Lie identities in symmetric elements in group rings. A survey

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To Professor César Polcino on the occasion of his 60th birthday.

Abstract. In this survey we gather some results concerning polynomial identities (resp. group identities) in the set of symmetric elements (resp. symmetric units), and when these identities are transferred to the whole group ring (resp. unit group). We consider several kinds of involutions, not only the classical involution.

1. Introduction

Let R be a ring with unity and $*$ a ring involution on R. The set of $*$-symmetric elements, or just symmetric when the involution $*$ is clear from the context, is $R_* = \{r \in R |\ r^* = r\}$. We are going to denote by $\mathcal{U}(R)$ the group of units of R and by $\mathcal{U}_*(R) = \{u \in \mathcal{U}(R) |\ u^* = u\}$ the set of symmetric units. In general the set R_* (respectively $\mathcal{U}_*(R)$) is not a ring (resp. a subgroup). In fact R_* (resp. $\mathcal{U}_*(R)$) is a ring (resp. subgroup) if and only if the symmetric elements (resp. units) commute.

Several papers have dealt with questions of how various algebraic properties of the set R_*, affect the structure of the whole ring. Similar question may be posed by making assumptions about the symmetric units or subgroup they generate.

Definition 1.1. *Let A be an R-algebra and $S \subset A$ be a subset. We say that S satisfies a polynomial identity (PI for short) if there exists a nonzero polynomial $f(z_1, \ldots, z_n)$ in the polynomial ring $R\langle z_1, \ldots, z_n\rangle$ in the non-commuting variables $z_1, \ldots, z_n$ with $f(s_1, \ldots, s_n) = 0$ for all $s_i \in S$.*

The set of units $S \cap \mathcal{U}(A)$ of the set S is said to satisfy a group identity (GI for short) if there exists a nontrivial word $w(x_1, \ldots, x_n)$ in the free group generated by $x_1, \ldots, x_n$ such that $w(u_1, \ldots, u_n) = 1$ for all $u_1, \ldots, u_n \in S \cap \mathcal{U}(A)$.

Since results about symmetric units in rings with involution seem to be difficult to obtain, one way to begin the study of these units is to try to mimic known results for the symmetric elements. A fundamental result of this kind is the following theorem of Amitsur [1].

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Theorem 1.2. [1] *If R is an algebra with involution whose symmetric elements S satisfy a polynomial identity, then R itself satisfies a polynomial identity.*

Most of the results concerning identities of the symmetric elements and symmetric units, and how these identities are transferred to the whole ring or its group of units respectively, have been stated in the context of group rings.

Let R be a commutative ring with unity and let G be a group. We denote by RG the group ring of G over R. This ring is a ring with involution. Any involution on the group G can be extended R-linearly to an involution on RG.

Let $*$ be an R-linear involution in RG. We define the set of symmetric elements in the group G by $G_* = \{g \in G | \; g^* = g\}$. Notice that an element α of RG is $*$-symmetric if and only if

$$\alpha^* = \left(\sum_{g \in G} \alpha_g g \right)^* = \sum_{g \in G} \alpha_g g^* = \sum_{g \in G} \alpha_g g = \alpha;$$

i.e., if and only if $\alpha_g = \alpha_{g^*}$, for all $g \in G$.

Thus, $(RG)_*$ is generated, as an $R-$module, by the set

$$\mathcal{S} = \{g + g^* \mid g \in G \text{ with } g^* \notin G_*\} \cup G_*.$$

Therefore $(RG)_*$ satisfy a multilinear polynomial identity if and only if $\mathcal{S}$ satisfy the same identity.

A complete classification of group algebras satisfying a polynomial identity was given by Isaacs and Pasmman [13] for characteristic zero and Passman in [26] for characteristic p. Taking into account Theorem 1.2 we resume these results in the following theorem.

Theorem 1.3. *Let R be a field of characteristic p and G be a group. Let $*$ be an involution of the group algebra RG. Then the following are equivalents.*

(i) *$(RG)_*$ satisfies a PI.*
(ii) *RG satisfies a PI.*
(iii) *G has a $p-$abelian subgroup of finite index.*

Recall that a group G is said to be p-abelian if the derived subgroup G' is a finite p-group. We say that G is 0-abelian if it is abelian.

