

# A Polynomial-Time Algorithm for Optimization of Quadratic Pseudo-Boolean Functions

Juan Ignacio Mulero-Martínez

*Departamento de Ingeniería de Sistemas y Automática. Universidad Politécnica de Cartagena. Campus Muralla del Mar. Cartagena. 30203. España*

---

## Abstract

We develop a polynomial-time algorithm to minimize pseudo-Boolean functions. The computational complexity is  $O\left(n^{\frac{15}{2}}\right)$ , although very conservative, it is sufficient to prove that this minimization problem is in the class  $P$ . A direct application of the algorithm is the 3-SAT problem, which is also guaranteed to be in the class  $P$  with a computational complexity of order  $O\left(n^{\frac{45}{2}}\right)$ . The algorithm was implemented in MATLAB and checked by generating one million matrices of arbitrary dimension up to 24 with random entries in the range  $[-50, 50]$ . All the experiments were successful.

---

## 1. Introduction

Pseudo-Boolean optimization also called nonlinear 0-1 optimization, is a term to refer to the unconstrained optimization of pseudo-Boolean functions over  $\mathcal{B}^n = \{0, 1\}^n$ . This field traces back to the late 60s of the 20th century with the works of Hammer and Rudeanu, [1]. From then, it has become an active research area in Discrete Mathematics and Complexity Theory (surveys in [2] and in [3], give a good account of this topic).

In this paper, we restrict to the unconstrained minimization problem

$$\text{minimize } f(x) \text{ subject to } x \in \mathcal{B}^n \quad (1)$$

where  $f$  is quadratic. The optimization problem of a pseudo-Boolean function is NP-hard, even when the objective function is quadratic; this is due to the fact that the problem encompasses several hard combinatorial problems such as max-cut, 0-1 linear programming, weighted stability, or MAX 2-SAT.

The problem of computing local minima of a quadratic pseudo-Boolean function  $f$  is in the class PLS-complete (these are the hardest problems in the class PLS (Polynomial Local Search)), [4], [5]. In general, local search procedures (with arbitrary pseudo-Boolean functions) are in the class EXP, [6], [7], [8], [9], [10], [11]. Global search procedures are NP-complete. Among others it is worth

mentioning the combinatorial variable elimination algorithm<sup>1</sup>, [1], [12], [13]; the linearization procedure, consisting in transforming the minimization problem into an equivalent linear 0–1 programming problem, [14], [15], [16], [17]; the continuous relaxation with linearization (where the integral requirement  $x_i \in \mathcal{B}$  is replaced by a weaker constraint  $x_i \in [0, 1]$ ), [18], [19]; the posiform transformations, [20], [21]; and the conflict graphs (the connection between posiform minimization problem and the maximum weighted stability), [22], [23], [24], [25], [26], [27].

In the literature, quadratic pseudo-Boolean functions have been intensely explored (the reader is referred to the surveys in [28], [2]). The importance of quadratic 0-1 optimization stems from both the fact that many applications are formulated in this form and the fact that the general case of pseudo-Boolean functions can be reduced to it. Essentially, the reduction can be made by a quadratization procedure, [29], or by a weighted stability problem enunciated as a 0-1 minimization problem.

Many researchers have extensively studied the optimization of quadratic pseudo-Boolean functions, however, up to the date nobody has succeeded in developing an algorithm running in polynomial time. We claim, and this is the main contribution of this work, that the problem (1) is in the complexity class P.

After a polynomial extension, the problem (1) is reduced to

$$\text{minimize } x^T Q x + b^T x + c \text{ subject to } x \in [0, 1]^n \quad (2)$$

where  $Q = Q^T \in \mathbb{R}^{n \times n}$ ,  $\text{diag}(Q) = (0, \dots, 0)$ ,  $b \in \mathbb{R}^n$ ,  $c \in \mathbb{R}$ . This is the kind of problem we are faced with. The main idea is to transform (2) into a linear programming (LP) problem, that is solved in polynomial time. We guarantee that the minimum of the LP problem is also the minimum of the problem (2). This procedure has been implemented in MATLAB (the source code is in the appendix) and checked with one million random matrices up to dimension 24, with entries in the range  $[-50, 50]$ . This algorithm is applied to the 3-SAT problem leading to a computational complexity of polynomial order both in time and in space. In sum, we claim that both the minimization of quadratic pseudo-Boolean functions and the 3-SAT problem are in the class P.

The paper is organized as follows. Section 2 introduces the main concepts related to quadratic pseudo-Boolean functions, such as quadratization and extension. The main result is dealt with in Section 3. We begin with a simple case,  $n = 3$ , to show the basic ingredients of the transformation into an LP problem. That case is generalized to the  $n$ -dimensional case introducing "consistency constraints". In Section 4, it is shown that the computational complexity of the algorithm is polynomial both in time and in space. The procedure is applied

---

<sup>1</sup>This algorithm is in the class EXP, and only runs in polynomial time for pseudo-Boolean functions associated with graphs of bounded tree-width.

to the 3-SAT problem in Section 5, where the computational complexity is analyzed. Finally, the concluding remarks are discussed in Section 6.

## 2. Background

Pseudo-Boolean functions (PBFs) are mappings from  $\mathcal{B}^n = \{0, 1\}^n$  to the reals  $\mathbb{R}$ . Any PBF  $f : \mathcal{B}^n \rightarrow \mathbb{R}$  can be uniquely represented as

$$f(x_1, \dots, x_n) = \prod_{S \in \mathcal{P}(\{1, 2, \dots, n\})} c_S \prod_{i \in S} x_i \quad (3)$$

where  $\mathcal{P}(\{1, 2, \dots, n\})$  is the power set of  $\{1, 2, \dots, n\}$ ,  $c_S \in \mathbb{R}$  (the reader is referred to Theorem 13.1 in [30] and it is attributed to T. Gaspar). Given a function a pseudo-Boolean function  $f : \mathcal{B}^n \rightarrow \mathbb{R}$  as in (3), we are interested in the following minimization problem:

$$(\mathbf{P}): \min_{x \in \mathcal{B}^n} f(x)$$

The above problem can be transformed into a quadratic minimization problem:

$$\min_{x \in \mathcal{B}^n} \min_{w \in \mathcal{B}^m} g(x, w)$$

where  $g : \mathcal{B}^n \times \mathcal{B}^m \rightarrow \mathbb{R}$  is a quadratic pseudo-Boolean function such that  $f(x) = \min_{w \in \mathcal{B}^m} g(x, w)$  for all  $x \in \mathcal{B}^n$ . For the quadratization, we follow the ideas presented by Ishikawa in [31] (also the reader is referred to [29] for a survey of quadratization methods). Let us consider a monomial  $ax_1x_2x_3 \dots x_d$  of degree  $d$ , and let  $S_1(x_1, \dots, x_d)$  and  $S_2(x_1, \dots, x_d)$  be the elementary symmetric polynomials

$$S_1 = \sum_{i=1}^d x_i, S_2 = \sum_{i=1}^d \sum_{j=i+1}^d x_i x_j = \frac{S_1(S_1 - 1)}{2}$$

The quadratization of negative monomials ( $a < 0$ ) was stated by Freedman and Drineas, [32], for arbitrary degree monomials:

$$ax_1x_2x_3 \dots x_d = \min_{w \in \mathcal{B}} w(S_1 - (d - 1)) \quad (4)$$

and a compact quadratization for positive monomials ( $a > 0$ ) is due to Ishikawa, [31]:

$$ax_1x_2x_3 \dots x_d = \min_{w_1, \dots, w_{n_d} \in \mathcal{B}} \sum_{i=1}^{n_d} w_i (c_{i,d}(-S_1 + 2i) - 1) + aS_2 \quad (5)$$

where

$$n_d = \left\lceil \frac{d-1}{2} \right\rceil, c_{i,d} = \begin{cases} 1, & \text{if } d \text{ is odd and } i = n_d \\ 2, & \text{otherwise} \end{cases}$$

Since quadratic pseudo-Boolean functions are the atomic elements of every optimization problem **(P)**, after a quadratization process, this is the class of functions we are interested in. Specifically, this work deals with the optimization of pseudo-Boolean functions of the form

$$f(x_1, \dots, x_n) = \prod_{\substack{S \in \mathcal{P}(\{1, 2, \dots, n\}) \\ |S| \leq 2}} c_S \prod_{i \in S} x_i \quad (6)$$

where  $|S|$  denotes the cardinality of the set  $S$ . For the sake of clarity, we abandon the Rudeanu and Hammer notation in (6) and adopt the standard matrix notation:

$$f(x) = x^T Q x + b^T x + c \text{ where } x \in \mathcal{B}^n \quad (7)$$

where  $Q = Q^T \in \mathbb{R}^{n \times n}$ ,  $\text{diag}(Q) = (0, \dots, 0)$ ,  $b \in \mathbb{R}^n$  and  $c \in \mathbb{R}$ .

