EMPTY INTERIOR RECURRENCE FOR CONTINUOUS FLOWS ON SURFACES

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Dedicated to the memory of Valery S. Melnik

In this paper we characterize topologically the empty interior subsets of a compact surface S which can be ω-limit sets of recurrent orbits (but of no nonrecurrent ones) of continuous flows on S. This culminates the classification of ω-limit sets for surface flows initiated in [Jiménez & Soler, 2001], [Soler, 2003], [Jiménez & Soler, 2004], and [Jiménez & Soler, 2004b]. We also show that this type of ω-limit sets can always be realized (up to topological equivalence) by smooth flows but cannot be realized by analytic flows.

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1. Introduction

Let S be a compact surface with empty combinatorial boundary. By a *(continuous) flow on* S we mean a continuous map $\Phi : \mathbb{R} \times S \to S$ satisfying $\Phi(0, u) = u$ for every $u \in S$ and $\Phi(t, \Phi(s, u)) = \Phi(t + s, u)$ for every $t, s \in \mathbb{R}$ and $u \in S$. Starting from the seminal works of Poincaré and Bendixson, the qualitative theory of surface flows was extensively studied in the twentieth century and still remains an area of intensive research, see [Aranson *et al.*, 1996], [Aranson & Zhuzhoma, 1998], [Nikolaev, 2001], [Nikolaev & Zhuzhoma, 1999] for recent accounts of the state of the art in this subject.

No doubt the jewel crown of this theory is the celebrated Poincaré-Bendixson theorem. Under some restrictions it provides a very simple description of ω-limit sets in the plane (or the sphere) setting. (The *ω-limit set* of a point u or its *orbit* $\Phi_u(\mathbb{R})$, $\Phi_u(t) := \Phi(t, u)$, is the set $\omega_\Phi(u) = \{v \in S : \exists t_n \to \infty; \Phi_u(t_n) \to v\}$. The *$\alpha$-limit set* $\alpha_\Phi(u)$ is analogously defined now taking $t_n \to -\infty$. We emphasize that all the results in this paper concerning ω-limit sets are true for α-limit sets as well; the proofs require no essential changes.) Original formulations by Poincaré [Poincaré, 1886] and Bendixson [Bendixson, 1901] required some differentiability properties for the flow, but since the thirties it is well known that this result is of a purely topological nature [Whitney, 1933], [Bohr & Frenchel, 1936], [Bebutov, 1939]. The presentation below can be found, for instance, in [Aranson *et al.*, 1996; Theorem 3.1, p. 63]:

Theorem 1.1 (Poincaré-Bendixson). *Let Φ be a flow on the sphere $\mathbb{S}^2$ and let $u \in \mathbb{S}^2$. If $\omega_\Phi(u)$ does not contain any singular point, then it is a periodic orbit.*

Recall that a *singular* (respectively, a *periodic*) orbit is that consisting of a single point (respectively, realized by a periodic solution $\Phi_u(t)$ of the flow).

The Poincaré-Bendixson theorem renders a precise description of the ω-limit sets containing no singular points for flows on the sphere. This description has simultaneously a topological and a dynamical character: from the topological point of view the theorem says that, in the prescribed conditions, the ω-limit set of u is a (topological) circle; from the dynamical point of view it says that the ω-limit set of u is a periodic orbit.

Now the question of describing ω-limit sets with arbitrarily many periodic points, in $\mathbb{S}^2$ and other compact surfaces, suggests itself. The literature provides many results in this regard. An especially beautiful one, due to A. J. Schwartz [Schwartz, 1963], says that the Poincaré-Bendixson theorem holds true for C^2 flows on arbitrary surfaces after just including an additional possibility: the ω-limit set may also be the whole torus $\mathbb{T}^2$ (when the flow is conjugate to an irrational rotation). We emphasize that the smoothness requirement is sharp [Denjoy, 1932]. For a nice converse of Schwartz's theorem see [Gutierrez, 1986]. A different trend of work, initiated by Solncev [Solncev, 1945] and Vinograd [Vinograd, 1952] in the forties and culminated in [Balibrea & Jiménez, 1998], aims to describe explicitly how a closed set and a family of curves must look like to become, respectively, the set of singular points and the set of nonsingular orbits of some ω-limit set for a sphere flow. In arbitrary surfaces and assuming that the set of singular points is finite, more concise (but less informative) descriptions are very well known, see for instance [Aranson *et al.*, 1996; Theorem 3.6, p. 86]:

Theorem 1.2. *Let Φ be a flow on S having finitely many singular points and finitely many orbits connecting these points. If $u \in S$, then $\omega_\Phi(u)$ is either a singular point, a periodic orbit, a polycycle, or a quasiminimal set.*

By a *polycycle* we mean the union of finitely many singular points (possibly repeated) and finitely many nonsingular orbits (without repetitions) connecting the singular points in a specific order: the time oriented j-th orbit connects the j-th and $(j+1)$-st singular points. A *quasiminimal set* is the ω-limit set of a *nontrivial recurrent orbit* (*recurrent* means the orbit is contained in its own ω-limit set; *nontrivial* means that it is neither a singular point nor a periodic orbit).

The above-mentioned results concern the dynamical side of affairs. If one is just interested in knowing the topological structure of ω-limit sets regardless the type of orbits they are made of, then the following theorem, essentially due to Vinograd [Vinograd, 1952] settles the question in $\mathbb{S}^2$.

Theorem 1.3 ([Vinograd, 1952]). *Let Φ be a flow on $\mathbb{S}^2$ and let $u \in \mathbb{S}^2$. Then $\omega_\Phi(u)$ is the boundary of a simply connected region O, $\emptyset \subsetneq O \subsetneq \mathbb{S}^2$.*

Conversely, if Ω is the boundary of a simply connected region O, $\emptyset \subsetneq O \subsetneq \mathbb{S}^2$, then there are a smooth (C^∞) flow Φ on $\mathbb{S}^2$ and $u \in \mathbb{S}^2$ such that $\Omega = \omega_\Phi(u)$.

In 1995 D. V. Anosov [Anosov, 1995] (see also [Nikolaev & Zhuzhoma, 1999; p. 39]) remarked that Vinograd's topological characterization is no longer true for the projective plane $\mathbb{P}^2$ and posed the problem of finding an appropriate characterization of ω-limit sets in this setting. This being our source of inspiration, we undertook the task of fully characterizing ω-limits for surface flows in a series of papers [Jiménez & Soler, 2001], [Soler, 2003], [Jiménez & Soler, 2004], [Jiménez & Soler, 2004b]. The present one is the last step towards such a characterization.

We answered Anosov's question in [Jiménez & Soler, 2001] by showing that ω-limit sets in $\mathbb{P}^2$ are the boundaries of regions with nonempty connected complementary. Next, the second author characterized ω-limit sets on the

Klein bottle $\mathbb{B}^2$ as boundaries of regions O with nonempty connected complementary such that either O is simply connected, or there is a non-null homotopic circle $C \subset O$ such that the boundary of O is contained in the boundary of one of the components of $O \setminus C$ [Soler, 2003].

Together with the sphere $\mathbb{S}^2$, the projective plane $\mathbb{P}^2$ and the Klein $\mathbb{B}^2$ are the easiest surfaces to work with because they admit no flows having nontrivial recurrent orbits [Aranson, 1969], [Thomas, 1970], [Markley, 1969], [Gutierrez, 1978]. In [Jiménez & Soler, 2004] we extended the previous results by characterizing ω-limit sets of nonrecurrent orbits in arbitrary surfaces. If by a *regular annulus* we mean an annulus (i.e., a space homeomorphic to $\mathbb{R}^2\setminus\{(0,0)\}$) with two boundary components, then the generalization can be stated as follows.

Theorem 1.4 ([Jiménez & Soler, 2004]). *Let Φ be a flow on S and let $u \in S$. Assume that (the orbit of) u is nonrecurrent or that $\operatorname{Int}\omega_\Phi(u) = \emptyset$ and $S \setminus \omega_\Phi(u)$ has a finite number of components. Then $\omega_\Phi(u)$ is a boundary component of a regular annulus in S.*

Conversely, if Ω is a boundary component of a regular annulus in S then there are a smooth flow Φ on S and $u \in S \setminus \Omega$ such that $\Omega = \omega_\Phi(u)$.

Any surface different from $\mathbb{S}^2$, $\mathbb{P}^2$ or $\mathbb{B}^2$ admits flows having some nontrivial recurrent orbits. The description of the ω-limit sets of this type of orbits (quasiminimal sets), is rather more complicated. Sometimes a quasiminimal set can also be the ω-limit set of a nonrecurrent orbit (see the Denjoy flow below), hence the previous theorem applies, but this is not often the case.

Quasiminimal sets have been studied in the literature in great depth. For instance, Chapter 2 from the recent monography [Aranson *et al.*, 1996] devotes Sections 3 (partially) and 4 (totally) to them. In this context the works of Hilmy [Hilmy, 1936], Cherry [Cherry, 1937], Maĭer [Maĭer, 1943], Gutierrez [Gutierrez, 1986], and Marzougui [Marzougui, 1996], among many others, deserve to be mentioned. Surprisingly enough, while these and other works involve a close understanding of the topology of quasiminimal sets, their precise topological characterization has almost never been addressed for. The present paper intends to fill this gap.

Obvious examples of quasiminimal sets that cannot be realized by nonrecurrent orbits are those with nonempty interior. It is easy to show that such an ω-limit set is the closure of some transitive region, that is, a region *invariant* by the flow (which means that is the union set of some orbits of the flow) with the property that all orbits in the region are dense in it. Transitive regions were partially characterized in [Smith & Thomas, 1988] and then, completely, in [Jiménez & Soler, 2004b].

Theorem 1.5 ([Jiménez & Soler, 2004b]). *Let Φ be a flow on S and let $u \in S$. Assume that $\operatorname{Int}\omega_\Phi(u) \neq \emptyset$. Then there exists a region $O \subset S$ such that $\operatorname{Cl} O = \omega_\Phi(u)$ and O is not homeomorphic to $\mathbb{S}^2$, $\mathbb{P}^2$, nor to any region in $\mathbb{B}^2$.*

Conversely, if $O \subset S$ is a region not homeomorphic to $\mathbb{S}^2$, $\mathbb{P}^2$, nor to any region in $\mathbb{B}^2$, then there are a smooth flow Φ on S and a point $u \in O$ such that $\omega_\Phi(u) = \operatorname{Cl} O$.

Thus, in order to complete the intended topological description of ω-limit sets for surface flows, we are bound to understand the structure of quasiminimal sets with empty interior that *cannot be* ω-limit sets of any nonrecurrent orbit. This is exactly what we do in this paper. In [Aranson *et al.*, 1996; p. 54] nontrivial recurrent orbits whose ω-limit set has empty interior are called *exceptional*. We are looking for even rarer ω-limits (those which cannot be realized by a nonrecurrent orbit) but it seems apt to keep using this word to describe them.

