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OPTIMAL DESIGN OF SHAPE AND REINFORCEMENT FOR CONCRETE SECTIONS

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Abstract. *This document presents a procedure for the optimal design of reinforced concrete sections of general shapes subject to a biaxial bending. The optimal design problem is formulated as a non-linear mathematical programming problem. The problem is formulated so that time-consuming searches for the precise location of the neutral axis are avoided through intermediate steps of the optimization process.*

There are three kinds of design variables: geometry variables, reinforcement variables and location of the neutral axis variables.

The objective function is the cost of a structural member per unit length. There are three kinds of constraints: strength constraints, minimal amount of steel constraints and bound constraints.

1 INTRODUCTION

The problem of ultimate strength analysis of reinforced concrete sections under biaxial bending appears in structural design frequently. Usually, the cross section has a simple rectangular geometry, but the shape is often more complex.

In common practice, the biaxial capacity of a concrete section is interpolated from its uniaxial capacities[1,2]. More specifically, the capacity against the axial force and bending moment acting simultaneously about the *x*-*x* and *y*-*y* axes is obtained by idealizing the M_x-M_y interaction curve.

However, there are several limitations on applying this method, which was developed originally for rectangular sections with symmetrical arrangement of reinforcement, in order to design irregular sections.

In this paper, to calculate the ultimate strength, the section is divided into fixed finite elements, and for approximate integration, the coefficients in equilibrium equations[3] are computed.

A procedure for the optimal design of shape and reinforcement arrangement for concrete sections of general shapes subject to a biaxial bending is presented and several examples have been tested.

The problem is formulated so that time-consuming searches for the precise location of the neutral axes are avoided through intermediate steps of the optimization process[4].

The optimization problem is formulated as a non-linear programming problem.

This work has been developed according to with the EH-91[5] Spanish design code.

2 ULTIMATE STRENGTH DETERMINATION OF REINFORCED CONCRETE SECCTIONS UNDER BIAXIAL BENDING

Consider the section shown in fig. 1.a. To calculate the ultimate strength of reinforced concrete sections under biaxial bending is necessary to know the precise location of the neutral axis, from the equilibrium and compatibility equations and stress-strain relationships of concrete and steel in compression and tension. These equations can't be expressed in analytic way where the variables are the parameters that fix the location of the neutral axis, because of the problem has not an analytic exact solution, so it's necessary to use approximate methods which are based on trial of several locations of the neutral axis.

The equilibrium equations for a reinforced concrete section of a given general shape subject to a biaxial bending:

$$\iint_s \sigma_c(\varepsilon_c)\, ds + \sum_{j=1}^{n} \sigma_s(\varepsilon_s)\, A_j = N \tag{1}$$

$$\iint_s \sigma_c(\varepsilon_c)\, y ds + \sum_{j=1}^{n} \sigma_s(\varepsilon_s)\, y_j\, A_j = N e_y = M_x$$

$$\iint_s \sigma_c(\varepsilon_c)\, x\, ds + \sum_{j=1}^{n} \sigma_s(\varepsilon_s)\, x_j\, A_j = N e_x = M_y$$

where

σ_c	stress at concrete;
σ_s	stress at steel;
ε_c	strain at concrete;
ε_s	strain at reinforcement;
N	axial load;
e_x, e_y	eccentricity about the *y*-*y* and *x*-*x* axis;
M_x, M_y	bending moment about the *x*-*x and y*-*y* axis;
A_j	area of *j*-th reinforcing bar;
x_j, y_j	coordinates of *j*-th reinforcing bar;
ds	area of an element of concrete, and
n	number of reinforcing bars.