We denote by  $\mathcal{H}_n$  the solid hypercube  $[0, 1]^n$  spanned by  $\mathcal{B}^n$ . The extension of the pseudo-Boolean function  $f : \mathcal{B}^n \rightarrow \mathbb{R}$  is a function  $g : \mathcal{H}_n \rightarrow \mathbb{R}$  that coincides with  $f$  at the vertices of  $\mathcal{H}_n$ . Here, we adopt the polynomial extension of  $f$ ,  $f^{pol} : \mathcal{H}_n \rightarrow \mathbb{R}$ . Rosenberg discovered an attractive feature regarding the multilinear polynomial extension  $f^{pol}$ , [33]: the minimum of  $f^{pol}$  is always attained at a vertex of  $\mathcal{H}_n$ , and hence, that this minimum coincides with the minimum of  $f$ . From this, our optimization problem is reduced to the following problem:

$$\textbf{(P): } \min_{x \in \mathcal{H}_n} f(x)$$

### 3. Main Result

In this section we prove that Problem (P) can be reduced to a Linear Programming Problem. Without loss of generality we focus on functions  $f(x) = x^T Q x + b^T x$  since

$$\min_{x \in \mathcal{H}_n} (x^T Q x + b^T x + c) = \min_{x \in \mathcal{H}_n} (x^T Q x + b^T x) + c$$

#### 3.1. A simple case

We begin with the simple case of minimization of a quadratic form  $f(x)$  in the cube  $\mathcal{H}_3 = [0, 1]^3$ . The minimization problem is stated as follows:

$$\textbf{(P}_3\textbf{): } \min_{x \in \mathcal{H}_3} f(x).$$

**Definition 1 (Primary variables).** For the triplet of variables  $(x_1, x_2, x_3) \in \mathcal{H}_3$  we define the primary variables

$$u_{12} = \frac{x_1 + 2x_1x_2 + x_2}{2}, u_{13} = \frac{x_1 + 2x_1x_3 + x_3}{2}, u_{23} = \frac{x_2 + 2x_2x_3 + x_3}{2} \quad (8)$$

$$v_{12} = \frac{x_1 - 2x_1x_2 + x_2}{2}, u_{13} = \frac{x_1 - 2x_1x_3 + x_3}{2}, u_{23} = \frac{x_2 - 2x_2x_3 + x_3}{2} \quad (9)$$

where  $u_{12}, u_{13}, u_{23} \in [0, 2]$  and  $v_{12}, v_{13}, v_{23} \in [0, \frac{1}{2}]$ .

3 – dimensional Space  $\mathcal{C}_3$

6 – dimensional Space  $\mathcal{H}_3$

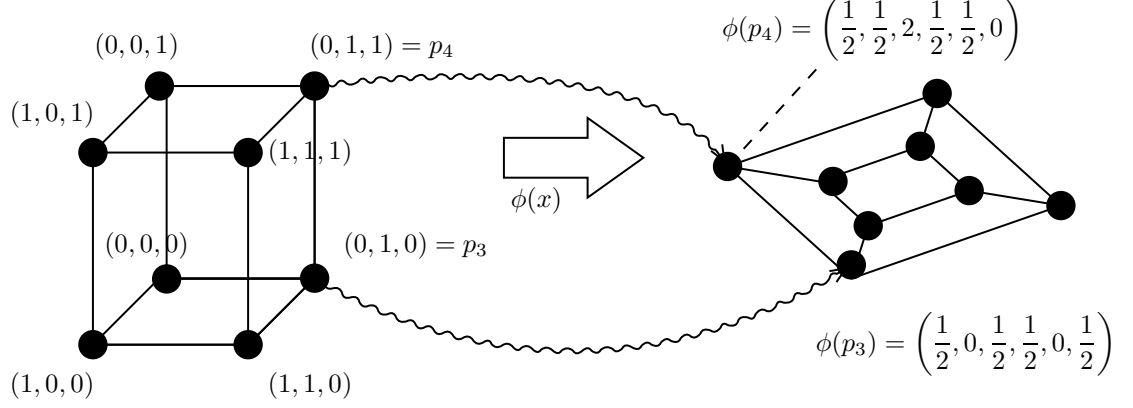


Figure 1: Map  $\phi$  between the cube  $\mathcal{H}_3$  and the 6-dimensional convex-hull  $\mathcal{C}_3$ .

For the sake of simplicity we use the notation  $w^T = (u^T, v^T)$  where  $u \in [0, 2]^3$  is given by (8) and  $v \in [0, \frac{1}{2}]^3$  by (9). The advantage of defining these variables is that they satisfy the following relationships:

(i) Cross-products in the objective function (corresponding to the off-diagonal entries in  $Q$ ):

$$x_i x_j = \frac{u_{ij} - v_{ij}}{2} \text{ for } 1 \leq i < j \leq 3 \quad (10)$$

(ii) Single variables (corresponding to the vector  $b$ ):

$$x_i = \frac{u_{ij} + v_{ij} + u_{ik} + v_{ik} - u_{jk} - v_{jk}}{2} \text{ for } 1 \leq i < j \leq 3 \quad (11)$$

An important fact is that the cube  $\mathcal{H}_3$  can be expressed as a convex hull of a finite set of vertices  $V = \{0, 1\}^3$ . For simplicity, we enumerate the vertices in  $V$  as  $p_1, p_2, \dots, p_8$  so that  $\mathcal{H}_3$  can be written as convex combinations of those vertices, i.e.  $\mathcal{H}_3 = \text{conv}(V)$ , where

$$\text{conv}(V) = \left\{ \sum_{i=1}^8 \alpha_i p_i : \alpha_i \geq 0, \sum_{i=1}^8 \alpha_i = 1 \right\}$$

Let  $\phi : \mathcal{H}_3 \rightarrow [0, 2]^3 \times [0, \frac{1}{2}]^2$  be the map  $\phi(x) = w$  according to the definitions of primary variables in (8) and (9). From this map, another convex hull is built  $\mathcal{C}_3 = \text{conv}(\phi(V))$ ; in Figure 1, the transformation between  $\mathcal{H}_3$  and  $\mathcal{C}_3$  through the map  $\phi$  is illustrated.

At this point, we need to introduce the maps  $\alpha : \mathcal{H}_3 \rightarrow \alpha(\mathcal{H}_3)$  and  $\beta : \alpha(\mathcal{H}_3) \rightarrow \mathcal{C}_3$ :

$$\alpha(x) = (x_1, x_1 x_2, x_1 x_3, x_2, x_2 x_3, x_3) \quad (12)$$

$$\beta(y) = E_3 y \text{ for every } y \in \alpha(\mathcal{H}_3) \quad (13)$$

where  $E_3$  is obtained from (8) and (9) as

$$E_3 = \frac{1}{2} \begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 1 & -2 & 0 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{pmatrix} \quad (14)$$

As a summary, the maps  $\phi$ ,  $\alpha$  and  $\beta$  are represented in the diagram of Figure 2.

**Lemma 1.** *The set  $\phi(\mathcal{H}_3)$  is convex.*

**Proof.** First of all note that  $\phi = \beta \circ \alpha$ . We will see that the map  $\alpha$  is built from  $\mathcal{H}_3$  as a selection of the Kronecker product  $\tilde{x} \otimes \tilde{x}$  where  $\tilde{x}^T = (1, x^T)$  and  $x \in \mathcal{H}_3$ . The set  $\tilde{\mathcal{H}}_3 = \left\{ \tilde{x} = \begin{pmatrix} 1 \\ x \end{pmatrix} : x \in \mathcal{H}_3 \right\}$  is convex: this is trivial simply by building a convex combination of two points  $\tilde{x}$  and  $\tilde{y}$  in  $\tilde{\mathcal{H}}_3$ ,

$$\lambda \tilde{x} + (1 - \lambda) \tilde{y} = \begin{pmatrix} 1 \\ \lambda x + (1 - \lambda) y \end{pmatrix} \in \tilde{\mathcal{H}}_3 \text{ with } \lambda \in [0, 1]$$

Since  $\mathcal{H}_3 = \text{conv}(V)$ , it follows that  $x = \sum_{i=1}^8 \lambda_i p_i$  where  $\sum_{i=1}^8 \lambda_i = 1$ ,  $\lambda_i \geq 0$ , and  $p_i \in V$ . The set of vertices of  $\tilde{\mathcal{H}}_3$  is

$$\tilde{V} = \left\{ \tilde{p}_i = \begin{pmatrix} 1 \\ p_i \end{pmatrix} : i = 1, \dots, 8 \right\}$$

So  $\tilde{\mathcal{H}}_3 = \text{conv}(\tilde{V})$  and  $\tilde{x} \otimes \tilde{x}$  is written as a convex combination:

$$\tilde{x} \otimes \tilde{x} = \sum_{i,j=1}^8 \lambda_i \lambda_j (\tilde{p}_i \otimes \tilde{p}_j) = \sum_{i,j=1}^8 \gamma_{ij} (\tilde{p}_i \otimes \tilde{p}_j)$$

Here  $\sum_{i,j=1}^8 \gamma_{ij} = 1$ ,  $\gamma_{ij} \geq 0$ , and the set  $\tilde{\mathcal{H}}_3^{\otimes} = \left\{ \tilde{x} \otimes \tilde{x} : \tilde{x} \in \tilde{\mathcal{H}}_3 \right\}$  is the convex hull generated by the vertices  $\tilde{p}_i \otimes \tilde{p}_j$ . There exists a matrix  $S_3$  given by

$$S_3 = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

such that  $\alpha(x) = S_3(\tilde{x} \otimes \tilde{x})$ . The set  $\phi(\mathcal{H}_3)$  is obtained from  $\alpha(\mathcal{H}_3)$  as follows:

$$\phi(\mathcal{H}_3) = E_3 \alpha(\mathcal{H}_3) = E_3 S_3 \tilde{\mathcal{H}}_3^{\otimes}$$

Owing to the convexity is preserved by a linear transformation we conclude that  $\phi(\mathcal{H}_3)$  is convex. ■

**Example 1.** The image  $\alpha(x)$  is built from  $\tilde{x} \otimes \tilde{x}$  via the selector  $S_3$  as

$$\alpha(x) = S_3(\tilde{x} \otimes \tilde{x}) = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \\ x_1 \\ x_1^2 \\ x_1x_2 \\ x_1x_3 \\ x_2 \\ x_1x_2 \\ x_2^2 \\ x_2x_3 \\ x_3 \\ x_1x_3 \\ x_2x_3 \\ x_3^2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1x_2 \\ x_1x_3 \\ x_2 \\ x_2x_3 \\ x_3 \end{pmatrix}$$

**Lemma 2.** The preimage of  $\mathcal{C}_3$  through  $\phi$  (denoted as  $\phi^{-1}[\mathcal{C}_3]$ ) is a subset of  $\mathcal{H}_3$ , i.e.  $\mathcal{C}_3 \subset \phi(\mathcal{H}_3)$ .