Definition 1.6. Let Φ be a flow on S. We say that $\Omega \subset S$ is an *exceptional ω-limit set (for Φ)* if $\operatorname{Int}\Omega = \emptyset$ and it is the ω-limit set of a nontrivial recurrent orbit, but of no nonrecurrent orbit, of Φ.

We say that $\Omega \subset S$ is an *exceptional set* if it is an exceptional ω-limit set for some flow on S, and it is not the ω-limit set of a nonrecurrent orbit for any flow on S (or equivalently, in view of Theorem 1.4, it is not a boundary component of a regular annulus).

The following example will help to clarify these notions. Our starting point is the irrational flow on the torus $\mathbb{T}^2$. After blowing up one of the orbits to a full band of orbits we can generate the well-known Denjoy C^1 flow Ψ [Denjoy, 1932], see

[Aranson *et al.*, 1996; pp. 24–27] for a modern presentation. If the interior of this band is denoted by O and we write $\Sigma = \mathbb{T}^2 \setminus O$, then the empty interior set Σ is *minimal*, that is, it is the ω-limit set of all orbits in Σ and, in fact, of all orbits of Ψ. Notice that although all orbits in Σ are exceptional, Σ is not an exceptional ω-limit set for the flow Ψ because it is also the ω-limit set of all orbits inside O, which are nonrecurrent. Of course O, even although it keeps spiralling around $\mathbb{T}^2$, is simply connected and hence its boundary Σ is one of the boundary components of the regular annulus $O\setminus\{p\}$, where p denotes an arbitrarily chosen point of O. This is in consonance with Theorem 1.4.

The next step is fixing a circle D transversal to the flow Ψ. The band O intersects D at consecutive (with respect to time) segments, open in D, with endpoints a_i, b_i, $i \in \mathbb{Z}$. We denote these segments by $(a_i; b_i)$. If the increasing sequence $(i_n)_{n=-\infty}^{\infty}$ is appropriately chosen, then the segments $(a_{i_n}; b_{i_n})$ can also be made to be consecutive in D, thus converging to respective points p and q of D when $n \to \pm\infty$. If f is the vector field associated to Ψ and we multiply it by a nonnegative scalar function vanishing exactly at $K = \{p, q\} \cup \bigcup_n [a_{i_n}; b_{i_n}]$, where $[a_i; b_i]$ denotes the closure of $(a_i; b_i)$, then it generates a new flow Ψ^* having K as its set of singular points. Moreover, K decomposes O into a family of consecutive simply connected regions $\{K_n\}_n$, and every orbit of Ψ^* contained in one of this regions has a singular point as is α-limit set and another, different one, as its ω-limit set. All orbits of Ψ outside O (except those ending at one of the points p, q or a_{i_n}, b_{i_n}) remain the same for Ψ^*. Hence Σ becomes an exceptional ω-limit set for Ψ^*. We emphasize that Σ is a quasiminimal set for Ψ^* but not a minimal set because it contains some singular points. On the other hand, recall that Σ is not an exceptional set because it is the ω-limit set of all nonrecurrent orbits of Ψ.

Finally, to create an exceptional set in $\mathbb{T}^2$ it suffices, starting from Ψ^*, to collapse each arc $[a_{i_n}; b_{i_n}]$ to a point to obtain a flow Φ having countably many singular points, all of them belonging to the circle E arising after the collapse of the arcs $[a_{i_n}; b_{i_n}]$. This flow has an exceptional ω-limit set Ω whose complement set is a pairwise disjoint union of simply connected regions $\{O_n\}_n$. The closure of each region O_n consists of a family R_n of parallelizable orbits and two singular points, which are the α- and the ω-limit set of all orbits in R_n. Thus Ω cannot be a boundary component of a regular annulus, that is, it is an exceptional set. What the main result of this paper (Theorem A) essentially says is that all exceptional sets have a topological-dynamical structure similar to that previously described. We emphasize that, in order Ω to be exceptional, it does not suffices that it is the boundary of a disjoint union of infinitely many simply connected regions: just think of the flow arising from the irrational torus rotation after blowing up an infinite number of orbits. The truly important point is that the closure of each of these regions is still (maybe after removing a few appropriately chosen boundary points) a simply connected surface, even although it may spiral around $\mathbb{T}^2$ for quite a long time.

2. Statements of the results

Before stating precisely our results some terminology is required.

By a *surface* we mean a second countable Hausdorff topological space such that every point has a neighbourhood homeomorphic to the unit disk $\mathbb{D}^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$; thus its *combinatorial boundary*, which is the set of points where the surface is not locally homeomorphic to the *open* unit disk $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, need not be empty. Throughout the paper, S will always denote a compact surface with empty combinatorial boundary. We recall that S admits an essentially unique smooth (C^∞) (even analytic) differential structure, which we fix from now on and to which we refer when speaking about *smooth* and *analytic* flows below.

We fix a distance $\operatorname{dist}(\cdot, \cdot)$ on S compatible with its topology. If $A \subset S$, then $\operatorname{Int} A$, $\operatorname{Cl} A$ and $\operatorname{diam} A$ denote the *interior*, the *closure* and the *diameter* of A. When we say that $R \subset S$ is a surface, we are referring to the induced topology in R, when the combinatorial boundary of R (as opposed to its *topological* boundary $\operatorname{Bd} R$) will be denoted by ∂R. If A, B are compact subsets of S, then $d_H(A, B)$ denotes the *Hausdorff distance* between A and B, that is,

$$d_H(A,B) = \max\{\max_{u\in A}\min_{v\in B}\operatorname{dist}(u,v), \max_{v\in B}\min_{u\in A}\operatorname{dist}(u,v)\}.$$

A *curve* B in S is the image $B=\varphi(I)$ of a continuous one-to-one map $\varphi: I\to S$, with I being a nondegenerate interval or the unit circle $\mathbb{S}^1=\{(x,y)\in\mathbb{R}^2 : x^2+y^2=1\}$. In the particular cases when I is a compact interval or $I=\mathbb{S}^1$, we call B, respectively, an *arc* (whose *endpoints* are the image by φ of the endpoints of I) and a *circle*. Sometimes we refer to an arc with endpoints u and v as $[u;v]$.

Let R be a surface. We say that a circle $C\subset R$ is *null homotopic* if there exists a continuous map $H:[0,1]\times[0,1]\to R$ satisfying $H(t,0)=\alpha(t)$, $H(t,1)=\alpha(0)$ and $H(0,s)=H(1,s)=\alpha(0)$ for every $t,s\in[0,1]$; here $\alpha:[0,1]\to C$ is a continuous onto map and $\alpha|_{[0,1)}$ is one-to-one. The surface R is *simply connected* if there is a homeomorphism $h:\mathbb{D}^2\backslash P\to R$ for a compact set $\emptyset\subset P\subset\mathbb{S}^1$. When $P=\mathbb{S}^1$ and $P=\emptyset$ we respectively say that R is an *open disk* and a *disk*. Simply connected surfaces are characterized by the property that all circles they contain are null homotopic, see Lemma 3.2 below. As it is well known every circle in S is either *orientable* or *nonorientable*, depending on whether it has a small neighbourhood which is an annulus or a Möbius band, respectively. (Recall that a *Möbius band* is a space homeomorphic to $\mathbb{D}^2\setminus\{(0,0)\}$ after identifying opposite points in $\mathbb{S}^1$).

Roughly speaking, exceptional sets will be characterized as the boundary of the union of infinitely many, pairwise disjoint, simply connected surfaces (we call them *feasible families* in Definition 2.4). The combinatorial boundaries of these surfaces, and the surfaces themselves, have some special properties. We describe them via the Definitions 2.1 and 2.2 below.

Definition 2.1. Let $\mathcal{B}$ be a family of curves in S, let $p\in B\in\mathcal{B}$ and let O be a neighbourhood of p. We say that O is a *Whitney regular neighbourhood* of p (with respect to $\mathcal{B}$) if for every $\epsilon>0$ there is $\delta>0$ such that:

(a) if $[p';q']\subset O\cap B'$ for some $B'\in\mathcal{B}$ and $\operatorname{dist}(p',q')<\delta$, then $\operatorname{diam}([p';q'])<\epsilon$;

(b) if $[p;q]\subset O\cap B$, $p'\in O\cap B'$ for some $B'\in\mathcal{B}$ and $\operatorname{dist}(p,p')<\delta$, then there is $[p';q']\subset O\cap B'$ such that $d_H([p;q],[p';q'])<\epsilon$.

We say that $\mathcal{B}$ is *Whitney regular* if every point from every $B\in\mathcal{B}$ has a Whitney regular neighbourhood.

Definition 2.2. Let $R\subset S$ be a simply connected surface and let $C\subset S$ be an orientable non-null homotopic circle. We say that R *twists around* C if there are a positive integer k, a subset N of $\{1/2,3/2,\dots,k-1/x2\}\times\{0,1\}$, and a homeomorphism $\phi:((-1/2,k+1/2)\times[0,1])\setminus N\to R$ such that:

(i) $C\cap\operatorname{Cl}R=\bigcup_{i=0}^k\phi(\{i\}\times[0,1])$;

(ii) if $\epsilon>0$ is very small and $i\in\{0,1,\dots,k-1\}$, then $\phi([i,i+\epsilon)\times[0,1])$ and $\phi((i+1-\epsilon,i+1]\times[0,1])$ lie at opposite sides of C;

(iii) there is $l\in\{1,\dots,k\}$ such that neither $(l-1/2,0)$ nor $(l-1/2,1)$ belong to N.

If l is as (iii), then we call the set $\phi([l-1,l]\times[0,1])$ a *twisting section* of R and the set $\phi([l-1,l]\times\{0,1\})$ its *twisting boundary.* We denote the union set of all twisting sections of R by $\Upsilon(R)$ and the union set of all their twisting boundaries by $\Upsilon(\partial R)$.

Remark 2.3. Of course all previous definitions depend on C, but in what follows the curve C will always be precisely stated, so this will not lead to confusion.

Definition 2.4. Let $\{R_n\}_{n=1}^\infty$ be a family of pairwise disjoint simply connected surfaces in S and let $C\subset S$ be an orientable non-null homotopic circle. We call $\{R_n\}_{n=1}^\infty$ a *feasible family for* C if the following properties hold:

(i) all R_n twist around C and, for every m, the set $R_m\cup\operatorname{Cl}(\bigcup_n\Upsilon(R_n))$ is a neighbourhood of ∂R_m;

(ii) the family of the components of all combinatorial boundaries ∂R_n is Whitney regular;

(iii) for every $u,v\in\bigcup_n\operatorname{Bd}R_n$ and every $\epsilon>0$ there is an arc $A\subset\bigcup_n\partial R_n$ satisfying $\operatorname{dist}(u,A)<\epsilon$, $\operatorname{dist}(v,A)<\epsilon$.