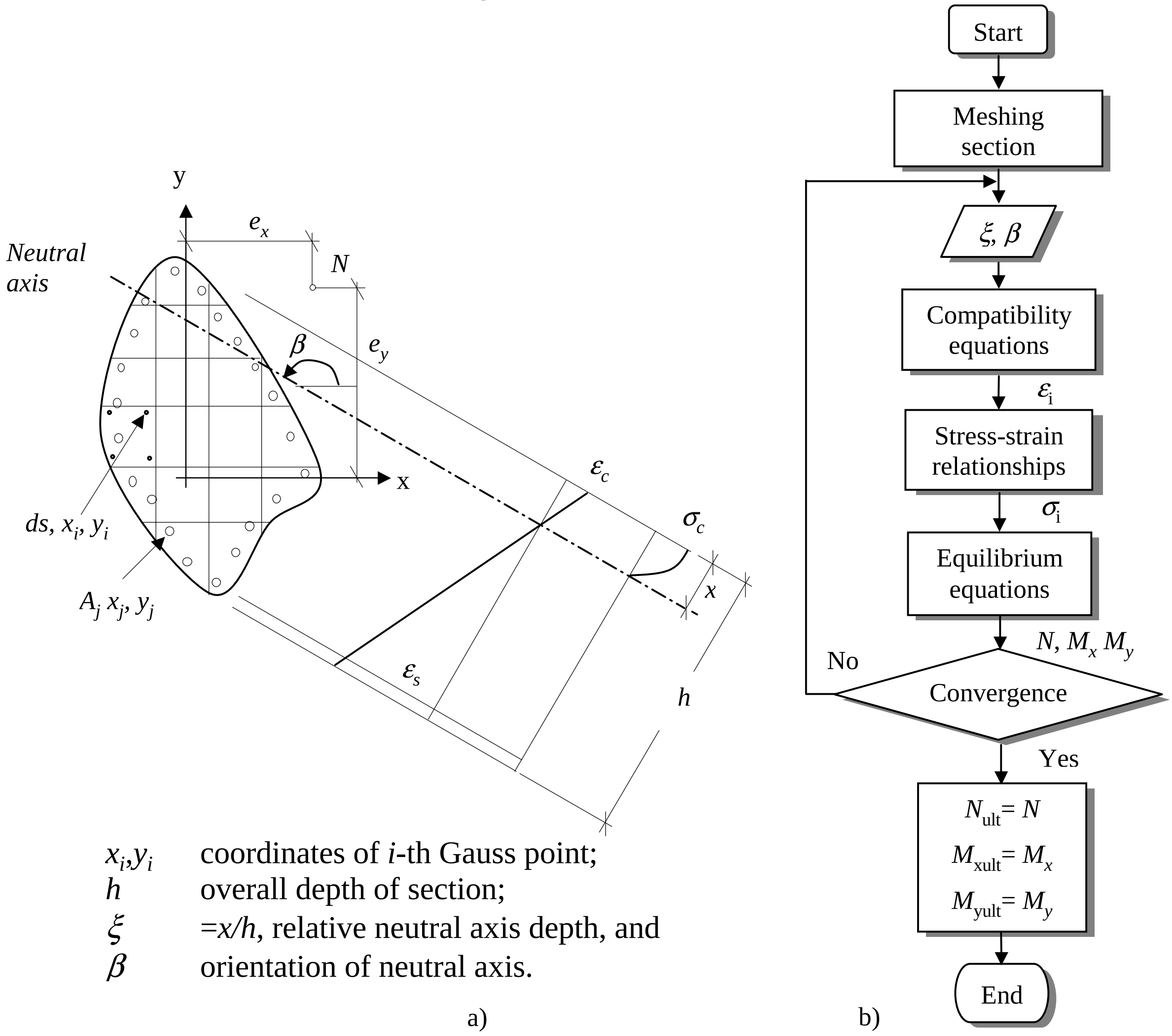

Figure 1: a) Reinforced concrete section. b) General flow chart to compute ultimate strength

In this work, to compute the ultimate strength of reinforced concrete sections, the section is divided in finite elements; several location of the neutral axis are tested and solved with an approximate integration of the equilibrium equations until the convergence of the problem. Figure 1.b shows the flow chart of the developed computer program for the ultimate strength analysis of reinforced concrete sections,

where

ε_i	strain at concrete or steel *i*-th element;
σ_i	stress at concrete or steel *i*-th element;
N_{ult}	ultimate axial load;
M_{xult}	ultimate bending moment about the *x-x, and*
M_{yult}	ultimate bending moment about the *y-y* axis.