**Proof.** This is straightforward because it was shown in Lemma 1 that  $\phi(\mathcal{H}_3)$  is convex, and we know that the vertices  $\phi(p_i)$  of  $\mathcal{C}_3$  are in  $\phi(\mathcal{H}_3)$ . Henceforth,  $\mathcal{C}_3 \subset \phi(\mathcal{H}_3)$ . ■

We define the linear transformation  $\varphi : \mathcal{C}_3 \rightarrow \mathbb{R}^6$  given by  $\varphi(w) = T_3w$ , with

$$T_3 = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ -1 & 1 & 1 & -1 & 1 & 1 \end{pmatrix}$$

We claim that  $\varphi \circ \beta = id$ ; this can be checked by inspection:

$$T_3E_3 = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ -1 & 1 & 1 & -1 & 1 & 1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 1 & -2 & 0 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (15)$$

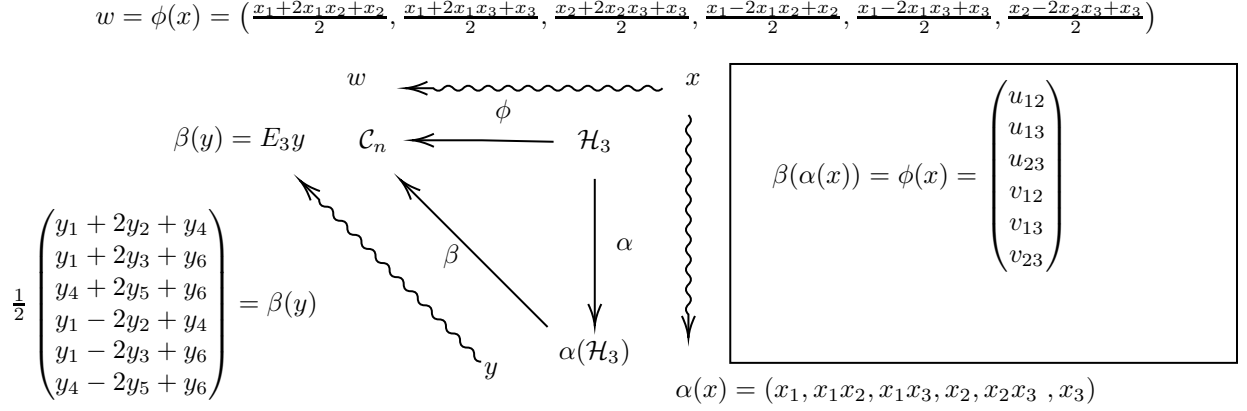


Figure 2: Diagram for the maps  $\phi$ ,  $\alpha$  and  $\beta$ .

**Remark 1.** According to the definition of  $\alpha$  there exists a vector  $c \in \mathbb{R}^6$  such that  $f(x) = c^T \alpha(x)$ . Additionally, always there exists a vector  $\tilde{c} \in \mathbb{R}^6$  such that  $f(x) = \tilde{c}^T \phi(x)$ , where

$$\tilde{c}^T = c^T T_3 \quad (16)$$

This is due to

$$\tilde{c}^T E_3 \alpha(x) = c^T T_3 E_3 \alpha(x)$$

and we know from (15) that  $T_3 E_3 = I$ .

Let  $\tilde{f}(w) = \tilde{c}^T w$ , we define the optimization problem:

$$(\mathbf{P}'_3): \min_{w \in \mathcal{C}_3} \tilde{f}(w).$$

The following theorem states that the minimum of  $f$  over  $\mathcal{H}_3$  is the minimum of  $\tilde{f}$  over  $\mathcal{C}_3$ .

**Theorem 3.** Let  $f(x^*)$  be the minimum of the problem  $(P_3)$ , and  $\tilde{f}(w^*)$  the minimum of the problem  $(P'_3)$ . Then  $f(x^*) = \tilde{f}(w^*)$ .

**Proof.** In virtue of Lemma 2,  $\phi^{-1}[\mathcal{C}_3] \subset \mathcal{H}_3$ . This means that there exists a point  $y \in \mathcal{H}_3$  such that  $\phi(y) = w^*$ . On the other hand, the minimum of  $f$  over  $\mathcal{H}_3$  is attained at  $x^* \in \mathcal{H}_3$ , so that

$$f(x^*) \leq f(y) \quad (17)$$

The connection between  $f$  and  $\tilde{f}$  yields:

$$\tilde{f}(\phi(x^*)) = \tilde{c}^T \phi(x^*) = (c^T T_3)(E_3 \alpha(x^*)) = c^T \alpha(x^*) = f(x^*) \quad (18)$$



and

$$f(y) = c^T \alpha(y) = (c^T T_3) (E_3 \alpha(y)) = \tilde{c}^T \phi(y) = \tilde{f}(\phi(y)) \quad (19)$$

According to (17) and (19),

$$f(x^*) \leq f(y) = \tilde{f}(w^*)$$

Since the minimum of  $\tilde{f}$  over  $\mathcal{C}_3$  is attained at  $w^* \in \mathcal{C}_3$ , and accounting for (18), we have that

$$f(x^*) = \tilde{f}(\phi(x^*)) \geq \tilde{f}(w^*)$$

Henceforth,  $f(x^*) = \tilde{f}(w^*)$ . ■

The problem (P'<sub>3</sub>): can be written as a linear programming problem:

$$(\mathbf{LP}_3): \begin{cases} \min \tilde{c}^T w \\ \text{such that} \begin{cases} B\lambda - w = 0 \\ u^T \lambda = 1 \\ w \geq 0, \lambda \geq 0 \end{cases} \end{cases}$$

where  $\lambda^T = (\lambda_1, \dots, \lambda_8)$ ,  $B = (\phi(p_1), \dots, \phi(p_8))$ , and  $u$  is an all-ones vector of appropriate dimension. Throughout this work the variables  $\lambda$  are called secondary variables.

**Example 2.** Let  $f(x) = x^T Qx + b^T x$  with

$$Q = \begin{pmatrix} 0 & -10 & -20 \\ -10 & 0 & -10 \\ -20 & -10 & 0 \end{pmatrix}$$

$$b = \begin{pmatrix} -2 \\ -2 \\ -26 \end{pmatrix}$$

This objective function is written as  $f(x) = c^T \alpha(x)$ , where  $c^T = (-2, -20, -40, -2, -20, -26)$ . The problem (LP<sub>3</sub>) has an objective function  $\tilde{f}(w) = \tilde{c}^T w$  with  $\tilde{c}^T = c^T T_3 = (1, -33, -23, 21, 7, -3)$ .

The matrix  $B$  in the constraints is given by:

$$B = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 2 & 2 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 2 & \frac{1}{2} & 2 \\ 0 & \frac{1}{2} & \frac{1}{2} & 2 & 0 & \frac{1}{2} & \frac{1}{2} & 2 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \quad (20)$$

The minimum of (P') is  $-110$  and is attained at  $w^* = (2, 2, 2, 0, 0, 0)^T$ .

### 3.2. The General Case

In this subsection, we generalize the simple case to the  $n$ -dimensional hypercube  $\mathcal{H}_n = [0, 1]^n$ . We state the quadratic minimization problem of an objective function  $f(x) = x^T Q x + b^T x$ ,  $Q \in \mathbb{R}^{n \times n}$ ,  $b \in \mathbb{R}^n$  over  $\mathcal{H}_n$ :

$$(\mathbf{P}_n): \min_{x \in \mathcal{H}_n} f(x).$$

In a similar way as done in (8) and (9) we define primary variables  $u_{ij}$  and  $v_{ij}$  for  $1 \leq i < j \leq n$ . In this case,  $u$  and  $v$  are  $\binom{n}{2}$ -dimensional vectors, and then  $w = (u^T, v^T)^T \in \mathbb{R}^{n(n-1)}$ . For the sake of clarity, we adopt the notation  $w_{i,j,k}$  to refer to the vector  $(u_{ij}, u_{ik}, u_{jk}, v_{ij}, v_{ik}, v_{jk})$  and  $x_{i,j,k}$  for the vector  $(x_i, x_j, x_k)$ . Also we generalize  $\mathcal{C}_3$  to  $\mathcal{C}_n$  by partitioning the variables  $(x_1, \dots, x_n)$  in triplets, leading to convex-hulls  $\mathcal{H}_3^{(i,j,k)}$  in the variables  $x_i, x_j$  and  $x_k$ . For each triplet  $(i, j, k)$ , we have a convex-hull  $\mathcal{C}_3^{(i,j,k)}$ , in the primary variables  $w_{ijk}$ , as defined for the simple case, and these basic convex-hulls are connected through consistency constraints:

**Definition 2.** For a point  $w \in \mathcal{C}_n$  and for  $1 \leq j < k \leq n$ , we add the consistency constraints:

$$g_{1,2,3}(w) = g_{1,j,k}(w) \text{ with } (j, k) \neq (2, 3) \quad (21)$$

$$g_{2,1,3}(w) = g_{2,j,k}(w) \text{ with } (j, k) \neq (1, 3) \quad (22)$$

$$g_{i,1,2}(w) = g_{i,j,k}(w) \text{ with } (j, k) \neq (1, 2), i \geq 3 \quad (23)$$

where  $g_{i,j,k}(w) = \frac{u_{ij} + v_{ij} + u_{ik} + v_{ik} - u_{jk} - v_{jk}}{2} = r^T w_{ijk}$  with  $r^T = \frac{1}{2}(1, 1, 1, 1, -1, -1)$ .