We are ready to state the main result of this paper:

Theorem A. *If $\Omega \subset S$ is an exceptional set, then there is a feasible family $\{R_n\}_{n=1}^\infty$ for some circle C such that $\Omega = \operatorname{Bd} \bigcup_n R_n$.*

Conversely, if $\{R_n\}_{n=1}^\infty$ is a feasible family for some circle C and $\Omega = \operatorname{Bd} \bigcup_n R_n$, then Ω is an exceptional set. Moreover, there is a homeomorphism $h : S \to S$ such that $h(\Omega)$ is an exceptional ω-limit set for a smooth flow on S.

In the example from the Introduction, the sets R_n are those defining the feasible family from Theorem A, but it is important to stress that the circle C is not the circle E there (in fact it contains all singular points of the flow), but a circle "parallel" to E containing no singular points. More precisely, if in this example R is any of the sets R_n, then (with the notation of Definition 2.2), it intersects C at the arcs $\phi(\{i\} \times [0,1])$, $0 \leq i \leq k$, and intersects E at the arcs $\phi(\{i + 1/2\} \times [0,1])$, $0 \leq i < k$.

Thus, in the example, the corresponding set N in Definition 2.2 is empty, but in general this need not be the case. To understand this, it suffices to replace some points $\phi(i+1/2, 0)$ or $\phi(i+1/2, 1)$ by singular points, and then "blow them up" to full disks of singular points, towards and from which spiral pairs of components of the combinatorial boundary of R. Observe that in such a case, $\operatorname{Cl} R$ is not even "contractible to one point", that is, $S \setminus \operatorname{Cl} R$ is not homeomorphic to S minus one point. We emphasize that such a surgery can be done (in an appropriate way) to infinitely many of the surfaces R_n without adding new singular points apart from those in the newly created disks.

Admittedly the formulation of Theorem A is rather complicated but it is difficult to suggest clear improvements. In fact there are families $\{R_n\}_{n=1}^\infty$ of pairwise disjoint simply connected surfaces in the torus $\mathbb{T}^2$ satisfying two of the three conditions in Definition 2.4 and such that $\operatorname{Bd} \bigcup_n R_n$ is not an ω-limit set for any flow on $\mathbb{T}^2$. More precisely, a counterexample can be found satisfying (ii), (iii) and even with all R_n twisting around some curve C, and also a counterexample (even in the sphere $\mathbb{S}^2$) satisfying (ii) and (iii) and such that $R_m \cup \operatorname{Cl}(\bigcup_n R_n)$ is a neighbourhood of ∂R_m for every m. Similarly, there are families satisfying (i), (iii) and with just (a) or (b) failing in Definition 2.1 for which the converse statement of Theorem A does not hold. One could interpret (iii) as a strong form of connectedness, because there is also a counterexample for the converse statement of Theorem A where (i) and (ii) are satisfied and the set $\operatorname{Bd} \bigcup_n R_n$ is connected. These counterexamples are constructed in full detail in [Soler, 2005; Chapter 3].

As explained two paragraphs above, there is no hope either to improve the topological description of the surfaces R_n. Still, it must be emphasized that if the set of singular points of the flow is totally disconnected, then the feasible family $\{R_n\}_n$ from the first part of Theorem A can be chosen so that $\operatorname{Cl} R_n$ is homeomorphic to a disk for every n. This can be done using ideas and techniques similar to those in [Balibrea & Jiménez, 1998].

Together with Theorems 1.4 and 1.5, Theorem A provides the following full topological characterization of ω-limit sets for surface flows, already announced in [Jiménez & Soler, 2006]:

Corollary B (Structure of ω-limit sets). *Let $\Omega \subset S$. Then Ω is an ω-limit set for some flow on S if and only if one of the following alternatives occurs:*

- *(i) Ω is one of the boundary components of a regular annulus;*
- *(ii) Ω is the closure of a region homeomorphic neither to $\mathbb{S}^2$, nor to $\mathbb{P}^2$ nor to any region in $\mathbb{B}^2$;*
- *(iii) Ω is the boundary of the union of the sets from some feasible family.*

Of course the smoothness part of Theorem A is not optimal because Ω is realized as an ω-limit set of a smooth flow via a homeomorphism. This cannot be helped, because it is easy to provide examples of exceptional sets which are ω-limit sets of no smooth flows. Notice also that, although an exceptional set can always be realized as an ω-limit set by a smooth flow up to homeomorphism, there are flows having exceptional ω-limit sets that are not topologically equivalent to a smooth (or, for that matter, a C^2) flow. (Here we say that two flows Φ, Ψ on S are *topologically equivalent* if there is a homeomorphism $h : S \to S$ mapping the orbits of Φ onto orbits of Ψ which preserves the induced orientations by the flows.) To devise a simple example we use, on the one hand, the exceptional set Ω we constructed at the end of the Introduction to this paper, modifying the flow Φ to include a circle of singular points in one of the components of $\mathbb{T}^2\backslash\Omega$. Similarly we include, on the other hand, a circle of singular points in the dense band for the Denjoy flow Ψ. After taking off the open disks enclosed by the circles and gluing both circles, we get a flow on the double torus, still having Ω as an exceptional ω-limit set. By Gutierrez's theorem [Gutierrez, 1986], this flow is not topologically equivalent to any C^2 flow.

In fact, as our second main result emphasizes, if an excepcional set is an ω-limit set for some flow, then this flow must be pretty "exceptional" as well.

Theorem C. *If $\Omega \subset S$ is exceptional and Φ is a flow on S realizing it as an ω-limit set, then the set of singular points of Φ has infinitely many components.*

It is well known that if the flow Φ is analytic, then its set of singular points has a finite number of components (for a proof see, e.g, [Jiménez & Llibre, 2007; Theorem 4.3]) so, by Theorem C, it cannot have an exceptional set as one of its ω-limit sets. It is important to emphasize that it is quite possible for an analytic flow to have an exceptional ω-limit set: Cherry's flow [Cherry, 1938] is a paradigmatic and remarkable example.

Thus in the analytic setting ω-limit sets admit the following simpler description:

Corollary D. *Let Φ be an analytic flow on S and let Ω be an ω-limit set for Φ. Then either Ω is the boundary component of a regular annulus or it is the closure of a region homeomorphic neither to $\mathbb{S}^2$, nor to $\mathbb{P}^2$ nor to any region in $\mathbb{B}^2$.*

Needless to say, Corollary D is far from providing a topological characterization of ω-limit sets for analytic flows. For instance in $\mathbb{S}^2$ there are many simply connected regions whose boundary cannot be an ω-limit set for any analytic flow. Up to homeomorphisms, the ω-limit sets of analytic flows have been recently characterized for the surfaces $\mathbb{R}^2, \mathbb{S}^2$ and $\mathbb{P}^2$ in [Jiménez & Llibre, 2007].

3. Proofs of the direct statement of Theorem A and of Theorem C

We begin by stating two well-known geometrical lemmas on surfaces, see, e.g., [Jiménez & Soler, 2004; Theorem 2.3 and Lemma 2.4] for the proofs. In the first one M_g (respectively, N_g) denotes a fixed orientable (respectively, nonorientable) connected surface of genus g. Since these surfaces are unique up to homeomorphisms, the sphere $\mathbb{S}^2$, the torus $\mathbb{T}^2$, the projective plane $\mathbb{P}^2$ and the Klein Bottle $\mathbb{B}^2$ are homeomorphic to M_0, M_1, N_1 and N_2, respectively. We denote by S_* and S_{**} the (noncompact) surface which results after removing one and two points, respectively, from S. Again, notice that S_* and S_{**} are uniquely defined up to homeomorphisms. By $R \cong T$ we mean that R and T are homeomorphic.

Lemma 3.1. *Assume that S is connected, let $C \subset S$ be a circle and let g be the genus of S.*

(i) If C is nonorientable (thus $S \cong N_g$), then either $S\backslash C \cong M_{(g-1)/2,*}$ or $S\backslash C \cong N_{g-1,*}$.

(ii) If C is orientable and $S\backslash C$ is connected, then $S\backslash C \cong M_{g-1,**}$ (if $S \cong M_g$), and $S\backslash C \cong M_{(g-2)/2,**}$ or $S\backslash C \cong N_{g-2,**}$ (if $S \cong N_g$).

(iii) If C is orientable and non-null homotopic, and $S\backslash C = O_1 \cup O_2$ for some pairwise disjoint open sets O_1 and O_2, then there are positive integers g_1, g_2 such that $g_1 + g_2 = g$ with $O_i \cong M_{g_i,*}$, $i = 1, 2$ (if $S \cong M_g$), and such that $2g_1 + g_2 = g$ with $O_1 \cong M_{g_1,*}$, $O_2 \cong N_{g_2,*}$, or $g_1 + g_2 = g$ with $O_i \cong N_{g_i,*}$, $i = 1, 2$ (if $S \cong N_g$).

Remark 3.1. If S and C are as in the previous lemma one more (trivial) possibility could arise:

(iv) If C is orientable and null homotopic then $S\backslash C = O_1 \cup O_2$ for some pairwise disjoint open sets O_1 and O_2 such that $O_1 \cong S_*$ and $O_2 \cong \mathbb{S}^2_*$.

Lemma 3.2. *Let R be a connected surface, $R \ncong \mathbb{S}^2$. Then R is simply connected if and only if it contains no non-null homotopic circles.*

Until the end of the section the surface S, the quasiminimal set Ω and the flow Φ on S satisfying $\Omega = \omega_\Phi(u)$ for some $u \in S$ will remain fixed. There is no loss of generality in assuming that S is connected.

Now we need some terminology. We say that a disk $N \subset S$ is a *flow box* for Φ if there is an embedding $\theta : [-1,1] \times [-1,1] \to N$ such that $\theta([-1,1] \times \{s\})$ is an orbit segment of Φ for every $s \in [-1,1]$. It is well known that every nonsingular point has a flow box neighbouring it [Whitney, 1933]. We say that a circle C is *transversal* to Φ if for every $y \in C$ there is a flow box N_y ($\theta_y : [-1,1] \times [-1,1] \to N_y$) such that $\theta_y(0,0) = y$ and $\theta_y(\{0\} \times [-1,1]) = N_y \cap C$. Since u is recurrent, there is an orientable transversal circle C to Φ intersecting $\Phi_u(\mathbb{R})$ [Gutierrez, 1978; Lemma 1.2]. Notice that C is non-null homotopic because u is recurrent and that $C \cap \Omega$ is a Cantor set because Ω is quasiminimal and by [Gutierrez, 1986; Structure Theorem, St. 4]. From now on we also fix the circle C.