Some specific PI's as Lie nilpotence, Lie n-Engel and commutativity have also been studied. We will call these identities *Lie-identities.* Recall that a set S of an R-algebra A is Lie nilpotent if, for some integer $n \geq 2$, $[x_1, x_2, \ldots, x_n] = 0$, for any $x_1, \ldots, x_n \in S$, where $[x, y] = xy - yx$ denote the *Lie bracket* and inductively $[x_1, \ldots, x_n] = [[x_1, \ldots, x_{n-1}], x_n]$. Also, recall that S is Lie n-Engel if $[x, \underbrace{y, y, \ldots, y}_{n}] = 0$, for all $x, y \in S$.

As we will see in the following sections, that the set of symmetric elements satisfies a Lie-identity does not imply that this Lie-identity is always satisfied by the whole algebra.

In this survey we are going to collect results concerning Lie-identities in the set of symmetric elements and symmetric units of a group ring and when these identities are transferred to the whole ring or the whole group of units respectively.

The rest of the paper is structured as follows. In Section 2 we collect the results concerning commutativity, Lie nilpotence and Lie n-Engel for the set of symmetric elements in group rings RG when the involution $*$ is the classical involution on the group G, that is $g^* = g^{-1}$. In Section 3 we deal with the commutativity of the symmetric elements when the involution $*$ is any group involution which is extended by linearity to the whole group ring. In Section 4 we present the results concerning the commutativity of the symmetric elements when the involution is not a group involution extended by linearity. The involutions treated in this section are the oriented involutions introduced by Novikov [23] in the context of K-theory and its extension by considering an arbitrary group involution. Finally in Section 5 we collect the results concerning GI property, nilpotence and commutativity of the set of symmetric units. The first two properties have been studied only for the classical involution, while the commutativity has been studied for any group involution extended R-linearly to the whole group ring and also for oriented involutions.

2. Symmetric elements under the classical involution

Let R be a ring with unity and G be a group. Throughout this section we are going to denote by $*$ the classical involution, which is defined by setting $g^* = g^{-1}$, for all $g \in G$, and extending by linearity to RG; i.e., $(\sum \alpha_g g)^* = \sum \alpha_g g^{-1}$. Notice that set of symmetric elements in the group G are the elements of order 2, that is $G_* = \{g \in G |\ g^2 = 1\}$.

In this section we gather some results concerning Lie identities of symmetric elements in group rings, when they are satisfied, and when these properties are transferred to the whole group ring RG. We start with the more simple identity, the commutativity of $(RG)_*$, or equivalently when $(RG)_*$, the set of symmetric elements, form a ring.

The following lemma states that if the polynomial identities like commutativity or Lie n-Engel are satisfied, then the symmetric group elements are central.

Lemma 2.1. [7, 15] *Let R be a commutative ring with unity and G be a group.*

(i) *If $(RG)_*$ is commutative then $G_* \subseteq Z(G)$.*

(ii) *If $char(R)$ is odd prime and $(RG)_*$ is Lie $n-$Engel then $G_* \subseteq Z(G)$.*

Thus, $(RG)_*$ is commutative if and only if $\mathcal{S} \setminus G_*$ is commutative. Furthermore, if $char(R)$ is odd prime then $(RG)_*$ is Lie nilpotent (resp. Lie $n-$Engel) if and only if $\mathcal{S} \setminus G_*$ is Lie nilpotent (resp. Lie $n-$Engel).

To study these Lie-identities in $\mathcal{S} \setminus G_*$ the following Lemma is very useful.

Lemma 2.2. [7, 15] *Let R be a commutative ring with unity and G be a group.*

(i) *If $(RG)_*$ is commutative then for any $g, h \in G$ one of the following condition holds:*

- (a) *gh is equal to either hg, hg^{-1} or $h^{-1}g$;*
- (b) *$char(R) = 2$, $gh = h^{-1}g^{-1}$ and the orders of g, h are both equal to 4.*

(ii) *If $char(R)$ is odd prime, $(RG)_*$ is Lie $n-$Engel and $g, h \in G$ such that $[g+g^{-1}, h+h^{-1}] = 0$ then gh is equal to either hg, hg^{-1} or $h^{-1}g$.*

Notice that Lemma 2.2 justifies the distinction between characteristic of the ring R equal to 2 and different from 2. The following results states when the symmetric elements $(RG)_*$ commute, if the characteristic of the ring R is different from 2.