Let  $w \in \mathcal{C}_n$ , and triplets  $(i, j, k)$ ,  $(i, j, l)$ , with  $l \neq k$ , there exist maps  $\phi_{i,j,k}$  and  $\phi_{i,j,l}$  (as shown in the simple case) such that

$$w_{i,j,k} = \phi_{i,j,k}(x_{i,j,k}), x_{i,j,k} \in \mathcal{H}_3^{(i,j,k)}$$

and

$$w_{i,j,l} = \phi_{i,j,l}(y_{i,j,k}), x_{i,j,l} \in \mathcal{H}_3^{(i,j,l)}$$

Consistency means that we expect that

$$\begin{aligned} x_i &= g_{i,j,k}(w) = g_{i,j,l}(w) = y_i \\ x_j &= g_{j,i,k}(w) = g_{j,i,l}(w) = y_j \end{aligned}$$

**Definition 3.** The set  $\mathcal{C}_n$  is built from the convex hulls  $\mathcal{C}_3^{(i,j,k)}$  with the consistency constraints:

$$\mathcal{C}_n = \left\{ w \in [0, 2]^n \times \left[0, \frac{1}{2}\right]^n : w_{i,j,k} \in \mathcal{C}_3^{(i,j,k)} \text{ and } w \text{ satisfies (21), (22) and (23)} \right\}$$

Analogously to the simple case we have that  $\mathcal{C}_n \subset \phi(\mathcal{H}_n)$ . This result is argued in the following lemma:

**Lemma 4.** *Given an arbitrary point  $w \in \mathcal{C}_n$ , there exists a  $y \in \mathcal{H}_n$  such that  $\phi(y) = w$ .*

**Proof.** For a triplet  $(i_1, j_1, k_1)$  there exists a point  $(x_{i_1}, x_{j_1}, x_{k_1}) \in \mathcal{H}_3$  and a map  $\phi_{i_1, j_1, k_1} : \mathcal{H}_3 \rightarrow \mathcal{C}_3$  such that  $\phi_{i_1, j_1, k_1}(x_{i_1}, x_{j_1}, x_{k_1}) = w_{i_1, j_1, k_1}$  (this was shown above in Lemma 2 for the simple case). Let  $(i_2, j_2, k_2)$  be another triplet with  $\phi_{i_2, j_2, k_2}(y_{i_2}, y_{j_2}, y_{k_2}) = w_{i_2, j_2, k_2}$ , such that  $\{i_1, j_1, k_1\} \cap \{i_2, j_2, k_2\} \neq \emptyset$ . Without loss of generality, let us assume that  $i_1 = i_2$  (otherwise we can make a permutation of the indices to get that configuration) then from the consistency constraints we have  $x_{i_1} = y_{i_1}$ :

$$x_{i_1} = g_{i_1, j_1, k_1}(\phi_{i_1, j_1, k_1}(x_{i_1}, x_{j_1}, x_{k_1})) = g_{i_1, j_2, k_2}(\phi_{i_1, j_2, k_2}(y_{i_1}, y_{j_2}, y_{k_2})) = y_{i_1}$$

Extending this idea to all the pairs of triplets, we conclude that for every  $w \in \mathcal{C}_n$ , there exists a point  $x \in \mathcal{H}_n$  such that  $\phi(x) = w$ . ■

**Example 3.** *For  $n = 4$ , the consistency constraints are*

*Consistency for  $x_1$ :*

$$\begin{cases} (u_{12} + v_{12} + u_{13} + v_{13} - u_{23} - v_{23}) - (u_{12} + v_{12} + u_{14} + v_{14} - u_{24} - v_{24}) = 0 \\ (u_{12} + v_{12} + u_{13} + v_{13} - u_{23} - v_{23}) - (u_{13} + v_{13} + u_{14} + v_{14} - u_{34} - v_{34}) = 0 \end{cases}$$

*Consistency for  $x_2$ :*

$$\begin{cases} (u_{12} + v_{12} + u_{23} + v_{23} - u_{13} - v_{13}) - (u_{12} + v_{12} + u_{24} + v_{24} - u_{14} - v_{14}) = 0 \\ (u_{12} + v_{12} + u_{23} + v_{23} - u_{13} - v_{13}) - (u_{23} + v_{23} + u_{24} + v_{24} - u_{34} - v_{34}) = 0 \end{cases}$$

*Consistency for  $x_3$ :*

$$\begin{cases} (u_{13} + v_{13} + u_{23} + v_{23} - u_{12} - v_{12}) - (u_{13} + v_{13} + u_{34} + v_{34} - u_{14} - v_{14}) = 0 \\ (u_{13} + v_{13} + u_{23} + v_{23} - u_{12} - v_{12}) - (u_{23} + v_{23} + u_{34} + v_{34} - u_{24} - v_{24}) = 0 \end{cases}$$

*Consistency for  $x_4$ :*

$$\begin{cases} (u_{14} + v_{14} + u_{24} + v_{24} - u_{12} - v_{12}) - (u_{14} + v_{14} + u_{34} + v_{34} - u_{13} - v_{13}) = 0 \\ (u_{14} + v_{14} + u_{24} + v_{24} - u_{12} - v_{12}) - (u_{24} + v_{24} + u_{34} + v_{34} - u_{23} - v_{23}) = 0 \end{cases}$$

*In Figure 3, we graphically show the consistency constraints for the variables  $x_1, x_2, x_3$  and  $x_4$ , generated from the convex-hulls  $\mathcal{C}_3^{(1,2,3)}$ ,  $\mathcal{C}_3^{(1,2,4)}$ ,  $\mathcal{C}_3^{(1,3,4)}$  and  $\mathcal{C}_3^{(2,3,4)}$  via the linear transformation  $C = (M, M)$  with*

$$M = \frac{1}{4} \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

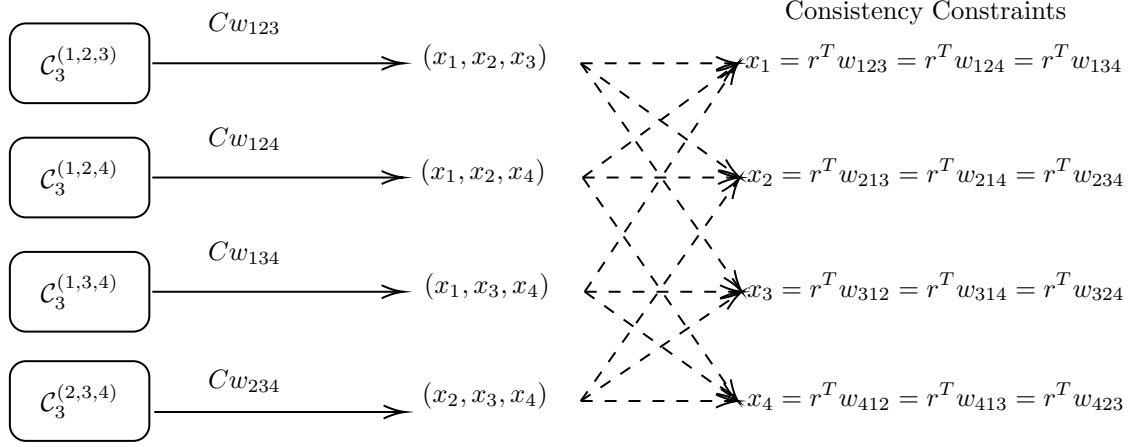


Figure 3: Consistency constraints for  $n = 4$

Note that

$$\begin{aligned}
 C\phi_{i,j,k}(x_{i,j,k}) &= \left( \frac{x_i + 2x_i x_j + x_j + x_i + 2x_i x_k + x_k - (x_j + 2x_j x_k + x_k) + x_i - 2x_i x_j + x_j + x_i - 2x_i x_k + x_k - (x_j - 2x_j x_k + x_k)}{4} \right. \\
 &\quad \left. \frac{x_i + 2x_i x_j + x_j + x_j + 2x_j x_k + x_k - (x_i + 2x_i x_k + x_k) + x_i - 2x_i x_j + x_j + x_j - 2x_j x_k + x_k - (x_i - 2x_i x_k + x_k)}{4} \right) \\
 &\quad \left. \frac{x_i + 2x_i x_k + x_k + x_j + 2x_j x_k + x_k - (x_i + 2x_i x_j + x_j) + x_i - 2x_i x_k + x_k + x_j - 2x_j x_k + x_k - (x_i - 2x_i x_j + x_j)}{4} \right) \\
 &= \begin{pmatrix} x_i \\ x_j \\ x_k \end{pmatrix}
 \end{aligned}$$

We extend the maps  $\alpha$ ,  $\beta$  and  $\varphi$  in a natural way: The map  $\alpha : \mathcal{H}_n \rightarrow \alpha(\mathcal{H}_n)$  is

$$\alpha(x) = \begin{pmatrix} x_1, x_1 x_2, \dots, x_1 x_n, x_2, x_2 x_3, \dots, \\ x_2 x_n, \dots, x_{n-1}, x_{n-1} x_n, x_n \end{pmatrix}^T \quad (24)$$

The map  $\varphi : \mathcal{C}_n \rightarrow \mathbb{R}^{\frac{n(n+1)}{2}}$  is given by  $\varphi(w) = T_n w$  where  $T_n$  is obtained by exploiting the relations of type  $x_i = \frac{u_{ij} + v_{ij} + u_{ik} + v_{ik} - u_{jk} - v_{jk}}{2}$  and  $x_i x_j = \frac{u_{ij} - v_{ij}}{2}$  in such a way that  $T_n E_n \alpha(x) = \alpha(x)$ .

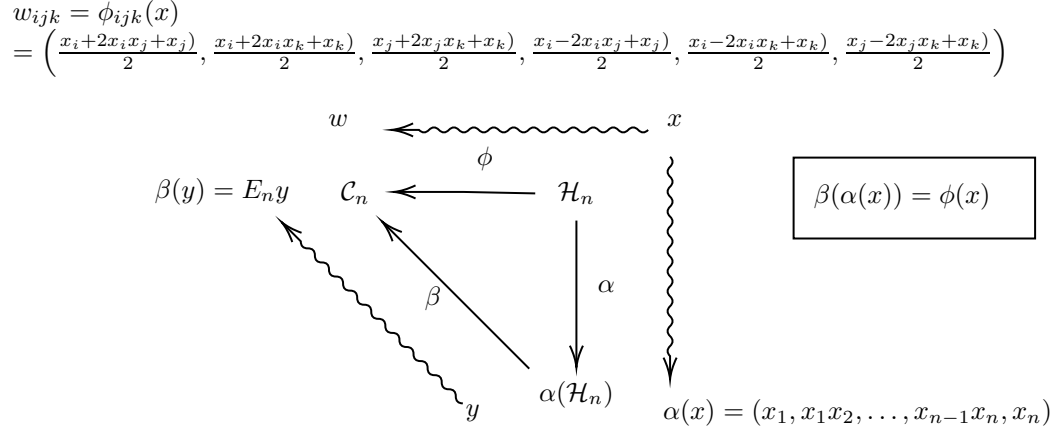


Figure 4: Diagram for the maps  $\phi$ ,  $\alpha$  and  $\beta$  in the general case.