The EH-91 design code specifies that, the ultimate strain in a section, according to loading conditions, the strain domains shown in fig. 2, the appropriate compatibility equations are (eq. 2 to 8).

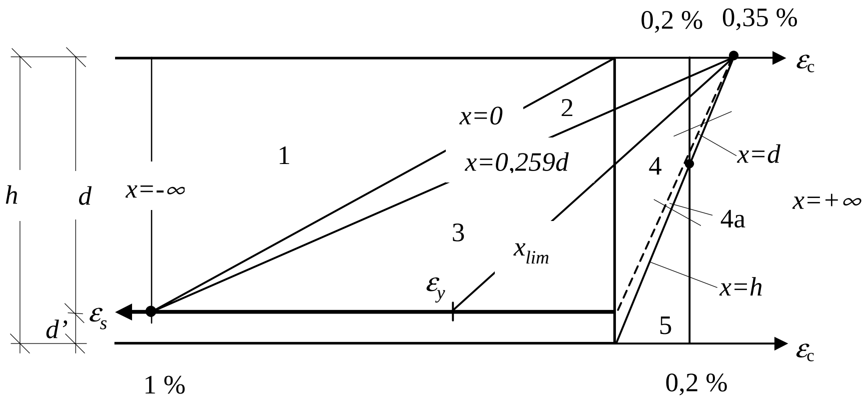

Figure 2: Strain domains

where

f_y	yield stress of steel;
E_s	modulus of elasticity of steel;
ε_y	$=f_y/E_s$;
d	effective depth;
d'	*h-d*;
x	neutral axis depth, and
x_{lim}	limit neutral axis depth in more tensioned reinforcement at yield stress.

$$\text{Domain}\,1\,(-\infty \le x \le 0)\begin{cases}\varepsilon_c = -0{,}01 \;\; (0 \text{ si } \; x=0)\\ \varepsilon_s = -0{,}01\end{cases} \tag{2}$$

$$Domain\ 2\ (0 \le x \le 0{,}259\ d\ ) \begin{cases} \varepsilon_c = -0{,}01 \dfrac{x}{d-x} \\ \varepsilon_s = -0{,}01 \end{cases} \tag{3}$$

$$Domain\ 3\ (0{,}259\ d \le x \le x_{lim}) \begin{cases} \varepsilon_c = 0{,}0035 \\ \varepsilon_s = 0{,}0035 \dfrac{x-d}{x} \end{cases} \tag{4}$$

$$Domain\ 4\ (\ x_{lim} \le x \le d) \begin{cases} \varepsilon_c = 0{,}0035 \\ \varepsilon_s = 0{,}0035 \dfrac{x-d}{x} \end{cases} \tag{5}$$

$$Domain\ 4a\ (\ d \le x \le h) \begin{cases} \varepsilon_c = 0{,}0035 \\ \varepsilon_s = 0{,}0035 \dfrac{x-d}{x} \end{cases} \tag{6}$$

$$Domain\ 5\ (\ h \le x \le +\infty\ ) \begin{cases} \varepsilon_r = 0{,}0035\ d' \\ \varepsilon_c = 0{,}0050 - 0{,}0015x \\ \varepsilon_s = \varepsilon_r + (0{,}0020 - \varepsilon_r)(x-1) \end{cases} \tag{7}$$

The concrete stress-strain relationships are shown in fig. 3.a (eq. 8), see fig 3.b. for steel.