Theorem 2.3. [7] *Let G be a non-abelian group and let R be a commutative ring of characteristic different from 2. Then, $(RG)_*$ is a commutative ring if and only if G is a Hamiltonian 2-group.*

Recall that a group G is a Hamiltonian 2-group if $G \cong Q_8 \times E$ where Q_8 is the quaternion group of order 8 and E is an elementary abelian 2-group. Thus, as a consequence of Theorem 2.3 one deduces that if $(RG)_*$ is a commutative ring then G is a torsion group.

In the case in which the characteristic of the ring R is 2 we have the following result

Theorem 2.4. [7] *Let G be a non-abelian group and let R be a commutative ring of characteristic 2. Then, $(RG)_*$ is a commutative ring if and only if G is the direct product of an elementary abelian 2-group and a group H, for which one of the following conditions holds.*

(i) *H has an abelian subgroup A of index 2 and an element b of order 4, such that conjugation by b inverts each element of A;*

(ii) *H is the direct product of the quaternion group of order 8 and the cyclic group of order 4, or the direct product of two quaternion groups of order 8;*

(*iii*) *H is the central product of the group* $\langle x, y \mid x^4 = y^4 = 1,\ x^2 = (y, x)\rangle$ *with the quaternion group of order 8, where the nontrivial common element to the two central factors is* x^2y^2*;*

(*iv*) *H is isomorphic to one of the groups* H_{32} *or* H_{245}*, where*

$$H_{32} = \langle x, y, u \mid x^4 = y^4 = 1, x^2 = (y, x), y^2 = u^2 = (u, x), x^2y^2 = (u, y)\rangle,$$

and

$$H_{245} = \langle x, y, u, v \mid x^4 = y^4 = (v, u) = 1, x^2 = v^2 = (y, x) = (v, y), \\ y^2 = u^2 = (u, x), x^2y^2 = (u, y) = (v, x)\rangle.$$

Notice that when $(RG)_*$ is a ring and the exponent of the group G is different from 4 then $G = H \times E$ where E is an elementary abelian 2-group and H is a group with the property (i) of Theorem 2.4. If the exponent of G is 4, then H can be any of the groups listed in Theorem 2.4. This list of groups was given by V. Bovdi, Kovács and Sehgal, in [4], answering the question of when the set of symmetric units of a modular group ring RG is a multiplicative group, assuming that R is a commutative ring of prime characteristic p and G is a locally finite p-group. In the more general context of Theorem 2.4 when the exponent of G is 4 the proof is reduced to verify the conditions of the result in [4] and to prove that the commutativity of the symmetric units implies the commutativity of the symmetric elements.

Another property in $(RG)_*$ which has been studied is the Lie nilpotence. ¿From now until the end of this section we assume that R is a field of characteristic $p \neq 2$.

The characterization of the Lie nilpotence of $(RG)_*$ was given by Giambruno, Sehgal and Lee in two different papers. In the first one [10], Giambruno and Sehgal considered the case in which the group G has no 2-elements. In the second one [15], Lee considered the case in which the group G contains 2-elements. These two results can be resumed in the following theorem.

Theorem 2.5 ([10, 15])**.** *Let R be a field of characteristic* $p \neq 2$ *and let G be a group. Then* $(RG)_*$ *is Lie nilpotent if and only if one of the following conditions holds:*

(*i*) $Q_8 \not\subseteq G$ *and G is nilpotent and p-abelian.*

(*ii*) $p > 2$ *and* $G \cong K \times P$ *where K is a Hamiltonian 2-group and P is a finite p-group.*

(*iii*) $p = 0$ *and G is a Hamiltonian* 2*-group.*

Recall that the Lie nilpotence of RG was characterized by Passi, Passman and Sehgal in [24] (see also [27]) in terms of the group G and the characteristic of the ring R. A consequence of this characterization is the following result.

Corollary 2.6. *Let R be a field of odd prime characteristic and let G be a group not containing a copy of* Q_8*. Then* $(RG)_*$ *is Lie nilpotent if and only if RG is Lie nilpotent.*

In [10], Giambruno and Sehgal also proved the following result in a general context, showing the strong relationship between Lie nilpotence and commutativity of the symmetric elements.

