Area preserving analytic flows with dense orbits

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Abstract

The aim of this paper is to give sufficient conditions on area-preserving flows that guarantee the existence of dense orbits. We also answer a question by M. D. Hirsch, [1]. The results of this work are a generalization of the ones in [1] and [2].

Key words: analytic flow, area-preserving flow, orbit, dense orbit, ω-limit set
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1. Introduction

Let S be a *surface*, that is, a second countable Hausdorff topological space which is locally homeomorphic to the plane. We recall that S admits an analytic structure which is unique up to diffeomorphisms, see [3, 4], [5, Example 3.1.6] and [6, p. 16]. A continuous map $\Phi : \mathbb{R} \times S \to S$ is a *flow* on S when the following properties hold:

(a) $\Phi(0, u) = u$ for any $u \in U$;

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(b) $\Phi(s, \Phi(t, u)) = \Phi(s + t, u)$ for any $s, t \in \mathbb{R}$.

If moreover Φ is C^r-class (resp. smooth, analytic) then Φ is said to be a *C^r (resp. smooth, analytic) flow.* Let $u \in S$, then the map $\Phi_u : \mathbb{R} \to S$ is defined by $\Phi_u(t) = \Phi(t, u)$ and the *orbit of* u is $\mathcal{O}_u = \Phi_u(\mathbb{R})$. When $\mathcal{O}_u = \{u\}$ u is called a *singular* point; otherwise u is *regular*. If Φ_u is a periodic nonconstant map the orbit of u is *periodic*. The set of singular points from the flow Φ is denoted by $\mathrm{Sing}(\Phi)$.

A set $A \subset S$ is said to be *invariant* if $\Phi(\mathbb{R} \times A) = A$. The *$\omega$-limit set of* u is defined by $\omega_\Phi(u) = \{v \in S : \exists (t_n)_{n=1}^{\infty} \to +\infty; (\Phi_u(t_n))_{n=1}^{\infty} \to v\}$ and the *α-limit set of* u is similarly defined by $\alpha_\Phi(u) = \{v \in S : \exists (t_n)_{n=1}^{\infty} \to -\infty; (\Phi_u(t_n))_{n=1}^{\infty} \to v\}$, both sets are closed and invariant and if S is compact they are also connected.

The Riemann structure on S induces a corresponding Lebesgue measure μ with a smooth density function (in particular, μ is positive on any open set of S). We assume that μ is normalized. Φ is an *area-preserving* flow (or μ is *Φ-invariant*) if for any measurable set $A \subset S$ we have $\mu(A) = \mu(\Phi(\{t\} \times A))$ for any $t \in \mathbb{R}$. Our interest is to give sufficient conditions on an area-preserving flow for having dense orbits. In particular we generalize the following result.

Theorem 1 (M.D. Hirsch and H. Marzougui,[1, 2]). *Let S be a compact connected surface and let $\Phi : \mathbb{R} \times S \to S$ be an area-preserving smooth flow such that* $\mathrm{Sing}(\Phi)$ *is totally disconnected. Then Φ has a dense orbit if and only if Φ has not periodic orbits and* $\mathrm{Sing}(\Phi) \cup \{x : \omega_\Phi(x) \cup \alpha_\Phi(x) \subset \mathrm{Sing}(\Phi)\}$ *does not separate S.*

As usually for any $U \subset S$, $\mathrm{Bd}\, U$ and $\mathrm{Int}\, U$ respectively denote the topological boundary and the interior of U. Our main results are the two following and generalize the previous one.