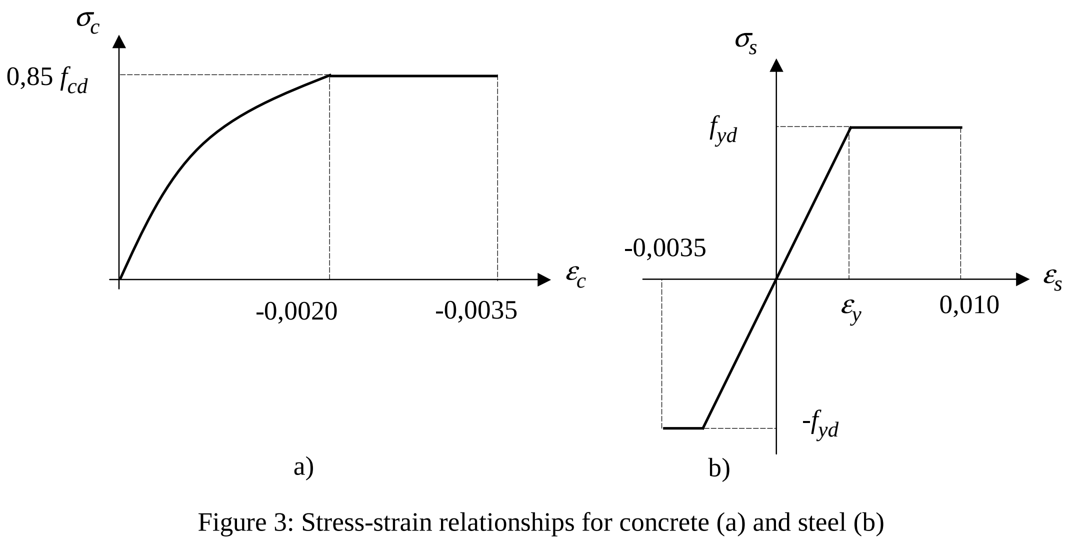

Figure 3: Stress-strain relationships for concrete (a) and steel (b)

$$\sigma_c = 0{,}85 f_{cd} \left[1 - \left(1 + \frac{\varepsilon_c}{0{,}0020} \right)^2 \right] \tag{8}$$

where

f_{cd}	cylinder strength of concrete, and
f_{yd}	calculus strength of steel.

3 OPTIMAL DESIGN PROBLEM

The most usual algebraic formulation for the general optimal design of structures and structural elements is:

To find a design variables vector $\boldsymbol{x}$ $(x_1,x_2,...,x_n)$ to:

Minimize the objective function $f(\boldsymbol{x})$

Satisfying the constraints:

$$h_j(\boldsymbol{x}) = 0 \qquad j = 1,2,...,m_i$$
$$g_j(\boldsymbol{x}) \geq 0 \qquad j = 1,2,...,m_a$$
$$x_i^I \leq x_i \leq x_i^S \qquad i = 1,2,...,n_b$$

where

$\boldsymbol{x}$	design variables n-dimensional vector;
$f(\boldsymbol{x})$	objective function;
$h_j(\boldsymbol{x})$	number j of equality constraints;
$g_j(\boldsymbol{x})$	number j of the inequality constraints;
x_i^I	lower limit of variable number i;
x_i^S	upper limit of variable number i;
m_j	number of equality constraints;
m_a	number of inequality constraints, and
n_b	number of bound constraints.

Usually, the objective function $f(\boldsymbol{x})$ and the equality $h_j(\boldsymbol{x})$ and inequality $g_j(\boldsymbol{x})$ constraints are non-linear functions. Then the problem is said to be *non-linear* optimization.

The optimization algorithm is a Sequential Quadratic Programming (SQP) method. In this method, a Quadratic Programming (QP) subproblem is solved at each iteration. An estimate of the Hessian of the Lagrangian is updated at each iteration using the BFGS formula.

4 OPTIMAL DESIGN PROBLEM FORMULATION

The optimal design problem has been formulated as a non-linear mathematical programming problem. A code has been written in MATLAB[6].

Figure 4 shows the flow chart for the optimization process.