**Example 4.** For  $n = 4$ :

$$T_4 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

The map  $\beta : \alpha(\mathcal{H}_n) \rightarrow \mathcal{C}_n$  is given by

$$\beta(y) = E_n y \text{ for every } y \in \alpha(\mathcal{H}_n) \quad (25)$$

Here  $E_n$  is not a square matrix but a rectangular  $n(n-1) \times \frac{n(n+1)}{2}$  and is defined by exploiting the relations:

$$u_{ij} = \frac{x_i + 2x_i x_j + x_j}{2}, u_{ik} = \frac{x_i + 2x_i x_k + x_k}{2}, u_{jk} = \frac{x_j + 2x_j x_k + x_k}{2} \quad (26)$$

$$v_{ij} = \frac{x_i - 2x_i x_j + x_j}{2}, v_{ik} = \frac{x_i - 2x_i x_k + x_k}{2}, v_{jk} = \frac{x_j - 2x_j x_k + x_k}{2} \quad (27)$$

Figure 4 illustrates the maps  $\phi$ ,  $\alpha$  and  $\beta$ , in a similar way as in the simple case ( $n = 3$ ).

**Example 5.** For  $n = 4$ , the matrix  $E_4$  is

$$\frac{1}{2} \begin{pmatrix} 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 1 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \end{pmatrix}$$

Using the matrix in example 4, it can be checked that  $T_4 E_4 = I_{10}$ .

Now we formulate the minimization problem in  $\mathcal{C}_n$ :

$$(\mathbf{P}'_n): \min_{w \in \mathcal{C}_n} \tilde{f}(w)$$

where the objective function  $\tilde{f} : \mathcal{C}_n \rightarrow \mathbb{R}$  is defined as  $\tilde{f}(x) = \tilde{c}^T w$  with  $\tilde{c}^T = c^T T_n$ . The following Theorem is an extension of Theorem 3, and proves that the minimum of the problem  $(P_n)$  is the same as that of  $(P'_n)$ .

**Theorem 5.** Let  $w^* \in \mathcal{C}_n$  be the point where the minimum of  $(P'_n)$  is attained and  $x^* \in \mathcal{H}_n$  be the point for the minimum of  $(P_n)$ , then  $\tilde{f}(w^*) = f(x^*)$ .

**Proof.** Owing to  $\mathcal{C}_n \subset \phi(\mathcal{H}_n)$  by virtue of Lemma 4, there exists a point  $y \in \mathcal{H}_n$  such that  $\phi(y) = w^*$ . On the other hand, the minimum of  $f$  over  $\mathcal{H}_n$  is attained at  $x^* \in \mathcal{H}_n$ , so that

$$f(x^*) \leq f(y) \tag{28}$$

The connection between  $f$  and  $\tilde{f}$  yields:

$$\tilde{f}(\phi(x^*)) = \tilde{c}^T \phi(x^*) = (c^T T_n)(E_n \alpha(x^*)) = c^T \alpha(x^*) = f(x^*) \tag{29}$$

and

$$f(y) = c^T \alpha(y) = (c^T T_n)(E_n \alpha(y)) = \tilde{c}^T \phi(y) = \tilde{f}(\phi(y)) \tag{30}$$

According to (28) and (30),

$$f(x^*) \leq f(y) = \tilde{f}(w^*)$$

Since the minimum of  $\tilde{f}$  over  $\mathcal{C}_n$  is attained at  $w^* \in \mathcal{C}_n$ , and accounting for (29), we have that

$$f(x^*) = \tilde{f}(\phi(x^*)) \geq \tilde{f}(w^*)$$

Henceforth,  $f(x^*) = \tilde{f}(w^*)$ . ■

$(\mathbf{P}'_n)$  can be written as a linear programming problem:

$$(\mathbf{LP}_n): \begin{cases} \min \tilde{c}^T w \\ \text{such that for } 1 \leq i < j < k \leq n: \\ \begin{cases} B\lambda^{(i,j,k)} - w_{i,j,k} = 0 \\ u^T \lambda^{(i,j,k)} = 1 \\ w^{(i,j,k)} \geq 0, \lambda^{(i,j,k)} \geq 0 \end{cases} \\ \text{and for } 1 \leq j < k \leq n: \\ \begin{cases} r^T (w_{123} - w_{1jk}) = 0 \text{ with } (j,k) \neq (2,3) \\ r^T (w_{213} - w_{2jk}) = 0 \text{ with } (j,k) \neq (1,3) \\ r^T (w_{i,1,2} - w_{i,j,k}) = 0 \text{ with } (j,k) \neq (1,2), i \geq 3 \end{cases} \end{cases}$$

where  $\lambda^{(i,j,k)} \in \mathbb{R}^8$ ,  $B = (\phi(p_1), \dots, \phi(p_8))$  is the matrix in (20),  $r^T = \frac{1}{2}(1, 1, 1, 1, -1, -1)$  and  $u$  is an all-ones vector of appropriate dimension.

Calling  $N = \binom{n}{3}$ ,  $N_1 = \binom{n}{2}$ , and  $N_2 = n \left( \binom{n-1}{2} - 1 \right)$ , the problem  $(\mathbf{LP}_n)$  can be written in matrix form:

$$(\mathbf{LP}_n): \begin{cases} \min \tilde{c}^T w \\ \text{such that :} \\ \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{pmatrix} \begin{pmatrix} \lambda \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} \\ w \geq 0, \lambda \geq 0 \end{cases} \quad (31)$$

where  $A_{11} \in \mathbb{R}^{6N \times 8N}$ ,  $A_{12} \in \mathbb{R}^{6N \times 2N_1}$ ,  $A_{22} \in \mathbb{R}^{N_2 \times 2N_1}$ ,  $A_{31} \in \mathbb{R}^{N \times 8N}$ ,  $u$  is an all-ones vector of dimension  $N$ , and  $A_{21}$  and  $A_{32}$  are zero matrices of dimensions  $N_2 \times 8N$  and  $N \times 2N_1$  respectively. The primary variables  $w$  and the secondary variables have appropriate dimensions. The equivalence of problems  $(\mathbf{P}_n)$ ,  $(\mathbf{P}'_n)$  and  $(\mathbf{LP}_n)$  is described in Figure 5, where

$$\tilde{A}_{11} = \begin{pmatrix} A_{11} \\ A_{21} \\ A_{31} \end{pmatrix}, \tilde{A}_{22} = \begin{pmatrix} A_{12} \\ A_{22} \\ A_{32} \end{pmatrix}, \tilde{b} = \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}$$

**Example 6.** Let  $Q \in \mathbb{Z}^{4 \times 4}$  be the symmetric matrix:

$$Q = \begin{pmatrix} 0 & -30 & 6 & -22 \\ -30 & 0 & 15 & -2 \\ 6 & 15 & 0 & -5 \\ -22 & -2 & -5 & 0 \end{pmatrix}$$

and  $b \in \mathbb{Z}^4$  the vector

$$b = \begin{pmatrix} -8 \\ -22 \\ 0 \\ -32 \end{pmatrix}$$

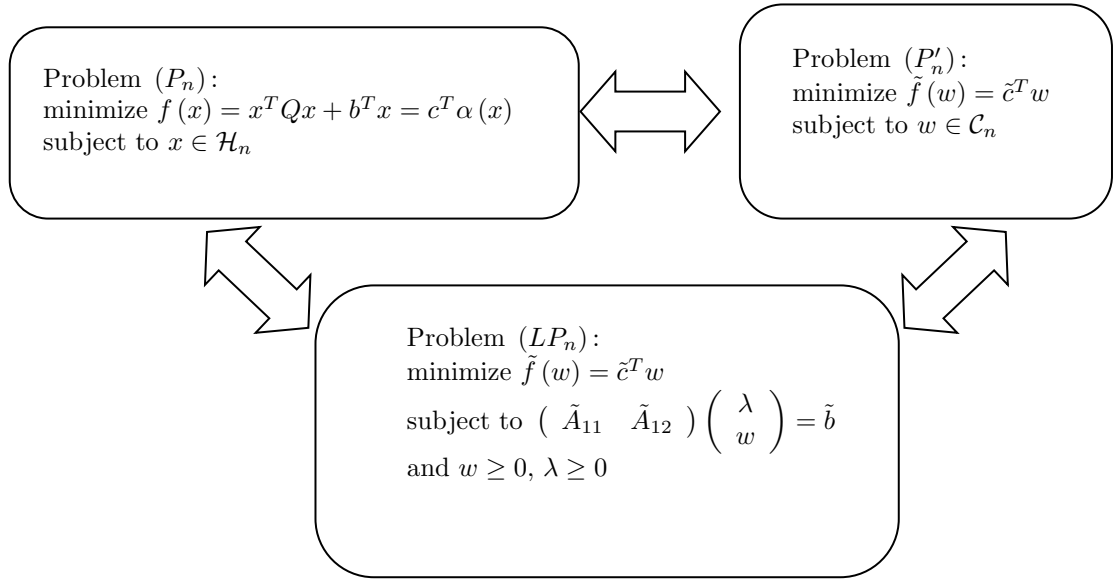


Figure 5: Equivalence of problems  $(P_n)$ ,  $(P'_n)$  and  $(LP_n)$

The consistency constraints were given in Example 3. The optimal value of  $f$  is  $-170$  and

$$\lambda^* = e_7 + e_{16} + e_{22} + e_{30}$$

$$w^* = \left( 2, \frac{1}{2}, 2, \frac{1}{2}, 2, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2} \right)^T$$

where  $e_k \in \mathbb{R}^{32}$  is the  $k$ -th vector of the standard basis (with an entry '1' at the position  $k$  and '0' for the rest of positions).