In what follows, if $a, b \in C$, then $[a;b]$ will always refer to an arc in C or an orbit segment of $\Phi_u(\mathbb{R})$ with endpoints a and b. What arc we are exactly referring to will be easily inferred from the context. We also mean $[a;a] = \{a\}$. We denote by $\mathcal{A}(C)$ the partition of C made up by all the (possibly degenerate) arcs $[a;b] \subset C$ satisfying one of the following statements:

- $[a;b]$ is the closure of a component of $C \backslash \Omega$;
- $a = b$ and it is not contained in the closure of any component of $C \backslash \Omega$.

Since Ω is both the α- and the ω-limit set of uncountably many orbits in Ω [Cherry, 1937], [Aranson *et al.*, 1996; Theorem 2.1, p. 57], we can assume that the orbit of u if one of them, and also that it does not contain any endpoint of some non-degenerate component of $C \backslash \Omega$. We emphasize that $\mathcal{A}(C) \cong C$ when $\mathcal{A}(C)$ is endowed with the quotient topology (because $C \cap \Omega$ is a Cantor subset of C), see for more details [Gutierrez, 1986; (3) in Proof of Lemma 3.9].

The key tool in the proofs of the direct statement of Theorem A and Theorem C is a paper by Gutierrez [Gutierrez, 1986] where quasiminimal sets were investigated to a great depth. We next detail the part of this work we are interested in.

For the point u having the properties listed above and the circle C, let $f : D \subset C \to C$ be the *forward Poincaré map* induced by the flow Φ, that is, $f(v) = \Phi_t(v)$ with t the first positive number such that $\Phi_t(v) \in C$ (whenever this makes sense). The *forward Gutierrez map* $f_C : \mathcal{D} \subset \mathcal{A}(C) \to \mathcal{A}(C)$ is defined by $f_C([a;b]) = [c;d]$ whenever one of the following two conditions is satisfied:

- $a = b \in \Phi_u(\mathbb{R})$ and $c = d = f(a)$;
- $[a;b] \cap \Phi_u(\mathbb{R}) = \emptyset$ and there are points $\{p_r\}_{r=1}^\infty, \{q_n\}_{r=1}^\infty$ in $\Phi_u(\mathbb{R})$ such that $\lim_{r\to\infty} p_r = a$, $\lim_{r\to\infty} q_r = b$, $\lim_{r\to\infty} f(p_r) = c$, $\lim_{r\to\infty} f(q_r) = d$; moreover, for every r we have that $[a;b] \subset [p_r;q_r]$, $[c;d] \subset [f(p_r); f(q_r)]$, and the circle
$$[p_r;q_r] \cup [p_r; f(p_r)] \cup [f(p_r); f(q_r)] \cup [q_r; f(q_r)]$$
is the boundary of a disk.

The *backward Poincaré map* $g : E \subset C \to C$ and the *backward Gutierrez map* $g_C : \mathcal{E} \subset \mathcal{A}(C) \to \mathcal{A}(C)$ are similarly defined. Notice that f and g, and similarly f_C and g_C, are inverse each other (whenever their composition makes sense). Also, observe that if $[a;b] \neq [c;d]$ are in $\mathcal{D}$, then $f_C([a;b]) \neq f_C([c;d])$ regardless these images belong to $\mathcal{D} \cup \mathcal{E}$ or not, and a similar statement holds for g. Hence we use the notation $g = f^{-1}$ and $g_C = f_C^{-1}$, when $f^i(p)$ and $f_C^i(A)$ have the obvious meaning for every $i \in \mathbb{Z}$.

Now we have:

Theorem 3.3 ([Gutierrez, 1986]). *The following statements hold:*

(i) $\mathcal{A}(C) \setminus (\mathcal{D} \cup \mathcal{E})$ *is finite;*

(ii) any forward or backward full orbit of the map f_C *is dense in* $\mathcal{A}(C)$.

By a *forward* (resp. *backward*) *full orbit* of f_C we mean the sequence $\{f_C^i(A)\}_{i=0}^\infty$ (respectively, $\{f_C^{-i}(A)\}_{i=0}^\infty$) whenever it is well defined.

From now on we assume additionally that Ω is a exceptional set. Then we can get some additional information about the orbit of u:

Lemma 3.4. *Let O be a component of $S \backslash \Omega$. Then* $\Phi_u(\mathbb{R}) \cap \operatorname{Bd} O = \emptyset$.

Proof. Assume the contrary. Then $\operatorname{Bd} O = \Omega$. If O is simply connected and p is an arbitrary point of O, then Ω is one of the boundary components of the regular annulus $O \backslash \{p\}$ (the other one is the point p itself). If O is not simply connected, then we use Lemma 3.1 and Lemma 3.2 finitely many times to find pairwise disjoint non-null homotopic circles $\{C_i\}_{i=1}^n$ in O and a compact surface N, $\partial N = 0$, such that $O \backslash \bigcup_{i=1}^n C_i$ is homeomorphic to a region $V \subset N$ with the property that $V \cup \{q_j\}_{j=1}^l$ is simply connected for some points q_j from N. Let $h : O \backslash \bigcup_{i=1}^n C_i \to V$ denote the corresponding homeomorphism, find a circle $D \subset V$ enclosing the points q_j and let U be the component of $O \backslash (h^{-1}(D) \cup \bigcup_{i=1}^n C_i)$ containing Ω in its boundary. Clearly, U is a regular annulus and Ω is one of its two boundary components (being $h^{-1}(D)$ the other one).

Thus, in any case, we have proved that Ω is one of the boundary components of a regular annulus in S. This is impossible because Ω is exceptional. ∎

The exceptionality of Ω also implies that the Gutierrez maps have some special properties. Namely, let $A \in \mathcal{D}$ (respectively, $A \in \mathcal{E}$) be a nondegenerate arc. We say that A is *forward consistent* (respectively, *backward consistent*) if $f_C(A) \in \mathcal{D}$ (respectively, $f_C^{-1}(A) \in \mathcal{E}$) is also nondegenerate and there is a component of $S \backslash \Omega$ intersecting both A and $f_C(A)$ (respectively A and $f_C^{-1}(A)$). Then we have:

Lemma 3.5. *The map f_C has neither forward nor backward full orbits of, respectively, forward and backward consistent arcs.*

Proof. Suppose not. Then some of the sequences $\{f_C^i(A)\}_{i=0}^\infty$ or $\{f_C^{-i}(A)\}_{i=0}^\infty$ is well defined and hence dense in $\mathcal{A}(C)$ (Theorem 3.3(ii)); moreover it consists exclusively of nondegenerate arcs. Since there is a component O of $S \setminus \Omega$ intersecting all arcs of this dense sequence, we get $\Phi_u(\mathbb{R}) \cap C \subset \operatorname{Bd} O$, which contradicts Lemma 3.4. ∎

Let $\{O_k\}_k$ denote the family of components of $S \backslash \Omega$ intersecting C at some arc from $\mathcal{D} \cap \mathcal{E}$, but at no arc from $\mathcal{A}(C) \setminus (\mathcal{D} \cup \mathcal{E})$. Let $O = O_k$ for some k and fix an arc $A \in \mathcal{D} \cap \mathcal{E}$ contained in O. According to Lemma 3.5, there are (minimal) nonnegative integers l, m such that $f_C^l(A)$ is not forward consistent and $f_C^{-m}(A)$ is not backward consistent. Notice that the sets $f_C^{-m}(A)$ and $f_C^l(A)$ also belong to $\mathcal{D} \cap \mathcal{E}$ and their definitions only depend on O. We denote them in the next lemma by, respectively, A_k and Z_k.

Lemma 3.6. *Let $O = O_k$ for some k. Then the following statements hold:*

(i) If $\mathcal{B}$ is the family of orbits of Φ in $\operatorname{Bd} O$ intersecting C, then $T = O \cup \bigcup_{B \in \mathcal{B}} B$ is a simply connected surface;

(ii) $\operatorname{Cl} O$ contains some singular point;

(iii) if $U = O_r$ for some $r \neq k$ and $A_k \neq Z_r$, $Z_k \neq A_r$, then there are disjoint disks D and E such that $O \subset D$, $U \subset E$, and $\operatorname{Bd} D \cup \operatorname{Bd} E$ does not contain any singular point.

Proof. Let A, l and m as before. For every $-(m+1) \leq i \leq l+1$ write $f_C^i(A) = [a_i; b_i]$ and find points $\{p_r\}_{r=1}^\infty, \{q_r\}_{r=1}^\infty$ in $\Phi_u(\mathbb{R})$ satisfying $\lim_{r\to\infty} f^i(p_r) = a_i$, $\lim_{r\to\infty} f^i(q_r) = b_i$, $[a_i; b_i] \subset [f^i(p_r); f^i(q_r)]$ for every r, and such that every circle

$$\begin{aligned}&[f^{-(m+1)}(p_r); f^{-(m+1)}(q_r)]\\ &\cup [f^{-(m+1)}(p_r); f^{l+1}(p_r)] \cup [f^{-(m+1)}(q_r); f^{l+1}(q_r)]\\ &\cup [f^{l+1}(p_r); f^{l+1}(q_r)]\end{aligned}$$

is the boundary of a disk D_r intersecting C exactly at the sets $[f^i(p_r); f^i(q_r)]$. Due to Lemma 3.4 and the specific properties of O (take also into account the transversality of C), $\operatorname{Cl} O$ does not intersect any of the circles $\operatorname{Bd} D_r$, hence $\operatorname{Cl} O \subset \operatorname{Int} D_r$ for every r.

We prove (i). We have that $\operatorname{Cl} T$ intersects the transversal C exactly at the arcs $[a_i; b_i]$, $-m \leq i \leq l$. Thus T is locally homeomorphic to a disk at all points a_i, b_i, $-m \leq i \leq l$ (and hence at all points from T because the definition of $\mathcal{B}$), that is, T is a surface. Furthermore, O (and then T) is simply connected because Ω (and then $S \setminus O$) is connected, so $D_1 \setminus O$ is connected as well.

We prove (ii). Let $v \in \operatorname{Bd} O$. If $\omega_\Phi(v)$ is a periodic orbit, then $\Omega = \omega_\Phi(u)$ contains a periodic orbit hence it is a periodic orbit itself (see, e.g., [Aranson *et al.*, 1996; Lemma 1.6, p. 41]), a contradiction because u is nontrivial recurrent. Then

$\omega_\Phi(v)$ must contain a singular point (recall that $\mathrm{Cl}\, O$ is contained in a disk so we can use a standard Poincaré-Bendixson argument) and we are done.

To prove (iii) we define, with respect to the region U, the sets $[c_j; d_j]$ and the disks E_s in similar fashion to the sets $[a_i; b_i]$ and the disks D_r with respect to O. By the hypothesis all sets $[a_i; b_i]$, $[c_j; d_j]$ are pairwise disjoint, hence there are r and s such that D_r and E_s are also disjoint. These are the disks D and E we are looking for. ■

We are ready to prove the first statement of Theorem A. Let $\{R_n\}_n$ denote the family of surfaces R such that $\mathrm{Int}\, R = O_k$ for some k, ∂R is the set of points from $\mathrm{Bd}\, R$ whose orbit intersects C, and $m \geq 1$ or $l \geq 1$ with the notation before Lemma 3.6. Notice that all R_n are well-defined simply connected surfaces by Lemma 3.6(i). Moreover, the condition on the numbers m and l ensures that they have twisting sections, that is, they twist around C. Furthermore, they are pairwise disjoint.