Theorem 2.7. [10] *Let R be a semiprime ring with involution* $*$ *such that* $2R = R$*. If* R_* *is Lie nilpotent then* R_* *is a commutative ring.*

Finally we consider a particular case of Lie nilpotence, the Lie n-Engel property. This property was characterized by Lee in [16]. The result is the following.

Theorem 2.8. [16] *Let R be a field of characteristic* $p \neq 2$ *and let G be a group. Then* $(RG)_*$ *is Lie n-Engel if and only if one of the following conditions holds:*

(1) $Q_8 \not\subseteq G$ *and either*

(*i*) *$p > 2$ and G is nilpotent and contains a normal p-abelian subgroup A with G/A a finite p-group or*
(*ii*) *$p = 0$ and G is abelian.*
(2) *$p > 2$ and $G \cong K \times P$ where K is a Hamiltonian 2-group and P is a nilpotent p-group of bounded exponent containing a normal subgroup A of finite index such that A' is also finite;*
(3) *$p = 0$ and G is a Hamiltonian 2-group.*

Recall that the Lie n-Engel property in group rings RG was characterized by Sehgal in [27]. A consequence of this characterization is the following corollary.

Corollary 2.9. *Let R be a field of characteristic prime odd and let G be a group not containing a copy of Q_8. Then $(RG)_*$ is Lie $n-$Engel if and only if RG is Lie $m-$Engel.*

3. Symmetric elements under an arbitrary group involution

Any group involution $\varphi : G \to G$ can be extended R-linearly to a ring involution $\varphi : RG \to RG$. In this section we only deal with the question of when $(RG)_\varphi$ is a commutative ring for an arbitrary group involution φ. Properties like the Lie nilpotence or Lie n-Engel of $(RG)_\varphi$ are open questions except in the semiprime case (see Theorem 2.7).

When φ is the classical involution the study of the commutativity of the symmetric elements was divided in two cases depending on the characteristic of the ring R, $char(R) = 2$ or $char(R) \neq 2$. The following theorem gives a characterization of when the symmetric elements commute in terms of the group G and the involution φ when the characteristic of the ring R is different from 2.

Theorem 3.1. [14] *Let φ be an involution on a non-abelian group G and let R be a commutative ring with $\mathrm{char}(R) \neq 2$. The following are equivalent:*

(*i*) *$(RG)_\varphi$ is commutative.*
(*ii*) *The group G has the LC property, a unique non-trivial commutator s and the involution $\varphi : G \to G$ is given by*

$$\varphi(g) = \begin{cases} g & \text{if } g \in Z(G) \\ sg & \text{if } g \notin Z(G) \end{cases}$$

(*iii*) *$G/Z(G) \cong C_2 \times C_2$ and $\varphi(g) = h^{-1}gh$ or $\varphi(g) = g$ for all $g, h \in G$.*

In this case, $(RG)_\varphi = Z(RG)$.

Groups with *lack of commutativity property* ("LC" for short) have been described in [12]. A group G has the LC property if for any pair of elements $g, h \in G$, it is the case that $gh = hg$ if and only if $g \in Z(G)$ or $h \in Z(G)$ or $gh \in Z(G)$.

When the characteristic of R is 2 we have the following result.

Theorem 3.2. [14] *Let R be a commutative ring with $char(R) = 2$ and let G be a non-abelian group with involution φ. Then $(RG)_\varphi$ is commutative if and only if one of the following conditions holds:*

(*i*) *G contains an abelian subgroup A of index 2 and $b \in G \setminus G_\varphi$ with $b^2 \in G_\varphi$ such that $\varphi(a) = b^{-1}ab$ for all $a \in A$.*
(*ii*) *G contains a central subgroup Z such that G/Z is an elementary abelian 2-group and the involution $\varphi : G \to G$ is given by $\varphi(g) = c_g g$ where $c_g \in Z$ and the following properties are satisfied: (1) $c_g^2 = 1$, (2) $c_g = 1$ if and only if $g \in Z$, (3) $c_{gh} = c_g c_h (g, h)$ and if $(g, h) \neq 1$ we have that $c_{gh} = c_g, c_h$ or (g, h).*

Under hypothesis of Theorem 3.2, if the group G has an element g with $g^2 \notin G_\varphi$ then the group is described by condition (i). In this case $G_\varphi = Z(G) \subseteq A$ and there exists an element $a_0 \in Z(G)$ of order two so that the involution φ is given by

$$\varphi(g) = \begin{cases} a_0 g & \text{if } g \notin A \\ b^{-1}gb & \text{if } g \in A \end{cases}$$

On the other hand if for all $g \in G$, $g^2 \in G_\varphi$ condition (ii) of Theorem 3.2 holds.