The optimal point  $x^*$  can be recovered from  $w$  by applying  $E_3$  to  $w_{ijk}$ . To illustrate this point we compute  $x^*$  for the previous example.

**Example 7.** From example 6:

$$w_{123}^* = \left( 2, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2} \right)^T$$

$$w_{234}^* = \left( \frac{1}{2}, 2, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2} \right)^T$$

We can recover the point  $x^*$  as follows:

$$E_3^{-1} w_{123}^* = (1, 1, 0, 1, 0, 0)^T = (x_1, x_1 x_2, x_1 x_3, x_2, x_2 x_3, x_3)^T$$



and this means  $x_1^* = x_2^* = 1$ ,  $x_3^* = 0$ . We proceed in a similar way for  $w_{234}^*$ :

$$E_3^{-1}w_{234}^* = (1, 0, 1, 0, 0, 1)^T = (x_2, x_2x_3, x_2x_4, x_3, x_3x_4, x_4)^T$$

which implies that  $x_2^* = 1$ ,  $x_3^* = 0$ ,  $x_4^* = 1$ . The optimal point is  $x^* = (1, 1, 0, 1)$ .

#### 4. Computational Complexity

The problem  $(\mathbf{LP}_n)$  requires an amount of memory given by the dimension of  $A$  in (31), that is  $(7N + N_2)(8N + 2N_1)$ . Henceforth, the space complexity is of order  $O(n^6)$ . Since both  $B$  (involved in the definition of the convex-hull  $\mathcal{C}_3^{(i,j,k)}$ ) and  $E_3$  (corresponding to the constraint  $\sum_{l=1}^8 \lambda_l^{(i,j,k)} = 1$  in the convex-hull  $\mathcal{C}_3^{(i,j,k)}$ ) are constant, the time complexity is also  $O(n^6)$  (assuming that the storing of an entry in a matrix is  $O(1)$ ). Note that the objective function  $\tilde{f}(w) = \tilde{c}^T w$  requires  $2N_1$  multiplications and  $2N_1 - 1$  sums, resulting in a time complexity of order  $O(n^2)$ .

Once generated the matrix  $A$  and the vector  $\tilde{c}$ , the problem  $(\mathbf{LP}_n)$  can be solved in polynomial-time via interior-point methods: (i) ellipsoid method due to Khachiyan ( $O(n^6L)$  where  $L$  denotes the number of bits in a binary representation of  $A$ ), (ii) projective algorithm of Karmarkar ( $O(n^{\frac{7}{2}}L)$ ), (iii) Vaidya's 87 algorithm ( $O(n^3)$ ), or (iv) Vaidya's 89 algorithm ( $O(n^{\frac{5}{2}})$ ), among others. Henceforth, the problem  $(\mathbf{P}_n)$  is solved in polynomial time. The number of variables in  $(\mathbf{PL}_n)$  is  $8N + 2N_1$  so that the problem can be solved with Vaidya's 89 algorithm in  $O(n^{\frac{15}{2}})$ .

#### 5. An Application to the 3-SAT Problem: P=NP

The 3-SAT problem is a satisfiability problem which can be enunciated as follows:

**(3-SAT):** Let  $f : \mathcal{B}^n \rightarrow \mathcal{B}$  be a boolean function in conjunctive normal form 3-CNF (a conjunction of disjunctive clauses of three literals):

$$f(x_1, \dots, x_n) = \bigwedge_{\substack{S \in \mathcal{P}(\{1, 2, \dots, n\}) \\ |S|=3}} c_S \bigvee_{i \in S} l_i(x_i)$$

where  $c_S \in \mathcal{B}$ , and the literal  $l(x_i)$  is either a variable,  $l_i(x_i) = x_i$  (positive literal), or the negation of a variable  $l_i(x_i) = \bar{x}_i$  (negative literal). The 3-SAT decides whether or not there exists a  $x^* \in \mathcal{B}^n$  such that  $f(x^*) = 1$ .

This problem is NP-complete since the general problem of satisfiability is NP-complete. Indeed SAT was the first known NP-complete problem found in 1973, as proved by Stephen Cook, [34], and [35].

The problem 3-SAT can be stated in disjunctive normal form (3-DNF) by applying the Morgan's laws (transformation of symbols  $x \mapsto \bar{x}$ ,  $\vee \mapsto \wedge$ ):

$$f(x_1, \dots, x_n) = 1 \text{ if and only if } \bar{f}(x) = \bigvee_{\substack{S \in \mathcal{P}(\{1,2,\dots,n\}) \\ |S|=3}} c_S \bigwedge_{i \in S} \bar{l}_i(x_i) = 0$$

Henceforth, **the 3-SAT problem is to decide when  $\bar{f}$  is not a tautology.**

A 3-DNF boolean function can be arithmetized by the transformations  $\vee \mapsto +$ ,  $\wedge \mapsto \cdot$ ,  $\bar{x} \mapsto 1 - x$ . This leads to a pseudo-Boolean function  $g : \mathcal{B}^n \rightarrow \mathbb{Z}$ :

$$g(x_1, \dots, x_n) = \prod_{\substack{S \in \mathcal{P}(\{1,2,\dots,n\}) \\ |S| \leq 3}} a_S \prod_{i \in S} x_i$$

where  $a_S \in \mathbb{Z}$ . Due to the construction procedure of  $g$ ,  $g(x) \geq 0$  for all  $x \in \mathcal{B}^n$ . According to Rosenberg's multilinear polynomial extension, we have the following equivalence:

$$\exists x^* \in \mathcal{B}^n \text{ such that } f(x^*) = 1 \text{ iff } g(x^*) = \min_{x \in \mathcal{H}_n} g(x) = 0$$

The third-order pseudo-Boolean function can be quadratized from the Freedman-Drineas-Ishikawa method. This results in a quadratic optimization problem:  $\min_{y \in \mathcal{H}_d} y^T Q y + b^T y + c$ , with  $Q \in \mathbb{Z}^{d \times d}$ ,  $b \in \mathbb{Z}^d$  and  $c \in \mathbb{Z}$ , where  $y$  includes the  $n$  variables  $x_i$  and the  $d-n$  auxiliary variables  $w_j$  in (4) and (5). We have reduced 3-SAT problem to an optimization problem of quadratic pseudo-Boolean functions, that it was shown above has a polynomial-time complexity.

### 5.1. Computational Complexity of 3-SAT

At first glance a 3-DNF formula depends on the number of clauses  $m$  and the number of Boolean variables  $n$ . Which of these two quantities should be considered as relevant in order to analyze the computational complexity? Are they completely independent? To answer these two questions we previously need to remove both duplicated clauses and inconsistent clauses, i.e. clauses including a pair of literals of the form  $x\bar{x}$ . After removing those clauses in  $\bar{f}$ , if  $m = 8\binom{n}{3}$ ,  $f$  is unsatisfiable in virtue of Whitehead's Theorem (see Theorem 5, Ch. 1, Section 3, pp. 27-28 in [1]). As a consequence, the assumption on the number of clauses,  $m < 8\binom{n}{3}$ , is very reasonable. Therefore  $m \in O(n^3)$  and the number of variables  $n$  turns out to be the interest variable.

#### 5.1.1. Arithmetization

A 3-DNF formula  $f(x)$  in  $n$  variables  $x_1, \dots, x_n$ , has conjunctive clauses of the types:  $x_i x_j x_k$ ,  $\bar{x}_i x_j x_k$ ,  $x_i \bar{x}_j x_k$  and  $\bar{x}_i \bar{x}_j \bar{x}_k$ . The maximum number of clauses in  $f(x)$ , for every type of clause, can be determined from Combinatorics, and is collected in Table 1. For each type of conjunctive clause in the variables  $x_i$ ,  $x_j$  and  $x_k$ , Table 2 shows the number of monomials (of degree 0, 1, 2 and

Type	Maximum number of clauses in $f(x)$
$x_i x_j x_k$	$\binom{n}{3}$
$\bar{x}_i x_j x_k$	$3 \binom{n}{3}$
$\bar{x}_i \bar{x}_j x_k$	$3 \binom{n}{3}$
$\bar{x}_i \bar{x}_j \bar{x}_k$	$\binom{n}{3}$

Table 1: Maximum number of clauses in  $f(x)$  according to the type of clause.

Type	Degree of the Monomial			
	0	1	2	3
$x_i x_j x_k$	0	0	0	1
$\bar{x}_i x_j x_k$	0	0	1	1
$\bar{x}_i \bar{x}_j x_k$	0	1	2	1
$\bar{x}_i \bar{x}_j \bar{x}_k$	1	3	3	1

Table 2: Number of monomials of degree 0, 1, 2, and 3, generated from the arithmetization of a type of conjunctive clause.

3) generated after arithmetizing. On the basis of the information in Tables 1 and 2, we have an upper bound on the number of monomials generated by the arithmetization process, see Table 3. Then the number of sums required to build the coefficients in  $g$  is overestimated to

$$\binom{n}{3} - 1 + 4 \cdot \binom{n}{3} - 1 + 6 \cdot \binom{n}{3} - 1 + 4 \cdot \binom{n}{3} - 1 = 15 \binom{n}{3} - 4$$

This bound is very conservative but it is sufficient to show that the arithmetization process is  $O(n^3)$ .