Observe that, by Theorem 3.3(i), there are at most finitely many components of $S \setminus \Omega$ intersecting C which also intersect some arc from $\mathcal{A}(C) \setminus (\mathcal{D} \cup \mathcal{E})$. The closure of the union of such components cannot intersect $\Phi_u(\mathbb{R})$ (Lemma 3.4). Then every arc from $\mathcal{A}(C)$ close enough to a given point of $\Phi_u(\mathbb{R}) \cap C$ is included in a twisting section of some R_n (here we also use the continuity of Φ and the transversality of C). This implies $\Omega = \mathrm{Bd} \bigcup_n R_n$ (and then, because of Lemma 3.4, that the family $\{R_n\}_n$ is infinite), that $R_m \cup \mathrm{Cl}\,(\bigcup_n \Upsilon(R_n))$ is a neighbourhood of ∂R_m for every m, and property (iii) in Definition 2.4. Furthermore, the existence of a flow box neighbouring every nonsingular point of S and the fact that every given component of ∂R_n is a set $\Phi_v(I)$ for some $v \in S$ and some open interval I, guarantee property (ii) in Definition 2.4. Hence $\{R_n\}_n$ is a feasible family for C and the proof of the direct part of Theorem A is finished.

Theorem C is a direct consequence of Lemma 3.6(ii) and (iii), because these statements, together with the infiniteness of the family $\{O_k\}_k$, immediately imply that the set of singular points of Φ has infinitely many components.

4. Proof of the converse statement of Theorem A

We recall that a curve B in S was defined in Section 2 as the image $B = \varphi(I)$ of a continuous one-to-one map $\varphi : I \to S$ for either an interval or a circle I. We call φ a *parametrization* of B. In the particular case when $I = [a, b]$ we get an arc with endpoints $\varphi(a) = u$ and $\varphi(b) = v$ to which we sometimes refer as $[u; v]$. We emphasize that if I is not compact, then the immersion $\varphi : I \to S$ need not be an embedding, that is, the restriction $\varphi : I \to B$ need not be a homeomorphism.

We say that two parametrizations $\varphi_i : I_i \to B$, $i \in \{1, 2\}$, *induce the same orientation* on B if $\varphi_2^{-1} \circ \varphi_1 : I_1 \to I_2$ is an increasing homeomorphism. This defines an equivalence relation in the family of parametrizations of B and each of its equivalence classes is called an *orientation* of B, any of the maps belonging to one such class being called a parametrization *compatible* with that orientation. After associating a curve one of its orientations we get an *oriented* curve. Notice that some curves (e.g. the "eight figure") admit more than two orientations; still, all curves we will use admit just two. If A is an oriented arc and $\varphi : [a, b] \to A$ is a parametrization of A compatible with its orientation, then we call $\varphi(a) = u$ and $\varphi(b) = v$, respectively, the *initial* and the *final* endpoint of A and use the notation $[u, v]$, rather than $[u; v]$, to represent A.

Recall also that if A, B are arcs, then the Hausdorff distance between A and B is defined by

$$d_H(A, B) = \max\{\max_{u \in A} \min_{v \in B} \mathrm{dist}(u, v), \max_{v \in B} \min_{u \in A} \mathrm{dist}(u, v)\}.$$

If in addition A and B are oriented, then the *Fréchet distance* between A and B is defined by

$$d_F(A, B) = \inf_{h \in \mathcal{H}} \max_{u \in A} \mathrm{dist}(u, h(u)),$$

where $\mathcal{H}$ is the family of all homeomorphisms $h : A \to B$ preserving the orientations of A and B. Here we mean that h *preserves the orientations* of A and B if whenever $\varphi : I \to A$ is compatible with the orientation of A, $h \circ \varphi$ is compatible with that of B.

To illustrate the difference between the Hausdorff and Fréchet distances consider an horizontal arc A, three horizontal arcs B_1, B_2, B_3 very close to

A and the "S"-shaped arc B arising after connecting the right endpoints of B_1 and B_2 and the left endpoints of B_2 and B_3 with small arcs. The number $d_H(A, B)$ is very small; the number $d_F(A, B)$ is not.

Also in Section 2 we introduced the notion of Whitney regular family of curves (Definition 2.1), which played an important role in the definition of feasible family and hence in the formulation of Theorem A. It was formulated in the early thirties by Whitney when looking for conditions to ensure that a curve foliation gives rise to a flow [Whitney, 1933]. Next we explain Whitney's approach in some detail.

Assume that all curves from the family $\mathcal{B}$ are immersions of $\mathbb{R}$ or $\mathbb{S}^1$, are pairwise disjoint and fill an open subset U of S (we then say that $\mathcal{B}$ is *full*). Then Whitney showed in [Whitney, 1933] that Whitney regularity is equivalent to the property that for every $p \in B \in \mathcal{B}$ there are a neighbourhood U of p in O and a homeomorphism $h : [0, 1] \times [0, 1] \to U$ such that every arc $h([0, 1] \times \{x\})$, $x \in [0, 1]$ is contained in some curve from $\mathcal{B}$. (Incidentally, he showed that property (a) in Definition 2.1 is redundant some years later [Whitney, 1941] — this need not be the case if the family $\mathcal{B}$ is not full.)

Furthermore, assume that in Definition 2.1 all curves from $\mathcal{B}$ are oriented (with their subarcs inheriting the corresponding orientations) and replace (b) by

(b)' if $[p, q] \subset O \cap B$ (respectively, $[q, p] \subset O \cap B$), $p' \in O \cap B'$ for some $B' \in \mathcal{B}$ and $\mathrm{dist}(p, p') < \delta$, then there is some $[p', q'] \subset O \cap B'$ (resp. $[q', p'] \subset O \cap B'$) such that $d_F([p, q], [p', q']) < \epsilon$ (respectively, $d_F([q, p], [q', p']) < \epsilon$)

(when we call $\mathcal{B}$ a *full Whitney regular orientable family*). Then Whitney proved in [Whitney, 1933] (see [Nikolaev, 2001; pp. 216–222] for a recent reference) that there is a flow realizing $\mathcal{B}$ as its set of nonsingular orbits. More precisely:

Theorem 4.1 ([Whitney, 1933]). *Let $\mathcal{B}$ be a full Whitney regular orientable family and let U denote the union set of the curves from $\mathcal{B}$. Then there is a flow Φ on S such that every curve from $\mathcal{B}$ is an orbit of Φ. Moreover, the points from $S \setminus U$ are exactly the singular points of Φ and the orientation of every curve from $\mathcal{B}$ coincides with that induced by Φ.*

Later on we will resort to Theorem 4.1 to construct the required flow in the converse statement of Theorem A. This construction is rather involved, so we have divided it into several steps. To begin with, we must still introduce another variation of Whitney regularity, its main feature described by Lemma 4.3.

Definition 4.2. Let $\mathcal{B}$ be a family of curves in S and let $D \subset S$. We say that $\mathcal{B}$ is *almost regular in D* if for every $\epsilon > 0$ there is $\delta > 0$ such that if $[p'; q'] \subset D \cap B'$ for some $B' \in \mathcal{B}$ and $\mathrm{dist}(p', q') < \delta$, then $\mathrm{diam}([p'; q']) < \epsilon$.

If $\mathcal{B}$ is almost regular in S, then we just say that $\mathcal{B}$ is *almost regular*.

Observe that this is just property (a) in Definition 2.1. In particular, if D is a Whitney regular neighbourhood for some family of curves $\mathcal{B}$, then $\mathcal{B}$ is almost regular in D (and the family of the intersection curves of $\mathcal{B}$ with D is almost regular).

Lemma 4.3. *Let $\mathcal{B}$ be an almost regular family of arcs. Then for every $B \in \mathcal{B}$ there are parametrizations $\varphi_B : [0, 1] \to B$ such that the family $\{\varphi_B\}_{B \in \mathcal{B}}$ is equicontinuous.*

Proof. Find finite decompositions $\mathcal{P}^r$ of S into disks with pairwise disjoint interiors and diameters at most $1/r$, $r \in \mathbb{N}$. We can suppose that each $\mathcal{P}^{r+1}$ refines $\mathcal{P}^r$, that is, if $T \in \mathcal{P}^{r+1}$, then there is $P \in \mathcal{P}^r$ such that $T \subset P$. Now, for every given arc $B \in \mathcal{B}$, it is easy to construct inductively partitions $\mathcal{Q}^r = \mathcal{Q}^r_B$ of the interval $[0, 1]$, with each $\mathcal{Q}^{r+1}$ refining $\mathcal{Q}^r$, and define correspondingly the map φ_B, in such a way that all intervals from $\mathcal{Q}^1$ have the same length, all intervals from $\mathcal{Q}^{r+1}$ contained in the same interval from $\mathcal{Q}^r$ have the same length, and, finally, for every interval $[a, b] \in \mathcal{Q}^r$ there is a disk $T \in \mathcal{P}^r$ such that $\{\varphi_B(a), \varphi_B(b)\} \subset T$ and $\varphi_B([0, 1] \setminus [a, b]) \cap T = \emptyset$. The equicontinuity of the maps φ_B follows from Definition 4.2 and the fact that each interval from a partition $\mathcal{Q}^r_B$ has at least length $1/(\Pi_{s=1}^r \operatorname{Card} \mathcal{P}^s)$. ∎

Throughout this section, $\{R_n\}_n$ denotes a feasible family for some circle C and $\mathcal{A}$ is the family of components of all combinatorial boundaries ∂R_n of the surfaces R_n. Also, K denotes the ternary Cantor set, with $\{(a_i, b_i)\}_{i=1}^{\infty}$ being the family of components of $[0,1]\backslash K$ and $N = \bigcup_{i=1}^{\infty}\{a_i, b_i\}$. When speaking about a *partition* of K we refer to the corresponding decomposition of K into pairwise disjoint compact sets after intersecting K with a finite family of pairwise disjoint compact intervals covering it. As before, if $\mathcal{P}, \mathcal{Q}$ are partitions of K, then we say that $\mathcal{Q}$ *refines* $\mathcal{P}$ if for every $L \in \mathcal{Q}$ there is $M \in \mathcal{P}$ such that $L \subset M$.