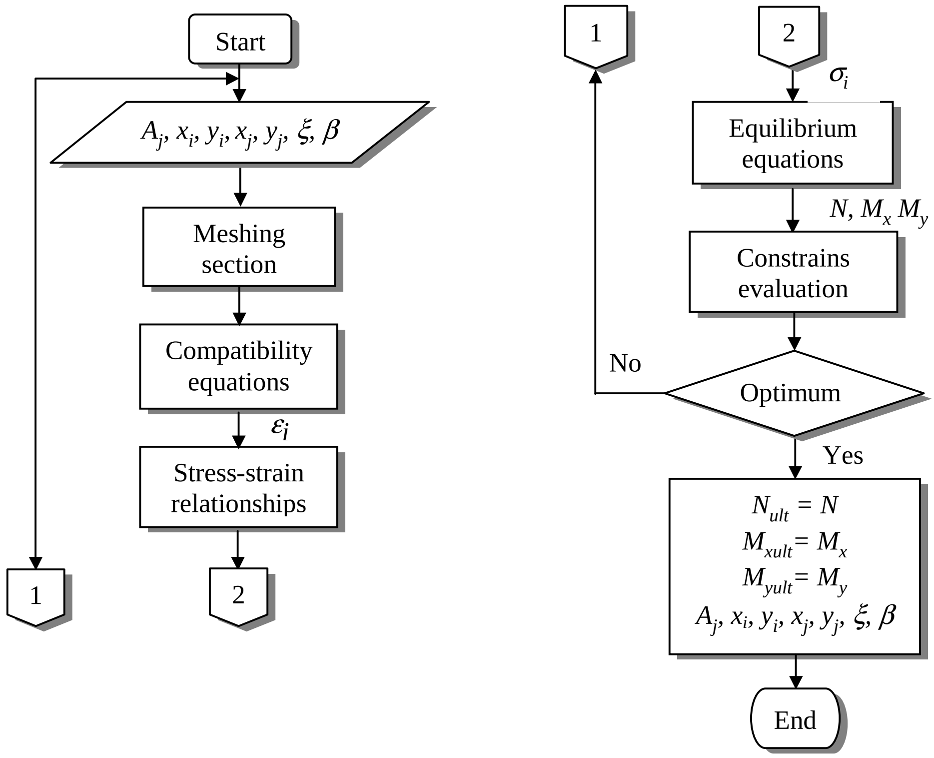

Figure 4: Flow chart for the optimization process

4.1 Design variables

There are three kinds of design variables: geometry variables, reinforcement variables and location of the neutral axis variables.

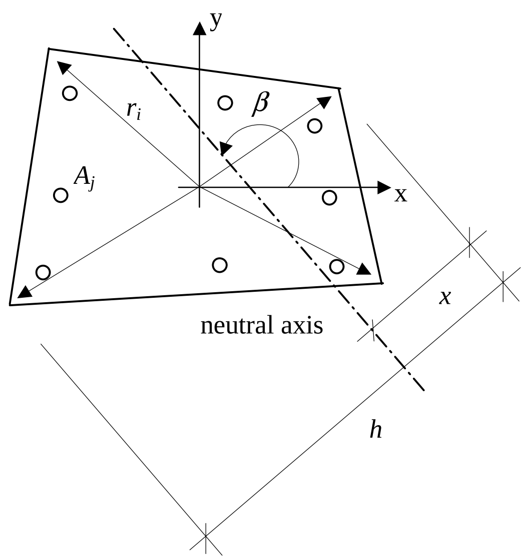

Figure 5: Design variables

The geometry variables used are the overall depth (*h*) and the width (*b*)of the section or also the modules of the vectors (r_i) which have their origins in a fixed point and their extremes in movable points, that define the location of every section vertexes (fig. 5). The direction of each vector is fixed in the process of optimization.

Reinforcement variables correspond to the areas of steel allocated in the section.

The variables of location of the neutral axis are ξ y β, above definited.

4.2 Objective Function

The objective function is the cost of structural member per unit length, which is the sum of the cost of concrete, reinforcing steel and formwork.

$$F = A_c\, C_c + S_p\, C_f + \rho_s C_s \sum_{j=1}^{n} A_j \qquad (9)$$

where

A_c	section area;
S_p	section perimeter;
C_c	cost of concrete (u.c./volume unit);
C_f	cost of formwork (u.c./area unit);
C_s	cost of reinforcing (u.c./weight unit), and
ρ_s	steel density.

4.3 Constraints

There are three kinds of constraints: strength constraints, minimal amount of steel constraints and bound constraints.

