\mid \mathbf{\Omega}_e \mid^2 \mid \boldsymbol{\lambda}^e \mid^2 - (\boldsymbol{\lambda}^e \cdot \mathbf{\Omega}_e)^2 = \frac{1}{g_1}(\boldsymbol{\lambda}^e \cdot (\boldsymbol{\nabla}_{\boldsymbol{\lambda}} \mathcal{V}_k)_e)$$

$$\mid \mathbf{\Omega}_e \mid^2 \mid \boldsymbol{\mu}^e \mid^2 - (\boldsymbol{\mu}^e \cdot \mathbf{\Omega}_e)^2 = \frac{1}{g_2}(\boldsymbol{\mu}^e \cdot (\boldsymbol{\nabla}_{\boldsymbol{\mu}} \mathcal{V}_k)_e)$$

The last two previous identities will be used to obtain necessary conditions for the existence of relative equilibria in this approximate dynamics.

We will study certain relative equilibria in the approximate dynamics supposing that the vectors $\mathbf{\Omega}_e$, $\boldsymbol{\lambda}^e$, $\boldsymbol{\mu}^e$ satisfy special geometric properties.

Definition 7. We say that $\mathbf{z}_e$ is a relative equilibrium of *planar rotation type*, in an approximate dynamics of order k, when $\boldsymbol{\Omega}_e$ is in the plane generated by $\boldsymbol{\lambda}^e$ and $\boldsymbol{\mu}^e$.

Next we obtain necessary and sufficient conditions for the existence of planar rotation type equilibria.

3 Relative equilibria of *planar rotation* type

In this section we study relative equilibria of planar rotation type. We obtain necessary and sufficient conditions for the existence of this type of solutions in different approximate dynamics.

3.1 Necessary condition of existence

Let us suppose that $\boldsymbol{\Omega}_e = a\boldsymbol{\lambda}^e + b\boldsymbol{\mu}^e$ being $a, b \in \mathbb{R}$ real constants to be determined. Then the equilibria $\mathbf{z}_e$ verify the following equations

$$\begin{aligned} |\boldsymbol{\Omega}_e|^2\,\boldsymbol{\lambda}^e - (\boldsymbol{\lambda}^e \cdot \boldsymbol{\Omega}_e)\boldsymbol{\Omega}_e &= \frac{1}{g_1}(\nabla_{\boldsymbol{\lambda}}\mathcal{V}^{(k)})_e \\ |\boldsymbol{\Omega}_e|^2\,|\boldsymbol{\mu}^e|^2 - (\boldsymbol{\mu}^e \cdot \boldsymbol{\Omega}_e)\boldsymbol{\Omega}_e &= \frac{1}{g_2}(\nabla_{\boldsymbol{\mu}}\mathcal{V}^{(k)})_e \end{aligned} \tag{3.1}$$

If we denote by

$$\begin{aligned} (\boldsymbol{\lambda}^e \cdot \boldsymbol{\Omega}_e) &= \widetilde{X} = a \mid \boldsymbol{\lambda}^e \mid^2 + b(\boldsymbol{\lambda}^e \cdot \boldsymbol{\mu}^e) \\ (\boldsymbol{\mu}^e \cdot \boldsymbol{\Omega}_e) &= \widetilde{Y} = b \mid \boldsymbol{\mu}^e \mid^2 + a(\boldsymbol{\lambda}^e \cdot \boldsymbol{\mu}^e) \end{aligned} \tag{3.2}$$

then from the equations (3.1) we deduce

$$\begin{aligned} g_1 b(\widetilde{Y}\boldsymbol{\lambda}^e - \widetilde{X}\boldsymbol{\mu}^e) &= (\nabla_{\boldsymbol{\lambda}}\mathcal{V}^{(k)})_e \\ g_2 a(\widetilde{X}\boldsymbol{\mu}^e - \widetilde{Y}\boldsymbol{\lambda}^e) &= (\nabla_{\boldsymbol{\mu}}\mathcal{V}^{(k)})_e. \end{aligned} \tag{3.3}$$

On the other hand we have

$$\begin{aligned} (\nabla_{\boldsymbol{\lambda}}\mathcal{V}^{(k)})_e &= (\widetilde{A}_{11})_e\boldsymbol{\lambda}^e + (\widetilde{A}_{12})_e\boldsymbol{\mu}^e \\ (\nabla_{\boldsymbol{\mu}}\mathcal{V}^{(k)})_e &= (\widetilde{A}_{12})_e\boldsymbol{\lambda}^e + (\widetilde{A}_{22})_e\boldsymbol{\mu}^e \end{aligned} \tag{3.4}$$

then

$$\begin{aligned} -g_1 b\widetilde{X} = (\widetilde{A}_{12})_e, &\quad g_1 b\widetilde{Y} = (\widetilde{A}_{11})_e \\ g_2 a\widetilde{X} = (\widetilde{A}_{22})_e, &\quad -g_2 a\widetilde{Y} = (\widetilde{A}_{12})_e. \end{aligned} \tag{3.5}$$