4.1. Regularizing the boundaries of the surfaces R_n

Subsection 4.1 is the most difficult part of our construction. Here we show that every arc A in the combinatorial boundary of each of the surfaces R_n is "regularizable". Essentially this means that, near to and together with A, the family of twisting boundaries of all twisting sections of the surfaces R_n is rectifiable. The union set of these twisting boundaries has empty interior, so there is no hope of embedding any square $[0,1]\times[0,1]$ in it; thus we are looking for an embedding of $K\times[0,1]$ instead. Indeed in such a case one gets more or less by force an embedding of $[0,1]\times[0,1]$ into the union set of A and the twisting sections of the surfaces R_n; once this is done it is relatively easy to define the desired flow by means of Theorem 4.1 (see Subsection 4.2). At this stage only properties (i) and (ii) in Definition 2.4 are used; we will need Definition 2.4(iii) at the very end of proof to show that the flow we have just constructed satisfies the required properties.

Definition 4.4. Let $A' \subset \partial R_m$ be an arc for some m. We say that A' is *regularizable* if there is an embedding $h : [0,1]\times[0,1] \to S$ such that:

(i) $h([0,1]\times\{0\}) = A'$ and $h([0,1]\times\{1\}) \subset \Upsilon(\partial R_l)$ for some l;

(ii) for every i there is R_k such that $h([0,1]\times(a_i,b_i)) \subset \Upsilon(R_k)\cap \operatorname{Int} R_k$ and $h([0,1]\times\{a_i,b_i\}) \subset \Upsilon(\partial R_k)$.

We call every such map h a *regularization* of A'.

If A is an arc and $p \in A$ is not an endpoint of A, then we call p an *inner point* of A. We call the union set of all inner points of A its *inner set* and denote it by $\operatorname{Inn} A$. If $A = [u;v]$, then we also write $\operatorname{Inn} A = (u;v)$.

Lemma 4.5. *Let $p \in \partial R_m$ for some m. Then there are arcs A, B, Y, Z, an open neighbourhood U of p, and a family of arcs $\mathcal{E}$ such that:*

(i) the inner sets of the arcs A, B, Y, Z are pairwise disjoint, their union set is the boundary of a disk D, and $\mathcal{E}$ is the family of arcs in $\bigcup_n \Upsilon(\partial R_n)$ lying in D and intersecting $\operatorname{Bd} D$ at exactly one point of Y and one point of Z; moreover:

- *$p \in \operatorname{Inn} A$ and $A \subset \partial R_m$,*
- *$B \subset \Upsilon(\partial R_l)$ for some l,*
- *$D \cap R_m = A$ and $D \cap R_l \cap W = B$ for some neighbourhood W of B,*
- *there is an arc $[p;p'] \subset U \cap D$ with $(p;p') \subset \operatorname{Int} D$ and $p' \in \operatorname{Inn} B$;*

(ii) if $u \in U \cap \operatorname{Int} D \cap \bigcup_n \Upsilon(\partial R_n)$, then there is $E \in \mathcal{E}$ such that $u \in E$;

(iii) $\bigcup_n \Upsilon(R_n)$ is dense in D;

(iv) $\mathcal{E}$ is almost regular in D; in particular, $\mathcal{E} \cup \{A, B\}$ is an almost regular family.

Proof. The main ideas of the proof are illustrated by Figure 1. Let O be a small open disk neighbouring p (whose closure is a disk) such that, by Definition 2.4(ii),

$$\mathcal{A} \text{ is almost regular in } \operatorname{Cl} O \tag{1}$$

and, by Definition 2.4(i),

$$O \subset R_m \cup \operatorname{Cl}\left(\bigcup_n \Upsilon(R_n)\right). \tag{2}$$

We also assume that there is an arc $A = [r;q] \subset \partial R_m$ with $p \in (r;q) \subset O$ and $r, q \in \operatorname{Bd} O$, which disconnects O into two open disks, one of them included in $\operatorname{Int} R_m$ and the other one, call it V, disjoint with R_m. In particular observe that if $u \in V \cap \operatorname{Bd} R_n$ for some $n \neq m$, then $u \in \Upsilon(\partial R_n)$ and there is an arc $[r_u;q_u] \in \operatorname{Cl} O \cap \Upsilon(\partial R_n)$ with $u \in (r_u;q_u) \subset O$ and $r_u, q_u \in \operatorname{Bd} V\backslash A$. Fix now a much smaller open neighbourhood U of p and denote by $\mathcal{D}$ the family of all arcs $[r_u;q_u]$ with the additional property that $u \in U$. Then

$$\inf_{X\in\mathcal{D}} \operatorname{diam} X > 0. \tag{3}$$

Let us define in $\mathcal{D}$ an equivalence relation $\approx$ as follows. If $X, X' \in \mathcal{D}$ then we say that $X \approx X'$ if either $X = X'$ or the following two conditions hold:

- either X is not contained in the disk in $\mathrm{Cl}\,V$ delimited by X' and A, or X' is not included in the disk in $\mathrm{Cl}\,V$ delimited by X and A;
- the disk in $\mathrm{Cl}\,V$ enclosed by X, X' and the (possibly degenerated) disjoint arcs $E, E' \subset \mathrm{Bd}\,V \backslash A$ connecting their endpoints contains no arc from $\mathcal{D}$ intersecting just one of the sets E, E'.

We denote by $[X]$ the equivalence class containing X. Also, we introduce in $\mathcal{D}/\approx$ a partial order relation $\prec$ by saying that $[X'] \prec [X]$ if $[X'] \neq [X]$ and $\mathrm{Inn}\,X'$ is contained in the component of $\mathrm{Cl}\,V \backslash X$ not including A. Clearly, both $\approx$ and $\prec$ are well defined.

We claim that there is a maximum for this ordering. Notice that (1) and (3) imply that every infinite subset of $\mathcal{D}/\approx$ contains some pair $[X'] \prec [X]$. In fact, if $\mathcal{D}/\approx$ is infinite, then it is easy to construct either an infinite increasing chain $[X_1] \prec [X_2] \prec [X_3] \prec \dots$ or an infinite decreasing chain $[X_1] \succ [X_2] \succ [X_3] \succ \dots$. On the other hand, observe that if $[X'] \prec [X]$, then there is $[X'']$ with $[X''] \prec [X]$ and $[X'] \not\prec [X'']$, $[X''] \not\prec [X']$, and we can use the previous chain to construct an infinite subset of $\mathcal{D}/\approx$ such that $[X] \not\prec [X']$ for each of its pairs of points $[X], [X']$, a contradiction. Thus $\mathcal{D}/\approx$ is finite and (2) implies that p is in the closure of the union set of all arcs from some $[X] \in \mathcal{D}/\approx$.

Next we show that $[X]$ is the maximum for $\approx$. To do this, take a sequence $X_r \in [X]$, $r \in \mathbb{N}$, approaching p, and such that any X_{r+1} lies between A and X_r. Since the arcs X_r are contained in $\mathrm{Cl}\,V \subset \mathrm{Cl}\,O$, (1) implies that the family $\{X_r\}$ is almost regular. By Lemma 4.3 and the Arzelá theorem, we can assume that the corresponding parametrizations $\varphi_{X_r} : [0,1] \to X_r \subset V$ converge uniformly to some continuous map $\varphi : [0,1] \to V$. Moreover, (1) implies that if $\varphi(a) = \varphi(b) = u$, then $\varphi([a,b]) = \{u\}$. Hence $\varphi([0,1])$ is an arc. Indeed, in view of Definition 2.4(ii), we can assume that there is a neighbourhood of $\mathrm{Cl}\,O$ which is a regular neighbourhood of p and then $A = \varphi([0,1])$ (this is the only moment in the whole proof where we need property (b) of regular neighbourhood in Definition 2.1). This clearly implies that $[X]$ is a maximum for $\prec$.

Say $X \subset \Upsilon(\partial R_l)$. Slightly modifying O if necessary, we can assume that either the open disk R enclosed by A, X and corresponding arcs in $\mathrm{Bd}\,V$ does not contain any points of $\mathrm{Int}\,R_l$ close enough to X, or that all points of R close enough to X belong to $\mathrm{Int}\,R_l$. It is not restrictive to assume that the first case occurs, since otherwise there is an arc $X' \in [X]$ in $\mathrm{Cl}\,R$ that can be connected to X via an arc F in $\Upsilon(R_l) \cap \mathrm{Cl}\,R \cap U$ with $\mathrm{Inn}\,F \subset \mathrm{Int}\,R_l$ (in particular $X' \subset \Upsilon(\partial R_l)$), and again possibly modifying O we can assume that the open disk enclosed by A, X' and the corresponding arcs in $\mathrm{Bd}\,V$ does not contain any points of $\mathrm{Int}\,R_l$ close enough to X'. We can also assume that there is a curve $(p; p') \subset U \cap R$ with $p' \in \mathrm{Inn}\,X$. Now it suffices to take $B = X$ and choose as Y and Z the abovementioned arcs in $\mathrm{Bd}\,V$, with $D = \mathrm{Cl}\,R$, and $\mathcal{E}$ being the family of all arcs from $[X] \backslash \{X\}$ included in D. Indeed, the definition of $[X]$ and its condition of maximum for $\prec$ guarantee that $\mathcal{E}$ is the family of arcs from $\mathcal{D}$ contained in $D \backslash (A \cup B)$, and then the family of arcs in $\bigcup_n \Upsilon(\partial R_n)$ lying in D and intersecting $\mathrm{Bd}\,D$ at exactly one point from Y and one point from Z (since any such arc must intersect $(p; p')$ and hence belong to $\mathcal{D}$); thus (i) and (ii) are satisfied. Properties (iii) and (iv) follow from (1) and (2). ∎

In the following four lemmas the point $p \in \partial R_m$ will remain fixed. Our immediate aim is to show that there is a regularizable arc $A' \subset \partial R_m$ with $p \in \mathrm{Inn}\,A'$ (Lemma 4.9). Until then we use the notation of Lemma 4.5 and assume that all arcs from $\mathcal{E}$ are oriented so that their initial endpoints lie in Y.

Let $\mathcal{F}$ denote the family of all oriented arcs in D with initial endpoint in Y and final endpoint in Z and endow it with a structure of metric space by using the Fréchet distance d_F. We introduce in $\mathcal{F}$ a partial order relation $<$ as follows: $F < F'$ if $F \neq F'$ and the sets in D enclosed by A and F and by F' and B have pairwise disjoint interiors. Observe that $<$ is a total ordering in $\mathcal{E}$.