4. **Symmetric elements under oriented involution**

One can wonder what happens with the commutativity of the symmetric elements if the involution on RG does not come from the group G. In this section we are going to consider an R-linear involution ψ on RG which does not come from the group G.

Let $\sigma : G \to \{\pm 1\}$ be a group homomorphism. Such a map is called an *orientation* of the group G. Novikov [23], introduced in the context of K-theory, an involution of RG, defined by

$$\left(\sum_{g\in G} \alpha_g g\right)^{\sigma} = \sum_{g\in G} \alpha_g \sigma(g) g^{-1} \tag{4.1}$$

which was subsequently studied by various authors (see [2], [6], [18] and [19]). Clearly, this involution does not come from a group involution.

The above involution can also be constructed in a more general context. Given both an orientation $\sigma : G \to \{\pm 1\}$ and a group involution $\varphi : G \to G$, an *oriented involution* ψ of RG is defined by

$$\psi\left(\sum_{g\in G} \alpha_g g\right) = \sum_{g\in G} \sigma(g)\alpha_g \varphi(g).$$

Notice that if σ is trivial then ψ is the kind of involutions studied in the previous section. Now we assume that σ is non-trivial and hence the characteristic of the ring R must be different from 2. Let N be the kernel of σ. Then N is a subgroup of index 2 in G. It is obvious that the involution ψ coincides on the subring RN with the ring involution φ. Thus, if $(RG)_\psi$ is commutative then $(RN)_\varphi$ is commutative and, by Theorem 3.1, we know the struture of N and the action of φ on N.

If $(RG)_\psi$ is a commutative ring, $[G : N] = 2$ and the structure of the group N and the action of φ on N are both known. Despite this it is not an easy task to describe G and the action of φ on G. The following theorem gives us this characterization.

Theorem 4.1. [8] *Let R be a commutative ring with unity and let G be a non-abelian group with involution φ and non-trivial orientation homomorphism σ with kernel N. Then $(RG)_\psi$ is a commutative ring if and only if one of the following conditions holds:*

- (i) *N is an abelian group and $(G \setminus N) \subset G_\varphi$;*
- (ii) *G and N have the LC property, and there exists a unique non-trivial commutator s such that the involution φ is given by*

$$\varphi(g) = \begin{cases} g & \text{if } g \in N \cap Z(G) \text{ or } g \in (G \setminus N) \setminus Z(G). \\ sg & \text{if otherwise.} \end{cases}$$

- (iii) *$char(R) = 4$, G has the LC property, and there exists a unique non-trivial commutator s such that the involution φ is given by*

$$\varphi(g) = \begin{cases} g & \text{if } g \in Z(G). \\ sg & \text{if } g \notin Z(G). \end{cases}$$

In the case of when ψ is the involution of Novikov (φ is the classical involution) we have the following characterization.

Theorem 4.2. *Let R be a commutative ring with unity and let G be a non-abelian group with non-trivial orientation homomorphism σ with kernel N. Then $(RG)_\sigma$ is a commutative ring if and only if one of the following conditions holds:*

(i) *N is an abelian group and $(G \setminus N)^2 = 1$;*

(ii) *$N \cong \langle x, y \mid x^4 = 1,\ y^2 = x^2,\ y^{-1}xy = x^{-1}\rangle \times E$ and*

$$G \cong \langle x, y, g \mid x^4 = 1,\ y^2 = x^2 = g^2,\ x^y = x^{-1},\ x^g = x,\ y^g = y\rangle \times E,$$

where E is an elementary abelian 2-group;

(iii) *$char(R) = 4$ and G is a Hamiltonian 2-group.*

5. Symmetric units

In this section we gather results concerning Lie-identities and nilpotence of the set of symmetric units, $\mathcal{U}_*(RG)$, and when these properties are transferred to the whole unit group $\mathcal{U}(RG)$. On the relation between group and polynomial identities Hartley made the following conjecture

Hartley's conjecture (1980). *If the units group $\mathcal{U}(RG)$ of the group algebra RG of a torsion group G over a field R satisfies a group identity, then RG satisfies a polynomial identity.*

An affirmative answer was proved first by Giambruno, Jespers, Sehgal and Valenti for infinite fields R in [9, 11] and next for finite fields by Liu in [20]. Passman [25] and Liu and Passman in [21] characterized when $\mathcal{U}(RG)$ satisfies a group identity for R a field of characteristic 0 and G a torsion group.