And if we eliminate the variables $\widetilde{X}$ and $\widetilde{Y}$ in the previous equations we obtain

$$\begin{aligned} bg_1(\widetilde{A}_{22})_e + ag_2(\widetilde{A}_{12})_e &= 0 \\ bg_1(\widetilde{A}_{12})_e + ag_2(\widetilde{A}_{11})_e &= 0 \end{aligned} \tag{3.6}$$

which are equivalent to the following ones

$$(\widetilde{A}_{11})_e(\widetilde{A}_{22})_e-(\widetilde{A}_{12})_e^2=0$$
$$a=-\frac{g_1}{g_2}\frac{(\widetilde{A}_{22})_e}{(\widetilde{A}_{12})_e}b. \tag{3.7}$$

In this case the angular velocity comes given by the expression

$$\mathbf{\Omega}_e=a(\boldsymbol{\lambda}^e-\frac{g_2(\widetilde{A}_{12})_e}{g_1(\widetilde{A}_{22})_e}\boldsymbol{\mu}^e)$$
$$\mid\mathbf{\Omega}_e\mid^2=\frac{(\widetilde{A}_{22})_e}{g_2}+\frac{(\widetilde{A}_{11})_e}{g_1}. \tag{3.8}$$

We summarize all these results in the following proposition.

Proposition 1. *Let* $\mathbf{z}_e=(\mathbf{\Pi}_e,\boldsymbol{\lambda}^e,\mathbf{p}_\lambda^e,\boldsymbol{\mu}^e,\mathbf{p}_\mu^e)$ *be a relative equilibrium verifying* $\mathbf{\Omega}_e=a\boldsymbol{\lambda}^e+b\boldsymbol{\mu}^e$ *being* $a,b\in\mathbb{R}$*, then the following relations are verified*

$$(\widetilde{A}_{11})_e(\widetilde{A}_{22})_e-(\widetilde{A}_{12})_e^2=0,\qquad a=-\frac{g_1}{g_2}\frac{(\widetilde{A}_{22})_e}{(\widetilde{A}_{12})_e}b$$
$$\mathbf{\Omega}_e=a(\boldsymbol{\lambda}^e-\frac{g_2(\widetilde{A}_{12})_e}{g_1(\widetilde{A}_{22})_e}\boldsymbol{\mu}^e),\quad\mid\mathbf{\Omega}_e\mid^2=\frac{(\widetilde{A}_{22})_e}{g_2}+\frac{(\widetilde{A}_{11})_e}{g_1} \tag{3.9}$$

where $\widetilde{A}_{ij}$ *come given by the expressions (2.9) and* $(\widetilde{A}_{ij})_e$ *denotes the evaluation of this function in the relative equilibrium.*

3.2 Necessary condition of existence for order zero and one

To this respect we have obtained the following result.

In an approximate dynamics of order zero such equilibria don't exist since $\widetilde{A}_{11}\widetilde{A}_{22}-\widetilde{A}_{12}{}^2\neq 0$. In an approximate dynamics of order one, calling $\mid\boldsymbol{\lambda}^e\mid=Z$, $\mid\boldsymbol{\mu}^e+\frac{m_2}{M_2}\boldsymbol{\lambda}^e\mid=Y$, $\mid\boldsymbol{\mu}^e-\frac{m_1}{M_2}\boldsymbol{\lambda}^e\mid=X$, $Y_1=\sum_{i=0}^{1}\frac{\beta_i}{Y^{2i+3}}$, $X_1=\sum_{i=0}^{1}\frac{\beta_i}{X^{2i+3}}$, so that $(\widetilde{A}_{11})_e(\widetilde{A}_{22})_e-(\widetilde{A}_{12})_e^2=0$ it should necessarily happen

$$Z^3=-(\frac{m_1}{X_1}+\frac{m_2}{Y_1}). \tag{3.10}$$

Therefore for $\beta_1\geq 0$ such relative equilibrium solutions don't exist.