Lemma 4.6. *Let $(E_r)_r \subset \mathcal{E}$ be an increasing (respectively, decreasing) sequence and, for every r, let $D_r \subset D$ be the disk delimited by A (resp. B) and E_r. Then $\mathrm{Cl}(\bigcup_r D_r)$ is a disk in D delimited by A and an arc $E \in \mathcal{F}$ and $(E_r)_r$ converges to E in $(\mathcal{F}, d_F)$.*

Proof. Assume for instance that $(E_r)_r$ is increas-

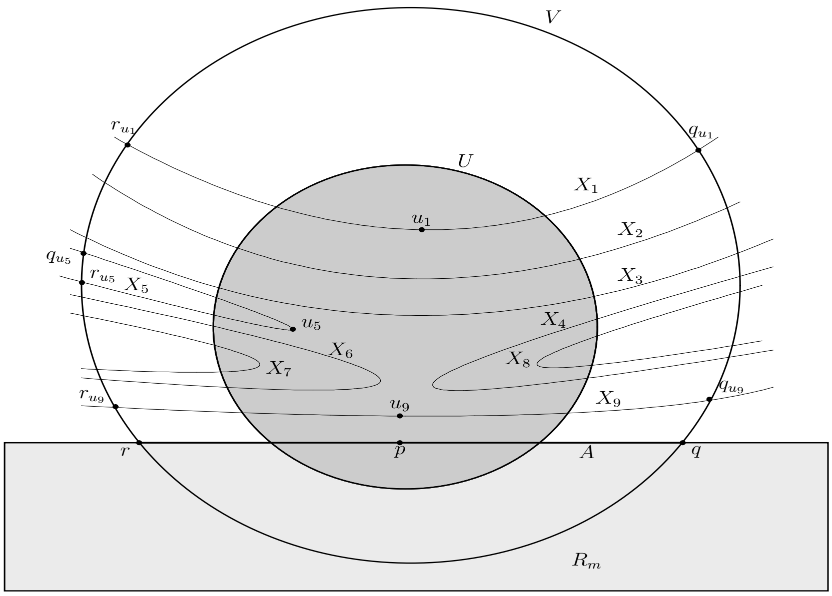

Fig. 1. In the above figure the only possible equivalences are $X_6 \approx X_7$, $X_4 \approx X_8$ and $X_1 \approx X_2 \approx X_3$; if this is so, then the only inequalities are $[X_1] \prec [X_9]$, $[X_4] \prec [X_9]$, $[X_5] \prec [X_9]$, $[X_6] \prec [X_9]$ (thus, for instante, $[X_1] \not\prec [X_4]$ and $[X_4] \not\prec [X_1]$).

ing. Using the equicontinuity of appropriate parametrizations $\varphi_r = \varphi_{E_r} : [0,1] \to E_r$ we can find a subsequence $(\varphi_{r_l})_l$ converging uniformly to some $\varphi : [0,1] \to D$ with $E = \varphi([0,1])$ an arc (Lemma 4.5(iv) and Lemma 4.3). Moreover $E \in \mathcal{F}$ and, because of the monotonicity, the disk in D delimited by A and E is $\mathrm{Cl}(\bigcup_{r_l} D_{r_l}) = \mathrm{Cl}(\bigcup_r D_r)$. A similar argument shows that every subsequence of $(E_r)_r$ has a sub-subsequence converging to E. This suffices to guarantee the convergence of the whole sequence $(E_r)_r$. ∎

Lemma 4.7. *There is an order preserving bijective map $\Sigma : N \to \mathcal{E}$ such that, for every i, there is $k = k(i)$ satisfying $\Sigma(a_i), \Sigma(b_i) \in \Upsilon(\partial R_k)$.*

Proof. Take an arbitrary arc $E \in \mathcal{E}$, say $E \subset \Upsilon(\partial R_k)$. Because of Lemma 4.5(i), (ii) and (iii) we can find an arc F in $\Upsilon(R_k) \cap D \cap U$ with its inner set in $\mathrm{Int}\, R_k$, and having one of its endpoints in E and the other one in some $E' \in \mathcal{E}$ with $E' \subset \Upsilon(\partial R_k)$. In particular, the disk in D delimited by E and E' encloses no other arc from $\mathcal{E}$. Say for instance $E < E'$. Then we define $\Sigma(a_1) = E$, $\Sigma(b_1) = E'$. Repeatedly using this procedure and taking Lemma 4.5(i) and (iii) into account we easily get the desired map Σ. ∎

In view of Lemmas 4.6 and 4.7, the arcs $F_x^-, F_x^+ \in \mathcal{F}$ given by $F_x^- = \lim_{z \to x, z \in N, z < x} \Sigma(z)$ and $F_x^+ = \lim_{z \to x, z \in N, z > x} \Sigma(z)$ are well defined for every $x \in K$. Here we mean $F_z^+ = \Sigma(z)$ (respectively, $F_z^- = \Sigma(z)$) if $z = a_i$ (respectively, $z = b_i$) for some i, and also $F_0^+ = A$, $F_1^- = B$. Clearly, we have $F_x^- \leq F_x^+ < F_y^- \leq F_y^+$ whenever $x < y$. Furthermore, because of Lemma 4.5(ii), (iii) we have for every x:

- $F_x^- \cap F_x^+ \neq \emptyset$;
- neither the first nor the last intersection point of F_x^- and F_x^+ lie in U;
- there are no arcs in F_x^- and F_x^+ exactly intersecting at their endpoints.

Thus F_x^- and F_x^+ intersect exactly at an arc F_x that contains all points from $(F_x^- \cup F_x^+) \cap U$. From the construction it is clear that the family $\{F_x^-, F_x^+\}_{x \in K}$ is almost regular. Hence, if we define in $\{F_x^-, F_x^+ : x \in K\} \subset \mathcal{F}$ an equivalence relation $\sim$ by identifying F_x^- and F_x^+ for any $x \in K$, we can reuse the argument from Lemma 4.6 to prove that the map $\Gamma : K \to \{F_x^-, F_x^+ : x \in K\}/\sim$ given by $\Gamma(x) = [F_x^-]_\sim = [F_x^+]_\sim$ is a homeomorphism.

Next we improve this result by introducing the family $\mathcal{G}$ of all subarcs of the arcs F_x, $x \in K$. After orienting the arcs from $\mathcal{G}$ in the natural way and endowing it with the Fréchet distance, $\mathcal{G}$ becomes a metric space. As it turns out:

Lemma 4.8. *There is a continuous map $\Theta : K \to \mathcal{G}$ such that $\Theta(x) \subset U$ and $\Theta(x)$ is a subarc of F_x for every $x \in K$, and $p \in \operatorname{Inn} \Theta(0)$.*

Proof. Let $[p; p']$ be as in Lemma 4.5(i) and let $\epsilon > 0$ be small enough such that all points of D which are at a distance less than 2ϵ of $[p; p']$ belong to U. Using the uniform continuity of Γ we can construct a sequence of partitions $\mathcal{K}^r$ of K, $r \in \mathbb{N}$, with every $\mathcal{K}^{r+1}$ refining $\mathcal{K}^r$, and arcs $G_x^r \subset F_x$, such that:

- every G_x^1 has some inner point in $[p; p']$ and its endpoints are at a distance at less $\epsilon/2$ and at most $3\epsilon/2$ from $[p; p']$;
- $d_F(G_x^r, G_y^r) < \epsilon/2^{r+1}$ for every pair of points x, y belonging to the same $L \in \mathcal{K}^r$;
- $G_x^r = G_x^{r+1}$ whenever x is simultaneously the smallest point of some $L \in \mathcal{K}^r$ and some $M \in \mathcal{K}^{r+1}$.

It is routine to check that every sequence $(G_x^r)_r$ converges to some nondegenerate arc $G_x \subset F_x$ and that the map $\Theta(x) := G_x$ is continuous. We emphasize that p is the inner point to G_0^1 belonging to $[p; p']$; hence $p \in \operatorname{Inn} G_0$. ■

Lemma 4.9. *There is a regularizable arc in ∂R_m having p as an inner point.*

Proof. First we explain how to define the map h at $[0,1] \times K$ so that its image is the union of all arcs from $\Theta(K)$, with Θ the map from Lemma 4.8; later on we extend it to the rest of points of $[0,1] \times [0,1]$.

Since Θ is continuous, there is $d_1 > 0$ such that $\operatorname{diam}(\Theta(x)) > d_1$ for every $x \in K$. We claim that there are an integer $l(1)$, a number $0 < \epsilon < \frac{1}{2}$ and finite sets $P_x \subset \Theta(x)$, with $\operatorname{Card} P_x = l(1) + 1$ and P_x containing the endpoints of $\Theta(x)$, such that the distance between consecutive points of any P_x is greater than ϵ and less than $1/2$.

To prove this we proceed as follows. Find $\epsilon' > 0$ small enough so that all subarcs of the arcs $\Theta(x)$ with endpoints at a distance at most $2\epsilon'$ have diameters less than $\min\{d_1, 1/2\}$: this is possible because the family $\{\Theta(x)\}_x$ is almost regular. Now it is easy to construct sets $P'_x \subset \Theta(x)$ containing the endpoints of $\Theta(x)$ such that the distance between every pair of consecutive points of P'_x is at most $2\epsilon'$ and the distance between *every pair* of points of P'_x is at least ϵ': for instance, the closest point of P'_x to the initial endpoint of $\Theta(x)$ is the last point in $\Theta(x)$ which is exactly at a distance ϵ' from its initial endpoint.

Since D is bounded, there is a number $l(1)$ such that every subset in D whose points are pairwise separated by a distance at least ϵ' has cardinality at most $l(1)$. Now, to construct the set P_x, we fix two consecutive points u and v of P'_x, add to P'_x a point w_1 in $\Theta(x)$ between u and v with $\operatorname{dist}(u, w_1) = \operatorname{dist}(v, w_1)$, then add a point w_2 between u and w_1 with $\operatorname{dist}(u, w_2) = \operatorname{dist}(w_1, w_2)$ and so on, until getting a set of the desired cardinality $l(1) + 1$. Clearly, the number $\epsilon = \epsilon'/2^{l(1)}$ does the job.

Because of the uniform continuity of Θ, there is a partition $\mathcal{L}^1$ of K such that if $x, y \in L \in \mathcal{L}^1$, then $d_F(\Theta(x), \Theta(y)) < \epsilon/4$. Now, starting from the points of the above sets P_x, it is simple to reason as in Lemma 4.8 to find continuous maps $\Theta_{j,L}^1 : L \to \mathcal{G}$ for every $L \in \mathcal{L}^1$ and every $1 \le j \le l(1)$, such that $\Theta_{j,L}^1(x) \subset \Theta(x)$ and, moreover, the initial endpoint of $\Theta_{1,L}^1(x)$ is the initial endpoint of $\Theta(x)$, the final endpoint of $\Theta_{l(1),L}^1(x)$ is the final endpoint of $\Theta(x)$, and the final endpoint of $\Theta_{j,L}^1(x)$ is the initial endpoint of $\Theta_{j+1,L}^1(x)$, $1 \le j < l(1)$. Furthermore, we can assume that $\operatorname{diam}(\bigcup_{x \in L} \Theta_{j,L}^1(x)) < 1$ for every j and every $L \in \mathcal{L}^1$.