Theorem A. *Let S be a compact connected surface and let $\Phi : \mathbb{R} \times S \to S$ be an area-preserving smooth flow. Φ has a dense orbit if and only if the following statements hold:*

1. *Φ has not periodic orbits.*
2. $\mathrm{Int}(\mathrm{Sing}(\Phi)) = \emptyset$.
3. $\mathrm{Sing}(\Phi) \cup \{x : \omega_\Phi(x) \cup \alpha_\Phi(x) \subset \mathrm{Sing}(\Phi)\}$ *does not separate S.*

Theorem B. *Let S be a compact connected surface and let $\Phi : \mathbb{R} \times S \to S$ be an area-preserving analytic flow such that* $\mathrm{Sing}(\Phi) \neq S$. *$\Phi$ has a dense orbit if and only if the following statements hold*

1. Φ *has not periodic orbits.*
2. $\mathrm{Sing}(\Phi) \cup \{x : \omega_\Phi(x) \cup \alpha_\Phi(x) \subset \mathrm{Sing}(\Phi)\}$ *does not separate* S*.*

The flow Φ is said to be *(topologically) transitive* if it admits a dense orbit. The study of transitive flows and what surfaces are transitive, i.e. admit transitive flows, has a long tradition in the literature (see for instance the bibliography in [7] and [8, 9, 10, 11]). It is worth emphasizing that a complete characterization of transitive surfaces was obtained in the following terms.

Two orientable circles on S are said to be a pair of *crossing circles* if they intersect transversally at exactly one point.

Theorem 2 (Th. A from [8]). *Let* S *be a connected surface. Then the following statements are equivalent:*

- S *is transitive;*
- S *is not homeomorphic to* $\mathbb{S}^2$ (the sphere)*,* $\mathbb{P}^2$ (the projective plane)*, nor to any surface in* $\mathbb{B}^2$ (the Klein bottle)*;*
- S *contains a pair of crossing circles.*

The flows constructed in the proof of the above theorem are smooth provided that any surface admits smooth structure. It is also interesting to remark that $\mathbb{S}^2$, $\mathbb{P}^2$ and $\mathbb{B}^2$ are not transitive.

If a flow Φ admits a dense orbit then there exists a dense G_δ set, $D \subset S$, so that $\mathcal{O}_v$ is dense for any $v \in D$, see [2, 8]. M. D. Hirsch [1] asked if this G_δ set is full measure or not; in the present note we answer this question as follows.

Theorem C. *Let* S *be a compact connected surface. Then the following statements hold:*

1. *Let* $\Phi : \mathbb{R} \times S \to S$ *be an analytical flow having a dense orbit. Then* $\mu(\{v : \mathcal{O}_v \text{ is dense}\}) = 1$.
2. *If* $S \neq \mathbb{S}^2, \mathbb{P}^2$ *nor* $\mathbb{B}^2$ *then there exists a smooth flow* $\Phi : \mathbb{R} \times S \to S$ *having a dense orbit and so that* $\mu(\{v : \mathcal{O}_v \text{ is dense}\}) < 1$.

The remainder of this work is divided in three sections. The first one introduces some theorems about singular points and dynamical properties of a flow. In Section 3 we prove Theorems A and B. Finally Section 4 is devoted to the proof of Theorem C.

2. Technical results

In this section we introduce notation and we prove some technical results.

2.1. Dynamic properties

A point $u \in S$ is said to be *recurrent* if $u \in \omega_\Phi(u)$ or $u \in \alpha_\Phi(u)$. Singular points and those from periodic orbits are examples of recurrent points, so called *trivial*. All other recurrent points u are *nontrivial*. The closure of the orbit from any nontrivial recurrent point is called *quasi-minimal* set. In the following $\mathrm{Rec}(\Phi)$ denotes the set of recurrent points of the flow Φ.

Theorem 3 (Poincaré recurrence theorem). *Let $\Phi : \mathbb{R} \times S \to S$ be an area-preserving flow, then $\mu(\mathrm{Rec}(\Phi)) = 1$*

Proof. See [12, p. 141]. □

Proposition 4. *Let Φ be a continuous flow on a compact connected surface S and let $u \in S$. If $\omega_\Phi(u)$ or $\alpha_\Phi(u)$ contains a periodic orbit then it reduces to this periodic orbit.*

In particular, If Φ is topologically transitive then it does not admit periodic orbits.