Let us see that it happens for $\beta_1<0$. Carrying out the appropriate calculations we obtain

$$Z^3=-\frac{m_1X^5(Y^2+\beta_1)+m_2Y^5(X^2+\beta_1)}{(Y^2+\beta_1)(X^2+\beta_1)}. \tag{3.11}$$

$$\mid\mathbf{\Omega}_e\mid^2=\frac{a_2\beta_1^2+2a_1\beta_1+a_0}{X^5Y^5[m_1X^5(Y^2+\beta_1)+m_2Y^5(X^2+\beta_1)]} \tag{3.12}$$

where the coefficients a_i come given by the following expressions

$$a_2 = (m_1 X^5 + m_2 Y^5)^2 + m_0(m_1 X^{10} + m_2 Y^{10})$$

$$a_1 = X^2 Y^2 [m_1(m_1 + m_0)X^8 + m_2(m_2 + m_0)Y^8 + m_1 m_2 (X^3 Y^5 + X^5 Y^3) \tag{3.13}$$

$$a_0 = (m_1 X^3 + m_2 Y^3)^2 + m_0(m_1 X^6 + m_2 Y^6).$$

The discriminant of the polynomial $a_2\beta_1^2 + 2a_1\beta_1 + a_0$ has the following expression

$$\Delta = -m_1 m_2 m_0 (m_1 + m_2 + m_0)(X^2 - Y^2)^2 \tag{3.14}$$

and it is negative for any value of X, Y, m_i therefore equilibria cannot exist with $\beta_1 < 0$ and $(Y^2 + \beta_1)(X^2 + \beta_1) \neq 0$.

Let us suppose now that $(Y^2 + \beta_1) = 0$, then we can deduce that

$$(X^2 + \beta_1) = 0, \quad \frac{a}{b} = -\frac{g_1}{g_2}\frac{(\widetilde{A}_{22})_e}{(\widetilde{A}_{12})_e} = \frac{m_0}{M_1}. \tag{3.15}$$

One also has

$$| \boldsymbol{\Omega}_e |^2 = G\frac{m_1 + m_2}{Z^3}$$

$$\boldsymbol{\Omega}_e = b(\frac{m_0}{M_1}\boldsymbol{\lambda}^e + \boldsymbol{\mu}^e) = a(\boldsymbol{\lambda}^e + \frac{M_1}{m_0}\boldsymbol{\mu}^e) \tag{3.16}$$

with $a, b \in \mathbb{R}$ calculated applying that $| \boldsymbol{\Omega}_e |^2 = G\frac{m_1 + m_2}{Z^3}$.

An interesting particular case is presented when the vector $\boldsymbol{\Omega}_e$ is perpendicular to $\boldsymbol{\lambda}^e$ and proportional to $\boldsymbol{\mu}^e$. If we impose these conditions, then $m_1 = m_2 = m$.

A simple calculation shows that

$$\boldsymbol{\Omega}_e = a\frac{M_1}{m_0}\boldsymbol{\mu}^e, \quad | \boldsymbol{\mu}^e | = \sqrt{r^2 - \frac{Z^2}{4}}, \quad a^2 = \frac{2Gmm_0^2}{M_1 Z^3 (r^2 - \frac{Z^2}{4})} \tag{3.17}$$

We summarize in the following proposition.

Proposition 2. *For the approximate dynamics of order zero relative equilibria of planar rotation type don't exist. For the approximate dynamics of order one, if* $\beta_1 \geqslant 0$ *(oblate gyrostat) equally relative equilibria of planar rotation type don't exist. If* $\beta_1 < 0$ *(prolate gyrostat) and we denote by* $| \boldsymbol{\lambda}^e | = Z$, $| \boldsymbol{\mu}^e + \frac{m_2}{M_2}\boldsymbol{\lambda}^e | = Y$, $| \boldsymbol{\mu}^e - \frac{m_1}{M_2}\boldsymbol{\lambda}^e | = X$, *when* $(Y^2 + \beta_1)(X^2 + \beta_1) \neq 0$ *such relative equilibria don't exist . If* $\beta_1 < 0$ *and* $(Y^2 + \beta_1) = 0$, *then* $(X^2 + \beta_1) = 0$ *and one has that*

$$| \boldsymbol{\Omega}_e |^2 = G\frac{m_1 + m_2}{Z^3}$$

$$\boldsymbol{\Omega}_e = b(\frac{m_0}{M_1}\boldsymbol{\lambda}^e + \boldsymbol{\mu}^e) = a(\boldsymbol{\lambda}^e + \frac{M_1}{m_0}\boldsymbol{\mu}^e) \tag{3.18}$$

with $a, b \in \mathbb{R}$ calculated applying that $\mid \mathbf{\Omega}_e \mid^2 = G\dfrac{m_1 + m_2}{Z^3}$*. In particular, if the masses $m_1 = m_2 = m$, the vector $\mathbf{\Omega}_e$ is perpendicular to $\boldsymbol{\lambda}^e$ and proportional to $\boldsymbol{\mu}^e$, the following relations are verified*

$$\mathbf{\Omega}_e = a\frac{M_1}{m_0}\boldsymbol{\mu}^e, \quad \mid \boldsymbol{\mu}^e \mid = \sqrt{r^2 - \frac{Z^2}{4}}, \quad a^2 = \frac{2Gmm_0^2}{M_1 Z^3 (r^2 - \frac{Z^2}{4})} \tag{3.19}$$

with $r = \sqrt{-\beta_1}$.