More generally, iterating this process it is possible to find a sequence $(\mathcal{L}^r)_{r=1}^\infty$ of partitions of K, an increasing sequence $(l(r))_r$ of positive integers with $l(r)$ dividing $l(r+1)$ for every r, and continuous maps $\Theta_{j,L}^r : L \to \mathcal{G}$ for every $1 \le j \le l(r)$ and $L \in \mathcal{L}^r$, having the following properties:

- $\Theta_{j,L}^r(x) \subset \Theta(x)$; moreover, the initial endpoint of $\Theta_{1,L}^r(x)$ is the initial endpoint of $\Theta(x)$, the final endpoint of $\Theta_{l(r),L}^r(x)$ is the final endpoint of $\Theta(x)$, and the final endpoint of $\Theta_{j,L}^r(x)$ is the initial endpoint of $\Theta_{j+1,L}^r(x)$, $1 \le j < l(r)$;
- if $x \in M \subset L$ with $L \in \mathcal{L}^r$, $M \in \mathcal{L}^{r+1}$, and $k = s + (j-1)l(r+1)/l(r)$ with $1 \le j \le l(r)$, $1 \le s \le l(r+1)/l(r)$, then $\Theta_{k,M}^{r+1}(x) \subset \Theta_{j,L}^r(x)$;

- $\operatorname{diam}(\bigcup_{x\in L}\Theta^r_{j,L}(x)) < 1/r$ for every $1\le j\le l(r)$ and $L\in\mathcal{L}^r$.

Hence, for any $t\in[0,1]$ and $x\in K$, we can construct a sequence of integers $1\le j_r\le l(r)$ and a sequence of sets $L^r_x\in\mathcal{L}^r$, such that $j_r/l(r)\to t$ as $r\to\infty$, $\{x\}=\bigcap_{r=1}^\infty L^r_x$, and $\Theta^{r+1}_{j_{r+1},L^{r+1}_x}(x)\subset\Theta^r_{j_r,L^r_x}(x)$ for any x. Clearly, the map $h(t,x)=\bigcap_{r=1}^\infty\Theta^r_{j_r,L^r_x}(x)$ is well defined and continuous in $[0,1]\times K$.

Now we must extend h to the rest of $[0,1]\times[0,1]$. To do this let us first emphasize the following. Since all arcs $\Theta(x)$ lie in U, Lemma 4.5(ii) implies that for every given $\rho>0$ there is $\delta>0$ such that if $u\in\Theta(a_i)$, $v\in\Theta(b_i)$ and $\operatorname{dist}(u,v)<\delta$, then there is an arc $[u;v]\in U$ with $\operatorname{diam}([u;v])<\rho$ and $(u;v)\subset\operatorname{Int}R_{k(i)}$ (see the notation in Lemma 4.7).

In particular, if (a_i,b_i) lies between $L,M\in\mathcal{L}^r$ and we take $0\le j\le l(r)$, then we can construct pairwise disjoint arcs X^i_j connecting the final endpoints of the arcs $\Theta_{j,L}(a_i)$ and $\Theta_{j,M}(b_i)$ (if $j=0$ then we mean the initial endpoints of the arcs $\Theta_{1,L}(a_i)$ and $\Theta_{1,M}(b_i)$), with inner sets in $\operatorname{Int}R_{k(i)}\cap\Upsilon(R_{k(i)})$, and such that if we denote by D^i_j the disk enclosed by X^i_{j-1}, X^i_j, $\Theta_{j,L}(a_i)$ and $\Theta_{j,M}(b_i)$, $1\le j\le l(r)$, then $\operatorname{diam}(D^i_j)\to 0$ as $i\to\infty$ uniformly with respect to j.

If we define h on $[0,1]\times[0,1]$ just taking care that it extends its previous definition on $[0,1]\times K$ and it maps homeomorphically any rectangle $[(j-1)/l(r),j/l(r)]\times[a_i,b_i]$ onto D^i_j, then we get the map we are looking for. ■

We are ready to finish this stage of the construction:

Proposition 4.10. *Let $A\subset\partial R_m$ be an arc. Then it is regularizable.*

Proof. Clearly it is sufficient to show that if $A',A''\subset\partial R_m$ are regularizable arcs intersecting in a proper subarc of both A' and A'', then $A'\cup A''$ is regularizable as well (for then we can use Lemma 4.9 and a simple compactness argument to prove that A is regularizable).

To do this take respective regularizations $h',h'':[0,1]\times[0,1]\to S$ of A' and A'' such that

- $h'([0,1]\times\{x\})\cap h''([0,1]\times\{x\})$ is a proper subarc of both $h'([0,1]\times\{x\})$ and $h''([0,1]\times\{x\})$ for any $x\in K$;
- for every component (a_i,b_i) of $[0,1]\setminus K$ there is R_k such that $h'([0,1]\times[a_i,b_i])\cup h''([0,1]\times[a_i,b_i])$ is a disk in $\Upsilon(R_k)$.

Clearly, such regularizations do exist: essentially we just need to take regularizations of A' and A'' whose images are very close to them and reparametrize in the second variable.

Observe that the function $t:K\to\mathbb{R}$ mapping each $x\in K$ to the corresponding $t=t(x)$ such that $h'(t,x)$ is the endpoint of $h'([0,1]\times\{x\})\cap h''([0,1]\times\{x\})$ closest to $h'(0,x)$ is continuous. Now it is easy to define a homeomorphism $h:[0,1]\times K\to h'([0,1]\times K)\cup h''([0,1]\times K)$ mapping $[0,1]\times\{x\}$ onto $h'([0,1]\times\{x\})\cup h''([0,1]\times\{x\})$ and such that $h(0,x)=h'(0,x)$, $h(1,x)=h'(1,x)$ for every x, and then extending it (as in the final part of the proof of Lemma 4.9) to a homeomorphism $h:[0,1]\times[0,1]\to h'([0,1]\times[0,1])\cup h''([0,1]\times[0,1])$ satisfying the requirements of Definition 4.4. ■

4.2. Construction of the flow

Let $\{A_l\}_{l=1}^\infty$ be the family of the components of all twisting boundaries of the twisting sections of all surfaces R_n and fix a "left side" and a "right side" for C. Using Proposition 4.10 it is easy to construct embeddings $h_l:[0,1]\times[-1,1]\to S$ such that, for every l, $h_l(\{0,1\}\times[-1,1])\subset C$, $h_l([0,1]\times[-1,0])$ is the twisting section including A_l, $h_l|_{[0,1]\times[0,1]}$ is a regularization of A_l, and, if $\epsilon>0$ is small enough, then $h_l((0,\epsilon)\times[-1,1])$ is on the right side of C and $h_l((1-\epsilon,1)\times[-1,1])$ is on the left side of C.

Next we construct for every l the family $\mathcal{B}_l$ of arcs $h_l([0,1]\times\{x\})$, $x\in I_l$, with I_l inductively defined by $I_1=[-1,1]$ and $I_l\subset[-1,1]$ being the maximal interval including 0 in its closure such that $h([0,1]\times I_l)\setminus C$ intersects no arc from $\mathcal{B}_m$ for $m<l$ (or $I_l=\emptyset$ if no such interval exists). Let $\mathcal{B}$ denote the family of all curves with are a maximal union of some arcs from $\bigcup_l\mathcal{B}_l$ (in the case that the endpoint of one of the arcs the curve consists of belongs to just that arc then we take it off from the curve).

After orienting the curves from $\mathcal{B}$ so that they always approach C from its left side and escape C from its right side, it is easy to check that $\mathcal{B}$ is a full Whitney regular orientable family of curves in S. Use Theorem 4.1 to obtain the corresponding flow Φ.

4.3. Finishing the proof

We show that $\Omega = \mathrm{Bd} \bigcup_n R_n$ is a quasiminimal set for Φ and that Φ is topologically equivalent to a smooth flow on S.

Let $\{V_r\}_{r=1}^{\infty}$ be a countable basis of neighbourhoods in S and let $\{V_{r_s}\}_s$ be the subfamily of neighbourhoods intersecting Ω. Notice that (after taking off its endpoints if necessary) every component of every $\Upsilon(\partial R_n)$ is an orbit of Φ. Now we use Definition 2.4(iii) (for the first and last time in the proof) and Proposition 4.10 to construct a family $\{D_s\}_s$ of disks with $D_{s+1} \subset D_s$ for every s, all intersecting Ω and with diameters tending to zero, such that the orbits of all points from D_s intersect V_{r_s}. Let $\{p\} = \bigcap_s D_s$. Then $p \in \Omega$ and the closure of its orbit contains Ω. Indeed it is exactly Ω because Ω is clearly invariant. Then either $\alpha_\Phi(x) = \Omega$ or $\omega_\Phi(x) = \Omega$. After reverting if necessary the orientation of Φ, we get $p \in \omega_\Phi(p) = \Omega$.

Now we prove that Φ is topologically equivalent to a smooth flow on S. According to [Gutierrez, 1986] we must prove that every minimal set of Φ is trivial, that is, it is either a singular point, a periodic orbit, or the whole surface $S = \mathbb{T}^2$ (the torus) with Φ being (topologically equivalent to) the irrational flow on $\mathbb{T}^2$. Assume that M is a nontrivial minimal set of Φ and take $q \in M$. We claim that $q \in \Omega$. Suppose not. Since q is not singular, $q \in \mathrm{Int}\, R_k$ for some k. Since Ω is invariant and $\mathrm{Bd}\, R_k \subset \Omega$, $\mathrm{Int}\, R_k$ is invariant. Then the restriction of Φ to $\mathbb{R} \times \mathrm{Int}\, R_k$ is a flow on $\mathrm{Int}\, R_k$ having a nontrivial recurrent point, which is impossible for planar flows (here we use that the surface R_k is simply connected).

Thus $\omega_\Phi(q) = M \subset \Omega = \omega_\Phi(p)$. As both p and q are nontrivial recurrent, a theorem of Maĭer's [Maĭer, 1943] (see also [Aranson *et al.*, 1996; Theorem 2.3, p. 65]) implies that $M = \Omega$. But Ω cannot be minimal; for instance, every set $\Upsilon(\partial R_n)$ consists of nonrecurrent orbits of Φ. We have arrived at a contradiction.

To conclude the proof we show that there is no flow $\Psi : \mathbb{R} \times S \to S$ satisfying $\omega_\Psi(u) = \Omega$ for some $u \notin \Omega$. In view of Proposition 4.10, we can assume that $u \in \mathrm{Int}\, R_k$ for some k. Since $\mathrm{Bd}\, R_k \subset \Omega$, the whole orbit of u is in $\mathrm{Int}\, R_k$, which means that $\mathrm{Bd}\, R_k = \Omega$. But this is impossible since, as Proposition 4.10 emphasizes, if $v \in \partial R_k$ then there are points from $\Omega \backslash \partial R_k$ as close to v as required (which cannot belong to $\mathrm{Bd}\, R_k$ because R_k is a surface).

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