Degree of the monomials	Number of monomials
0	$\binom{n}{3}$
1	$4 \cdot \binom{n}{3}$
2	$6 \cdot \binom{n}{3}$
3	$4 \cdot \binom{n}{3}$

Table 3: Upper-bound on the number of monomials generated by arithmetization

### 5.1.2. Quadraticization

The quadraticization of a 3-degree monomial  $ax_ix_jx_k$  can be carried out from (4) and (5):

$$ax_ix_jx_k = \min_{w \in \mathcal{B}} w(S_1 - 2), \text{ for } a < 0$$

$$ax_ix_jx_k = \min_{v \in \mathcal{B}} v(1 - S_1) + aS_2, \text{ for } a > 0$$

As shown above, for every 3-degree monomial, the quadraticization adds an auxiliary variable  $w \in \mathcal{B}$  regardless of the sign of the monomial. Since the number of 3-degree monomials in  $g$  is not greater than  $8\binom{n}{3}$ , the number of variables is upper-bounded by  $n + 8\binom{n}{3}$ . For positive 3-degree monomials we add new monomials in  $w(S_1 - 2)$ , not generated by other monomials (since are multiplied by the auxiliary variable  $w$  uniquely attached to  $ax_ix_jx_k$ ). For negative 3-degree variables, we add new monomials in  $v(1 - S_1)$ , again not generated by other monomials, and monomials in  $aS_2$  that should be combined with those 2-degree monomials yet existing from the arithmetization. Every monomial  $ax_ix_j$  appears in  $(n - 2)$  3-degree monomials, and in the worst case this means  $n - 2$  sums (including the addition of the monomial  $x_ix_j$  previously generated in the arithmetization process. As a result, we have an overestimation of  $\binom{n}{2}(n - 2)$  additional sums. Henceforth, the quadraticization process is  $O(n^3)$ .

After the quadraticization we have a minimization problem of a quadratic pseudo-Boolean function:

$$\min_{y \in B^d} y^T Q y + b^T y + c \quad (32)$$

The space complexity is given by the dimension of  $Q$  and is  $O(n^6)$  (recall that the number of variables is upper-bounded by  $n + 8\binom{n}{3}$ ).

### 5.1.3. 3-SAT Complexity

Applying Vaidya's 89 algorithm, (32) can be solved in  $O(n^{\frac{45}{2}})$ ; This order, although very conservative, represents a polynomial-time order. It is well known that 3-SAT problem belongs to the class NP-complete. This means P=NP.

## 6. Conclusion

In this paper, we have developed an algorithm to find the global minimum of a quadratic pseudo-Boolean function in polynomial time. Specifically, the computational complexity is  $O(n^{\frac{15}{2}})$ ; this bound is very conservative but sufficient to prove that the problem is in the class  $P$ . In the future this bound might be substantially reduced improving the efficiency of the algorithm.

A related problem is the 3-SAT problem, which is one of Karp's 21 NP-complete problems. This problem is proved to have a computational complexity  $O(n^{\frac{45}{2}})$ , again very conservative, but also sufficient from a theoretical viewpoint to prove that P=NP. The algorithm has been implemented in MATLAB and checked by

generating one million matrices of arbitrary dimension up to 24 with random entries in the range  $[-50, 50]$ . All the experiments were successful which conjures us up to think that the procedure is correct (the source code is included in the appendix).

## 7. Appendix

### 7.1. Source Code in MATLAB

In this appendix we include the source code to create a Linear Program equivalent to the 0-1 Quadratic program:

```
% Quadratic Pseudo-Boolean Minimization
%
% Computes the global minimum of a quadratic pseudo-Boolean function
in
% polynomial time. The result is compared with the exact minimum
obtained
% by brute force.
%
% Author: Juan Ignacio Mulero-Martínez
% Date: 13-06-2020.

% Definition of the Objective Quadratic Function  $f(x)=x'*Q*x+b'x$ 
clear all; % We clear memory before running
n=4; % We fix the dimension of Q.
lv=-20; % Lower value for the range of values
uv=20; % Upper value for the range of values

%P = randi([lv,uv],n); % Create a random matrix of integers in a
% specified range [lv,uv].

P=(uv-lv)*rand(n)+lv; % If you prefer to work with coefficients
in the
% Reals you should activate this line.
Q=P+P'; % Create the symmetric matrix Q from P.
% We set the diagonal entries to zero to avoid quadratic terms  $(x_i)^2$ 
Q=Q-diag(diag(Q));
% A random vector b of dimension nx1 is generated.
%
b=(uv-lv)*rand(n,1)+lv;
```

```

N=nchoosek(n,3); % Number of 3-combinations in the set {1,2,...,n}
N1=nchoosek(n,2); % Number of 2-combinations in the set {1,2,...,n}

if n>3,
    N2=(nchoosek(n-1,2)-1)*n;
else N2=0;
end;

% Transformation of the objective function into a linear form cc'*w

%Vector c from f(x)=c'*alpha(x)

c=zeros(n*(n+1)/2,1);

cont=0;
for i=1:n
    cont=cont+1;
    c(cont)=b(i);
    for j=i+1:n
        cont=cont+1;
        c(cont)=2*Q(i,j);
    end;
end;
% Generate Linear Transformation T
T=zeros(n*(n+1)/2,2*N1);
cont=0;
for i=1:n,
    S=1:n;
    cont=cont+1;

    % Generate xi^2

    switch i
    case 1
        T(cont,8*N+ind(1,2,n))=1/2;
        T(cont,8*N+ind(1,3,n))=1/2;
        T(cont,8*N+ind(2,3,n))=-1/2;

        T(cont,8*N+N1+ind(1,2,n))=1/2;
        T(cont,8*N+N1+ind(1,3,n))=1/2;
        T(cont,8*N+N1+ind(2,3,n))=-1/2;
    end;
end;

```

```

case 2
T(cont,8*N+ind(1,2,n))=1/2;
T(cont,8*N+ind(2,3,n))=1/2;
T(cont,8*N+ind(1,3,n))=-1/2;

T(cont,8*N+N1+ind(1,2,n))=1/2;
T(cont,8*N+N1+ind(2,3,n))=1/2;
T(cont,8*N+N1+ind(1,3,n))=-1/2;
otherwise
T(cont,8*N+ind(1,2,n))=-1/2;
T(cont,8*N+ind(1,i,n))=1/2;
T(cont,8*N+ind(2,i,n))=1/2;

T(cont,8*N+N1+ind(1,2,n))=-1/2;
T(cont,8*N+N1+ind(1,i,n))=1/2;
T(cont,8*N+N1+ind(2,i,n))=1/2;

end;

combos = combntns(S,2);
for j=i+1:n,
cont=cont+1;
T(cont,8*N+ind(i,j,n))=1/2;
T(cont,8*N+N1+ind(i,j,n))=-1/2;
end;
end;
cc=c'*T;
% Definition of the basic block B
B=zeros(6,8);
i=0;
for x1=0:1,
for x2=0:1,
for x3=0:1,
i=i+1;
B(1,i)=((x1+x2)^2)/2;
B(2,i)=((x1+x3)^2)/2;
B(3,i)=((x2+x3)^2)/2;
B(4,i)=((x1-x2)^2)/2;
B(5,i)=((x1-x3)^2)/2;
B(6,i)=((x2-x3)^2)/2;

end;
end;
end;

```

```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Consistency Constraints
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
if (n>3),
M=zeros(N2,2*N1);
cont=0;
for i=1:n,
    S=1:n;
    S(i)=[];
    combos = combntns(S,2);
    j1=combos(1,1);
    k1=combos(1,2);
    for l=2:length(combos),
        cont=cont+1;
        j=combos(l,1);
        k=combos(l,2);
        % Parte de u
        M(cont,ind(i,j1,n))=M(cont,ind(i,j1,n))-1;
        M(cont,ind(i,k1,n))=M(cont,ind(i,k1,n))-1;
        M(cont,ind(j1,k1,n))=M(cont,ind(j1,k1,n))+1;

        M(cont,ind(i,j,n))=M(cont,ind(i,j,n))+1;
        M(cont,ind(i,k,n))=M(cont,ind(i,k,n))+1;
        M(cont,ind(j,k,n))=M(cont,ind(j,k,n))-1;
    end;
end;

% Here we repeat for the variable v the same matrix computed above
for u
M(:,N1+1:2*N1)=M(:,1:N1);
end;

% MATRIX A FOR THE EQUALITIES IN THE LINEAR PROGRAMMING PROBLEM
%%%

Aeq=zeros(7*N+N2,8*N+2*N1);
cont=0;
for i=1:n,
    for j=i+1:n,
        for k=j+1:n,
            cont=cont+1;

% First Block Row

```



```

Aeq((cont-1)*6+1:cont*6,(cont-1)*8+1:cont*8)=B;
% Variables u
Aeq((cont-1)*6+1,8*N+ind(i,j,n))=-1;
Aeq((cont-1)*6+2,8*N+ind(i,k,n))=-1;
Aeq((cont-1)*6+3,8*N+ind(j,k,n))=-1;
% Variables v
Aeq((cont-1)*6+4,8*N+N1+ind(i,j,n))=-1;
Aeq((cont-1)*6+5,8*N+N1+ind(i,k,n))=-1;
Aeq((cont-1)*6+6,8*N+N1+ind(j,k,n))=-1;

% Second Block Row

if n>3, Aeq(6*N+1:6*N+N2,8*N+1:8*N+2*N1)=M;
end;

% Third Block Row

for l=1:N,

Aeq(6*N+N2+1,(l-1)*8+1:l*8)=ones(1,8);
end;

end;
end;
end;

beq=[zeros(6*N+N2,1);ones(N,1)];
lb=zeros(8*N+2*N1,1);
[x,fval]=linprog(cc,[],[],Aeq,beq,lb)

% EXACT SOLUTION BY BRUTE FORCE (Efficient Code)

%// Sample data
bit = [0,1]; %// Set of possible letters
%// Create all possible permutations (with repetition) of letters
stored in x
C = cell(n, 1); %// Preallocate a cell array
[C{:}] = ndgrid(bit); %// Create K grids of values
a = cellfun(@(bit){bit(:)}, C); %// Convert grids to column vectors
a = [a{:}];
minimo=inf;

```

```

for i=1:length(a),
    fobj=a(i,:)*Q*a(i,:)'+b'*a(i,:)';
    if fobj<minimo,
        minimo=fobj;
    end;
end;

% BRUTE FORCE: This code is inefficient
%minimo=inf;
%for i=0:(2^n)-1,
% y=de2bi(i,n,'left-msb');
% fobj=y*Q*y';
% if fobj<minimo,
% minimo=fobj;
% end;
%end;
%if round(fval)==minimo, %Use this condition for matrices in the
%Integers

```

```

if abs(cc*x-minimo)<1e-3,
    disp('SUCCESS');
else
    disp('UNSUCCESS');
end;