Proof. This follows from [13, p. 67, Proposition 7.11]. □

Proposition 5. *Let Φ be a continuous flow on a compact connected surface S and let $u \in S$ so that $\mathcal{O}_u$ is nontrivial recurrent. If $v \in \omega_\Phi(u)$ one of the two following possibilities occurs:*

- $\omega_\Phi(v) = \omega_\Phi(u)$;
- $\omega_\Phi(v) \subset \mathrm{Sing}(\Phi)$.

Proof. This result is equivalent to [14, Proposition 2.1] for orientable surfaces. It remains valid for non orientable ones by pulling-back the flow to the orientable 2-covering. □

2.2. Singularities of an analytic flow

By an *r-star* we mean a topological space R homeomorphic to $\{z \in \mathbb{C} : z^r \in [0,1]\}$, the homeomorphism maps 0 to a point p which is called the *vertex* of the star and maps the r-roots of the unity to the *endpoints* of the star. Any single point will be said to be a 0-star.

Theorem 6 (Th. 4.3 from [15]). *Let $\Phi : \mathbb{R} \times S \to S$ be an analytic flow on a compact connected surface S such that* $\mathrm{Sing}(\Phi) \neq S$ *and let* $u \in \mathrm{Sing}(\Phi)$. *Then* $\mathrm{Sing}(\Phi)$ *is locally a $2n$-star having u as its vertex for some nonnegative integer n.*

Corollary 7. *Let S be a compact connected surface and Φ an analytic flow on S. Then either* $\mathrm{Sing}(\Phi) = S$ *or* $\mu(\mathrm{Sing}(\Phi)) = 0$.

Proof. If $\mathrm{Sing}(\Phi) \neq S$, use that S is compact and $\mathrm{Sing}(\Phi)$ is closed to find a finite open covering $\{U_i\}_{i=1}^k \bigcup \{V_i\}_{i=1}^l$, $\{k,l\} \subset \mathbb{N}$, such that (i) $\mathrm{Sing}(\Phi) \cap U_i$ is a $2n_i$-star for any $1 \leq i \leq k$ and some $n_i \in \mathbb{N}$; (ii) $\mathrm{Sing}(\Phi) \cap V_i = \emptyset$ for any $1 \leq i \leq l$.

Since $\mathrm{Sing}(\Phi) \cap U_i$ is a $2n_i$-star, $\mu(\mathrm{Sing}(\Phi) \cap U_i) = 0$. Thus

$$\mu(\mathrm{Sing}(\Phi)) \leq \sum_{i=1}^{k} \mu(\mathrm{Sing}(\Phi) \cap U_i) = 0.$$

□

3. Proofs of Theorems A and B

3.1. Proof of Theorem A

We begin by assuming that $\mathcal{O}_u$ is dense for some $u \in S$, then by Proposition 4, Φ has not periodic orbits. Moreover it is clear that $\mathrm{Int}\,\mathrm{Sing}(\Phi) = \emptyset$ and $S_1 = \mathrm{Sing}(\Phi) \cup \{x : \omega_\Phi(x) \cup \alpha_\Phi(x) \subset \mathrm{Sing}(\Phi)\}$ is invariant. Assume that S_1 separates S, then $S \backslash S_1$ has at least two (invariant) components, S_1^1 and S_1^2, the first one containing $\mathcal{O}_u$; then $\overline{S_1^2} = \overline{\mathcal{O}_u} \cap \overline{S_1^2} \subset \overline{S_1^1} \cap \overline{S_1^2}$, a contradiction. Therefore S_1 does not separate S.