The strength constraints are:

$$\begin{gathered} N_{ult} \geq N \\ \left|e_{2ult}\right| \geq \left|e_2\right| \\ \frac{M_{2ult}}{M_{1ult}} = \alpha \end{gathered} \qquad (10)$$

where

$\left|M_1\right| = \mathrm{Max}\left\{\left|M_x\right|, \left|M_y\right|\right\}$. 1-axis corresponds to the largest acting bending moment, while 2-axis is the other one;

M_{1ult} maximum ultimate bending;

M_{2ult} minimal ultimate bending;

$$e_{2ult} = \frac{M_{1ult}}{N_{ult}};$$

$$e_2 = \frac{M_1}{N}; \text{ and}$$

$$\alpha = \frac{M_2}{M_1}.$$

The minimal amount of steel in tension, given for the EH-91 design code is:

$$A_s \geq 0{,}25 \frac{f_{cd}}{f_{yd}} \frac{W_1}{h} \tag{11}$$

where

A_s	area of reinforcing bars in tension;
W_1	$I/(d\text{-}x)$, and
I	section moment of inertia.

5 NUMERICAL EXAMPLES

Consider the section of fig. 6 taken from reference 4. Bar location 1 to 4 are mandatory. The design parameters and variables are shown in fig. 6. Table 1 shows the five cases studied with the reinforcing bars areas, geometry and location of the neutral axis variables, and table 2 the minimal, initial and maximum design variables values.

The objective function is the cost of the structural member per unit length.

The considered constraints are: strength constraints, minimal amount of steel constraints and bound constraints.

The load parameters are: axial load (N) 1135 kN; bending about x-x axis (M_x) 92,25 kN m and bending about y-y axis (M_y) 115,32 kN m.

The materials parameters are: calculus strength of steel (f_{yd}) 420 Mpa; strength of concrete in axial compression (f_{cd}) 20 Mpa; steel density (ρ_s) 78,5 kN/m^3; modulus of elasticity of steel (E_s) 2,1 10^5 MPa and modulus of strain of concrete (E_c) 2,5 10^4 MPa.

The cost parameters are: cost of concrete (C_h) 10865 u.c./volume unit; cost of formwork (C_f) 4000 u.c./area unit and cost of reinforcement (C_s) 14,7 u.c./weight unit.

The section has been divided into 9 elements (3x3 mesh) and it has been used 2x2 Gauss points in numerical integration.

Table 1: Cases studied. Design variables

Value	Reinforcement variables			Geometry variables			Location of the n.a.
	Minimal	Initial	Maximum	Minimal	Initial	Maximum	
	0,0	3,142e-4	3,142e-4	0,177	0,247	0,353	
Case 1	$A_1=A_4=A_7=A_{10}$ $A_2\ A_3\ A_5\ A_6\ A_8\ A_9\ A_{11}\ A_{12}$			-			$\xi\ \beta$
Case 2				$b\ h$			
Case 3				$r_1\ r_2\ r_3\ r_4$			
Case 4				$r_1 = r_2\ \ r_3 = r_4$			
Case 5				$r_1 = r_4\ \ r_2 = r_3$			

Table 2: Design variables. Minimal, initial and maximum values

Variable	Values		
	Minimal	Initial	Maximum
$A_1\ A_4\ A_7\ A_{10}$ (m^2)	3,142e-4	3,142e-4	3,142e-4
$A_2\ A_3\ A_5\ A_6\ A_8\ A_9\ A_{11}\ A_{12}$ (m^2)	0,0	3,142e-4	3,142e-4
b h (m)	0,25	0,35	0,50
$r_1\ r_2\ r_3\ r_4$ (m)	0,177	0,247	0,353
ξ	-1	0,625	2
β (º)	0,0	51,342	360

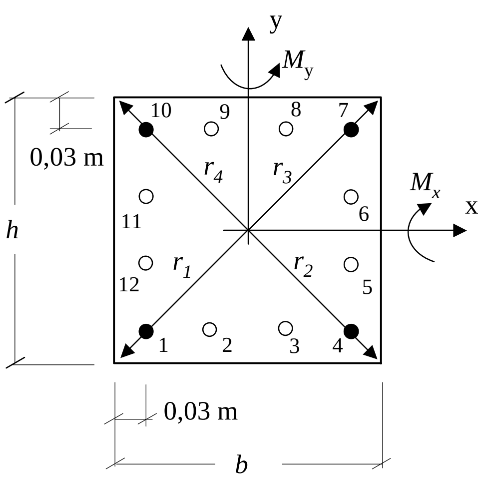

Figure 6: Numerical example. Reinforced concrete section

5 Results

First of all the table 3 shows the optimal designs obtained for the five cases, and the fig. 7 shows the initial and optimal sections and the neutral axis location for each one case.