3.3 Sufficient condition of existence

On the other hand, it is possible build explicitly relative equilibria of planar rotation type verifying the previously mentioned properties.

We suppose that the centers of mass of the gyrostat S_0 and the bodies S_i form an isosceles triangle whose same sides measure $Y = \sqrt{-\beta_1}$, with base given by the magnitude Z, that indicates us the separation distance among S_1 and S_2.

Denoting by $\theta = \widehat{S_1 S_0 S_2}$, we have

$$\cos\theta = \frac{-2\beta_1 - Z^2}{2\beta_1^2} \quad , \quad \sin\theta = \frac{Z\sqrt{-4\beta_1 - Z^2}}{2\beta_1^2} \tag{3.20}$$

and

$$\begin{gathered} \boldsymbol{\lambda}^e = (\sqrt{-\beta_1}(1 - \cos\theta), -\sqrt{-\beta_1}\sin\theta, 0) \\ \boldsymbol{\mu}^e = (\frac{\sqrt{-\beta_1}(m_1 + m_2\cos\theta)}{M_2}, \frac{m_2\sqrt{-\beta_1}\sin\theta}{M_2}, 0). \end{gathered} \tag{3.21}$$

On the other hand

$$\mathbf{\Omega}_e = a(\boldsymbol{\lambda}^e + \frac{M_1}{m_0}\boldsymbol{\mu}^e), \quad \mid \mathbf{\Omega}_e \mid^2 = G\frac{m_1 + m_2}{Z^3} \tag{3.22}$$

being

$$a^2 = \frac{GM_2}{Z^3 \mid \boldsymbol{\lambda}^e + \dfrac{M_1}{m_0}\boldsymbol{\mu}^e \mid^2} \tag{3.23}$$

And the vectors $\mathbf{p}_\lambda^e$ and $\mathbf{p}_\mu^e$ come given by the well-known relations

$$\mathbf{p}_\lambda^e = g_1(\mathbf{\Omega}_e \wedge \boldsymbol{\lambda}^e), \quad \mathbf{p}_\mu^e = g_2(\mathbf{\Omega}_e \wedge \boldsymbol{\mu}^e) \tag{3.24}$$

If $m_1 = m_2 = m$ and $\mathbf{\Omega}_e$ perpendicular to $\boldsymbol{\lambda}^e$ and proportional to $\boldsymbol{\mu}^e$. Then we have

$$\begin{aligned}
&\boldsymbol{\lambda}^e = (\sqrt{-\beta_1}(1-\cos\theta), -\sqrt{-\beta_1}\sin\theta, 0) \\
&\boldsymbol{\mu}^e = (\frac{\sqrt{-\beta_1}(1+\cos\theta)}{2}, \frac{\sqrt{-\beta_1}\sin\theta}{2}, 0) \\
&a^2 = \frac{2Gmm_0^2}{M_1 Z^3 (r^2 - \dfrac{Z^2}{4})}, \quad \mathbf{\Omega}_e = a\frac{M_1}{m_0}\boldsymbol{\mu}^e \\
&\mathbf{p}_\lambda^e = g_1(\mathbf{\Omega}_e \wedge \boldsymbol{\lambda}^e), \quad \mathbf{p}_\mu^e = g_2(\mathbf{\Omega}_e \wedge \boldsymbol{\mu}^e)
\end{aligned} \tag{3.25}$$

We summarize all these results in the following proposition.

Proposition 3. *Some results for different approximate dynamics:*

- *Order zero: relative equilibria of planar rotation type don't exist.*
- *Order one:*
  - *If $\beta_1 \geqslant 0$ (oblate gyrostat) equally relative equilibria of planar rotation type don't exist.*
  - *If $\beta_1 < 0$ (prolate gyrostat), with additional hypotheses, it is possible to find this type of equilibria.*

4 Conclusions

- The approximate dynamics of a gyrostat (or rigid body) in Newtonian interaction with two spherical or punctual rigid bodies is considered.
- For the approximate dynamics of order zero and one, we obtain necessary and sufficient conditions for the existence of planar rotations.
- We give explicit expressions of some of these relative equilibria, useful for the later study of their stability.
- Numerous problems are open, and among them it is necessary to consider the study of stability of the planar rotations for order one.

They also deserve to be considered as object of a later study the "inclined" relative equilibria, that is to say study of relative equilibria in approximate dynamics when $\mathbf{\Omega}_e$ form an angle $\alpha \neq 0$ and $\pi/2$ with the vector $\boldsymbol{\lambda}^e \times \boldsymbol{\mu}^e$.

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