We now prove the converse, we assume that Φ is an area-preserving and satisfies the three conditions in the theorem. Use Theorem 3, that $\mathrm{Int}\,\mathrm{Sing}(\Phi) = \emptyset$ and the absence of periodic orbits to deduce the existence of nontrivial recurrent points. Assume that Φ admits only empty interior quasiminimal sets, since these quasiminimal sets are only a finite number [14], we label them by $K_1, K_2, \ldots, K_l$. Let $S' = S\backslash(K_1 \cup K_2 \cup \cdots \cup K_l)$, then it is clear that S' is open, non empty (otherwise, one of K_i have a non empty interior, absurd), invariant and decomposes in connected components $\{U_i\}_{i\in\mathcal{I}}$. Use hypothesis (1), (2) and Theorem 3 to assure the existence of nontrivial recurrent points in any U_i generating nonempty interior quasiminimals. Take $u \in S$ so that $\omega_\Phi(u)$ has nonempty interior, then by [8, Lemma 2.2] $O = \operatorname{Int}\omega_\Phi(u)$ is non empty connected, invariant and $\overline{O} = \omega_\Phi(u)$. Assume that $\omega_\Phi(u) \neq S$, so $\operatorname{Bd} O \neq \emptyset$ and there is $w \notin \omega_\Phi(u)$. Then $S\backslash \operatorname{Bd} O$ contains at least two components, one containing u and the other one w, in other words, $\operatorname{Bd} O$ separates the surface S. Let $v \in \operatorname{Bd} O$ then, by Proposition 5, $\omega_\Phi(v) \subset \operatorname{Sing}(\Phi)$ and therefore $\operatorname{Bd} O \subset S_1$. As $\operatorname{Bd} O$ separates S, S_1 also separates S, a contradiction. Hence $\omega_\Phi(u) = S$, so $\mathcal{O}_u$ is dense in S and the theorem follows. $\square$

3.2. Proof of Theorem B

The proof of this theorem follows immediately by applying Theorem A and Corollary 7.

4. Proof of Theorem C

4.1. Proof of statement (1)

The proof is a consequence of technical results from [11] which can be stated in the following way.

Proposition 8 (Corollary 12 and Proposition 13 from [11]). *Let S be a compact connected surface and let $\Phi : \mathbb{R} \times S \to S$ be a transitive analytical flow. Let u be such that $\mathcal{O}_u$ is dense, then:*

1. $S\backslash\{v : \mathcal{O}_v \text{ is dense}\}$ *is the union of* $\operatorname{Sing}(\Phi)$ *and nonrecurrent points. If w is nonrecurrent, then $\omega_\Phi(w)$ (and also $\alpha_\Phi(w)$) is a singular point.*
2. *The number of nonrecurrent orbits is finite.*

We now prove the first statement of Theorem C. Let us write

$$T_1 := \{w : w \text{ is regular}, \ \omega_\Phi(w) \textit{ and } \alpha_\Phi(w) \textit{ are singular points}\},$$

then $\{v : \mathcal{O}_v \text{ is dense}\} = S\backslash(T_1 \cup \operatorname{Sing}(\Phi))$ by Proposition 8. Moreover $\mu(\{v : \mathcal{O}_v \text{ is dense}\}) = 1 - \mu(T_1 \cup \operatorname{Sing}(\Phi)) = 1 - \mu(T_1) - \mu(\operatorname{Sing}(\Phi))$. Finally as $\mu(\operatorname{Sing}(\Phi)) = 0$ by Corollary 7 and $\mu(T_1) = 0$ by Proposition 8(2), the first statement of the theorem is proved. $\square$

4.2. Proof of statement (2)

Let A and B a pair of crossing circles on S, let D be a disk on $S \backslash (A \cup B)$ and take a compact set $\mathcal{K} \subset D$ homeomorphic to $\mathcal{C} \times [0,1]$ where $\mathcal{C}$ is a Cantor set and $\mu(\mathcal{K}) > 0$.

The surface $T = S \backslash \mathcal{K}$ is connected and contains $A \cup B$, a pair of crossing circles. Then, by Theorem 2, T admits a smooth topologically transitive flow, $\Psi : \mathbb{R} \times T \to T$. Finally we apply [16, Lemma 2.1] to obtain a smooth topologically transitive flow, $\Phi : \mathbb{R} \times S \to S$, so that $\mathcal{K} \subset \mathrm{Sing}(\Phi)$ and the orbits from Ψ coincide with those of Φ contained in $S \backslash \mathcal{K}$. Now it is clear that Φ is topologically transitive and since $\mu(\mathcal{K}) > 0$, $\mu(\{v : \mathcal{O}_v \text{ is dense}\}) \leq 1 - \mu(\mathcal{K}) < 1$. $\square$

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