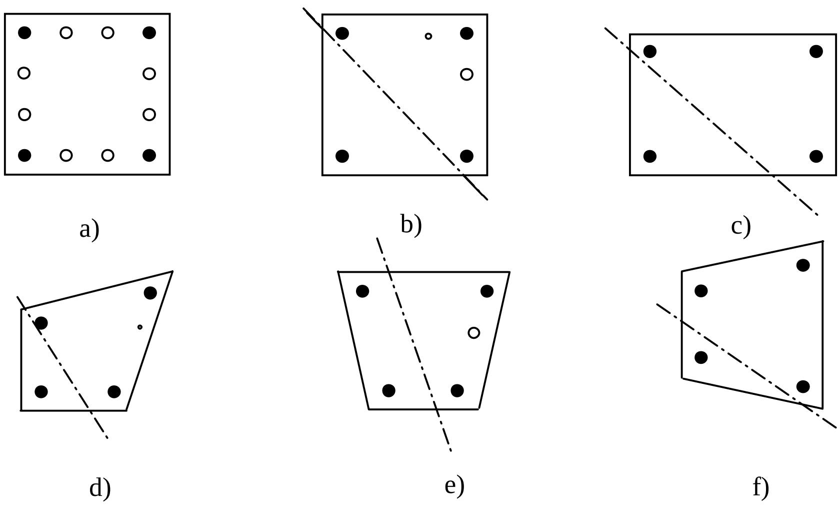

Figure 7: a) Initial design, b) Case 1, c) Case 2, d) Case 3, e) Case 4, f) Case 5

Table 3: Optimization results

Variable	Initial design	Optimal design				
		Case 1	Case 2	Case 3	Case 4	Case 5
$A_{1,4,7,10}$ (m^2)	3,142e-4	3,142e-4	3,142e-4	3,142e-4	3,142e-4	3,142e-4
A_2 (m^2)	3,142e-4	0,0	0,0	0,0	0,0	0,0
A_3 (m^2)	3,142e-4	0,0	0,0	0,0	0,0	0,0
A_5 (m^2)	3,142e-4	0,0	0,0	0,0	0,0	0,0
A_6 (m^2)	3,142e-4	3,142e-4	0,0	0,0	1,742e-4	0,0
A_8 (m^2)	3,142e-4	1,624e-4	0,0	0,0	0,0	0,0
A_9 (m^2)	3,142e-4	0,0	0,0	0,0	0,0	0,0
A_{11} (m^2)	3,142e-4	0,0	0,0	0,0	0,0	0,0
A_{12} (m^2)	3,142e-4	0,0	0,0	0,0	0,0	0,0
r_1 (m)	0,247	-	-	0,177	0,177	0,177
r_2 (m)	0,247	-	-	0,177	0,177	0,295
r_3 (m)	0,247	-	-	0,331	0,290	0,295
r_4 (m)	0,247	-	-	0,177	0,290	0,177
h (m)	0,350	-	33,505	-	-	-
b (m)	0,350	-	41,587	-	-	-
ξ	0,625	0,555	0,558	0,765	0,715	0,710
β (º)	38,659	310,346	320,983	298,233	290,141	325,447
Object. F. (u.c.)	11281	8931	8831	7429	8191	8071

The figure 8 shows a screen image, during an optimal design session in the developed program.