```

This program requires the function ind:

```

function y=ind(i,j,N)
if i<j,
    y=(i-1)*(2*N-i)/2+(j-i);
else
    y=(j-1)*(2*N-j)/2+(i-j);
end

```

## References

- [1] P. L. Hammer, S. Rudeanu, Boolean Methods in Operations Research and Related Areas, Springer Berlin Heidelberg, 1968.  
URL <https://doi.org/10.1007/978-3-642-85823-9>
- [2] E. Boros, P. L. Hammer, Pseudo-boolean optimization, Discrete Applied Mathematics 123 (1-3) (2002) 155–225.  
URL [https://doi.org/10.1016/s0166-218x\(01\)00341-9](https://doi.org/10.1016/s0166-218x(01)00341-9)

- [3] P. Hansen, B. Jaumard, V. Mathon, State-of-the-art survey—constrained nonlinear 0–1 programming, *ORSA Journal on Computing* 5 (2) (1993) 97–119.  
URL <https://doi.org/10.1287/ijoc.5.2.97>
- [4] P. M. Pardalos, S. Jha, Complexity of uniqueness and local search in quadratic 0–1 programming, *Operations Research Letters* 11 (2) (1992) 119–123.  
URL [https://doi.org/10.1016/0167-6377\(92\)90043-3](https://doi.org/10.1016/0167-6377(92)90043-3)
- [5] A. A. Schäffer, Simple local search problems that are hard to solve, *SIAM Journal on Computing* 20 (1) (1991) 56–87.  
URL <https://doi.org/10.1137/0220004>
- [6] C. A. Tovey, Hill climbing with multiple local optima, *SIAM Journal on Algebraic Discrete Methods* 6 (3) (1985) 384–393.  
URL <https://doi.org/10.1137/0606040>
- [7] C. A. Tovey, Low order polynomial bounds on the expected performance of local improvement algorithms, *Mathematical Programming* 35 (2) (1986) 193–224.  
URL <https://doi.org/10.1007/bf01580647>
- [8] P. L. Hammer, B. Simeone, T. M. Liebling, D. de Werra, From linear separability to unimodality: A hierarchy of pseudo-boolean functions, *SIAM Journal on Discrete Mathematics* 1 (2) (1988) 174–184.  
URL <https://doi.org/10.1137/0401019>
- [9] K. W. Hoke, Completely unimodal numberings of a simple polytope, *Discrete Applied Mathematics* 20 (1) (1988) 69–81.  
URL [https://doi.org/10.1016/0166-218x\(88\)90042-x](https://doi.org/10.1016/0166-218x(88)90042-x)
- [10] M. Emamy-K., The worst case behavior of a greedy algorithm for a class of pseudo-boolean functions, *Discrete Applied Mathematics* 23 (3) (1989) 285–287.  
URL [https://doi.org/10.1016/0166-218x\(89\)90018-8](https://doi.org/10.1016/0166-218x(89)90018-8)
- [11] C. A. Tovey, 3. local improvement on discrete structures, in: E. Aarts, J. K. Lenstra (Eds.), *Local Search in Combinatorial Optimization*, Princeton University Press, 2003, pp. 57–90.  
URL <https://doi.org/10.1515/9780691187563-006>
- [12] I. R. P.L. Hammer, S. Rudeanu, On the determination of the minima of pseudo-boolean functions (in romanian), *Studii si Cercetari Matematice* 14 (1963) 359–364.
- [13] Y. Crama, P. Hansen, B. Jaumard, The basic algorithm for pseudo-boolean programming revisited, *Discrete Applied Mathematics* 29 (2-3) (1990) 171–185.  
URL [https://doi.org/10.1016/0166-218x\(90\)90142-y](https://doi.org/10.1016/0166-218x(90)90142-y)

- [14] G. B. Dantzig, On the significance of solving linear programming problems with some integer variables, *Econometrica* 28 (1) (1960) 30.  
URL <https://doi.org/10.2307/1905292>
- [15] R. Fortet, L'algebre de boole et ses applications en recherche operationnelle, *Trabajos de Estadistica* 11 (2) (1960) 111–118.  
URL <https://doi.org/10.1007/bf03006558>
- [16] F. Glover, E. Woolsey, Technical note—converting the 0-1 polynomial programming problem to a 0-1 linear program, *Operations Research* 22 (1) (1974) 180–182.  
URL <https://doi.org/10.1287/opre.22.1.180>
- [17] P. Hansen, Methods of nonlinear 0-1 programming, in: *Discrete Optimization II, Proceedings of the Advanced Research Institute on Discrete Optimization and Systems Applications of the Systems Science Panel of NATO and of the Discrete Optimization Symposium co-sponsored by IBM Canada and SIAM Banff, Aha. and Vancouver, Elsevier, 1979*, pp. 53–70.  
URL [https://doi.org/10.1016/s0167-5060\(08\)70343-1](https://doi.org/10.1016/s0167-5060(08)70343-1)
- [18] E. Balas, J. B. Mazzola, Nonlinear 0–1 programming: I. linearization techniques, *Mathematical Programming* 30 (1) (1984) 1–21.  
URL <https://doi.org/10.1007/bf02591796>
- [19] E. Balas, E. Zemel, Facets of the knapsack polytope from minimal covers, *SIAM Journal on Applied Mathematics* 34 (1) (1978) 119–148.  
URL <https://doi.org/10.1137/0134010>
- [20] P. L. Hammer, P. Hansen, B. Simeone, Roof duality, complementation and persistency in quadratic 0–1 optimization, *Mathematical Programming* 28 (2) (1984) 121–155.  
URL <https://doi.org/10.1007/bf02612354>
- [21] J.-M. Bourjolly, P. L. Hammer, W. R. Pulleyblank, B. Simeone, BOOLEAN-COMBINATORIAL BOUNDING OF MAXIMUM 2-SATISFIABILITY, in: *Computer Science and Operations Research, Elsevier, 1992*, pp. 23–42.  
URL <https://doi.org/10.1016/b978-0-08-040806-4.50007-1>
- [22] P. Hammer, The conflict graph of a pseudo-boolean function, Tech. rep., Bell Laboratories (August 1978).
- [23] A. H. (alias P.L. Hammer), Stories of the one-zero-zero-one nights: Abu boul in graphistan, in: P. Hansen, D. de Werra (Eds.), *Regards sur la Theorie des Graphes, Presses Polytechniques Romandes, Lausanne, 1980*.
- [24] C. Ebenegger, P. Hammer, D. de Werra, Pseudo-boolean functions and stability of graphs, in: *Algebraic and Combinatorial Methods in Operations*

- Research, Proceedings of the Workshop on Algebraic Structures in Operations Research, Elsevier, 1984, pp. 83–97.  
 URL [https://doi.org/10.1016/s0304-0208\(08\)72955-4](https://doi.org/10.1016/s0304-0208(08)72955-4)
- [25] G. Alexe, P. L. Hammer, V. V. Lozin, D. de Werra, Struction revisited, *Discrete Applied Mathematics* 132 (1-3) (2003) 27–46.  
 URL [https://doi.org/10.1016/s0166-218x\(03\)00388-3](https://doi.org/10.1016/s0166-218x(03)00388-3)
- [26] P. L. Hammer, N. V. R. Mahadev, D. de Werra, The struction of a graph: Application to CN-free graphs, *Combinatorica* 5 (2) (1985) 141–147.  
 URL <https://doi.org/10.1007/bf02579377>
- [27] A. Hertz, On the use of boolean methods for the computation of the stability number, *Discrete Applied Mathematics* 76 (1-3) (1997) 183–203.  
 URL [https://doi.org/10.1016/s0166-218x\(96\)00124-2](https://doi.org/10.1016/s0166-218x(96)00124-2)
- [28] P. L. Hammer, B. Simeone, Quadratic functions of binary variables, in: *Lecture Notes in Mathematics*, Springer Berlin Heidelberg, 1989, pp. 1–56.  
 URL <https://doi.org/10.1007/bfb0083461>
- [29] E. Boros, A. Gruber, On quadratization of pseudo-boolean functions, in: *International Symposium on Artificial Intelligence and Mathematics, ISAIM 2012, Fort Lauderdale, Florida, USA, January 9-11, 2012, 2012*.  
 URL [http://www.cs.uic.edu/pub/Isaim2012/WebPreferences/ISAIM2012\\_Boolean\\_Boros\\_Gruber.p](http://www.cs.uic.edu/pub/Isaim2012/WebPreferences/ISAIM2012_Boolean_Boros_Gruber.p)
- [30] Y. Crama, P. Hammer, *Boolean Functions: Theory, Algorithms, and Applications*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 2011.  
 URL [https://books.google.es/books?id=3KmyKpw\\_pbUC](https://books.google.es/books?id=3KmyKpw_pbUC)
- [31] H. Ishikawa, Transformation of general binary MRF minimization to the first-order case, *IEEE Transactions on Pattern Analysis and Machine Intelligence* 33 (6) (2011) 1234–1249.  
 URL <https://doi.org/10.1109/tpami.2010.91>
- [32] D. Freedman, P. Drineas, Energy minimization via graph cuts: Settling what is possible, in: *2005 IEEE Computer Society Conference on Computer Vision and Pattern Recognition, IEEE, 2005*.  
 URL <https://doi.org/10.1109/cvpr.2005.143>
- [33] I. G. Rosenberg, Brèves communications. 0-1 optimization and non-linear programming, *RAIRO - Operations Research - Recherche Opérationnelle* 6 (V2) (1972) 95–97.
- [34] S. A. Cook, The complexity of theorem-proving procedures, in: *Proceedings of the Third Annual ACM Symposium on Theory of Computing, STOC '71, ACM, New York, NY, USA, 1971, pp. 151–158*.  
 URL <http://doi.acm.org/10.1145/800157.805047>

- [35] B. A. Trakhtenbrot, A survey of russian approaches to perebor (brute-force searches) algorithms., *IEEE Annals of the History of Computing* 6 (4) (1984) 384-400.  
URL <http://dblp.uni-trier.de/db/journals/annals/annals6.html#Trakhtenbrot84>