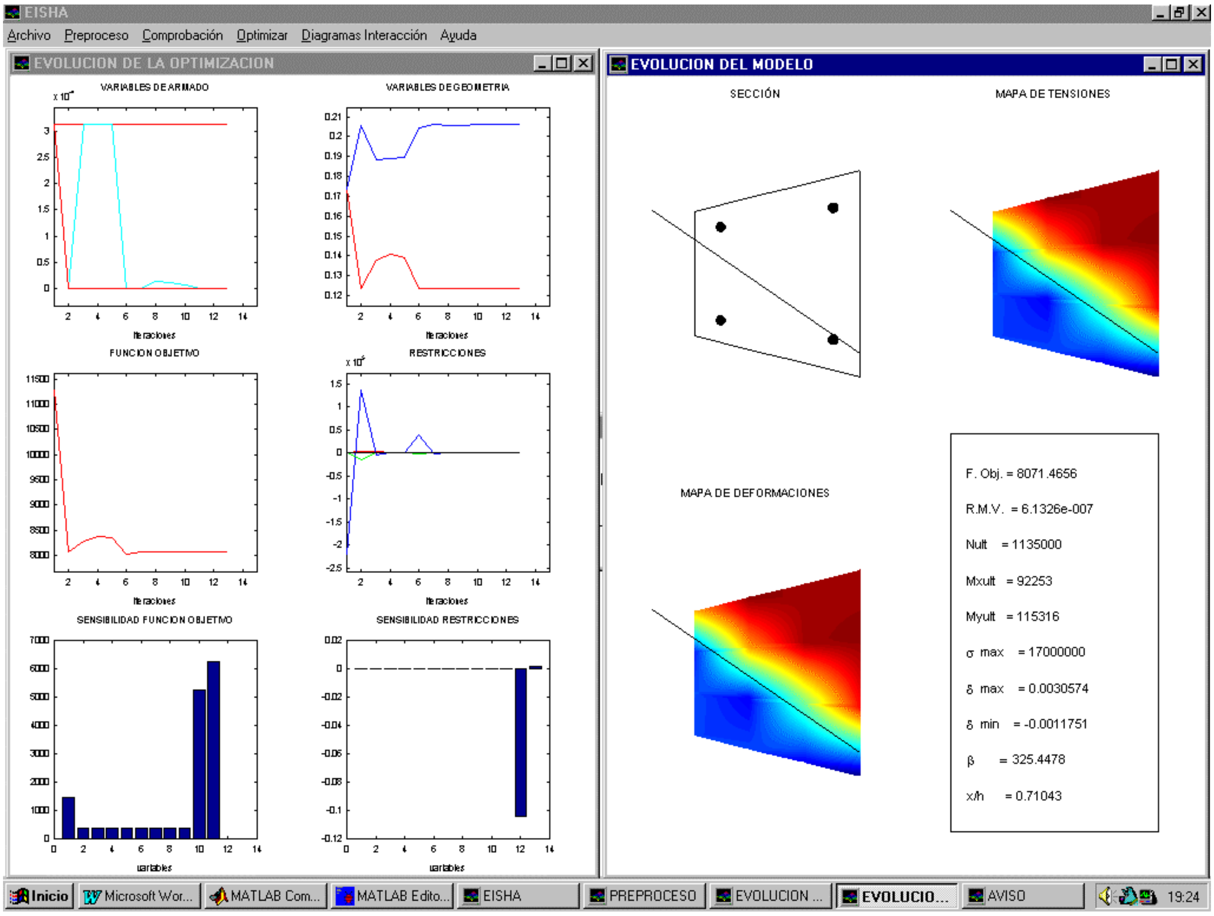

Figure 8: Screen image, during an optimal design session for case 5

5 CONCLUSIONS

An iteration procedure to compute the ultimate strength for general shape reinforced concrete sections is described.

The optimal design problem of shape and reinforcement for reinforced concrete sections has been formulated. The design variables are the reinforcing bars areas, geometry variables, and location of the neutral axis variables. The objective function is the cost of the structural member per unit length. The considered constraints are: strength constraints, minimal amount of steel constraints and bound constraints.

A code in Matlab to solve the problem written above has been developed.

Several test examples have been solved so as to prove the accuracy and efficiency of the techniques.

6 REFERENCES

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