We can ask if the identities at the unit level are also controlled by symmetric elements. We have the analogue of Hartley's conjecture:

Theorem 5.1. [11] *Let R be an infinite field of characteristic $\neq 2$, let G be a torsion group and let $*$ be the classical involution. If $\mathcal{U}_*(RG)$ satisfies a group identity then RG satisfies a polynomial identity.*

In fact, one can say more. In the following theorem Giambruno, Sehgal and Valenti characterized when the set of symmetric units, $\mathcal{U}_*(RG)$, satisfies a group identity when R is a infinite field and G is a non abelian torsion group.

Theorem 5.2. [11] *Let R be a infinite field and G a nonabelian torsion group. If $char(R) = 0$, $\mathcal{U}_*(RG)$ satisfies GI if and only if G is a Hamiltonian 2-group. If $char(R) = p > 2$, then $\mathcal{U}_*(RG)$ satisfies GI if and only if RG satisfies PI and G has a p-abelian subgroup of finite index and one of the following conditions holds:*

(i) *$Q_8 \not\subseteq G$ and G' is of bounded exponent p^k,*

(ii) *$Q_8 \subseteq G$ and the p-elements of G form a subgroup P and G/P is a Hamiltonian 2-group, or*

(iii) *$Q_8 \subseteq G$ and G is of bounded exponent $4p^s$.*

The nilpotence of $\mathcal{U}_*(RG)$ was characterized in terms of the group G by Lee in [17], dependent on whether or not Q_8 is contained in G.

Theorem 5.3. *Let R be a field of characteristic $p \neq 2$ and G a torsion group.*

(i) *If $Q_8 \not\subseteq G$, then $\mathcal{U}_*(RG)$ is nilpotent if and only if $\mathcal{U}(RG)$ is nilpotent.*

(ii) *If $Q_8 \subseteq G$ then $\mathcal{U}_*(RG)$ is nilpotent if and only if either*

  a) *$p > 2$ and $G \cong Q_8 \times E \times P$ where E is an elementary abelian 2-group and P is a finite p-group or* *b)* $p = 0$ *and* $G \cong Q_8 \times E$ *where* E *is an elementary abelian 2-group.*

As a consequence of this result and Theorem 2.5 we have the following corollary.

Corollary 5.4. *Let* R *be a field of characteristic* $p \neq 2$ *and* G *a torsion group. Then* $\mathcal{U}_*(RG)$ *is nilpotent if and only if* $(RG)_*$ *is Lie nilpotent.*

Now we are left with the most simple property, the commutativity. When one wants to characterize the commutativity in the set of symmetric units one needs to construct these units. This is a very difficult task.

Under some additional condition on the group G and the ring R, and when $*$ is the classical involution on G, we have the following result which gives us, in a very simple manner, how to construct symmetric units.

Lemma 5.5. [7] *Let* G *be a torsion group,* R *a commutative ring with unity of prime characteristic* $p \neq 2$ *and* $*$ *the classical involution on* G*. Assume that* $\mathcal{U}_*(RG)$ *is commutative. Then:*

(*i*) *If* g *is a 2'-element of* G *then* $g + g^{-1} \in \mathcal{U}_*(RG)$.
(*ii*) *If* $p \neq 3$ *and* g *is a 2-element of* G *then* $1 + g + g^{-1} \in \mathcal{U}_*(RG)$.
(*iii*) *If* $p \neq 5$ *and* g *is a 2-element of* G *then* $1 + g + g^{-1} + g^2 + g^{-2} \in \mathcal{U}_*(RG)$.

Recall [3] that a ring R is called *G-favourable* if for any $g \in G$, of finite order $|g|$, there is a nonzero $\alpha_g \in R$ such that $1 - \alpha_g^{|g|}$ is invertible in R. Notice that every infinite field is G-favourable. When the ring R is G-favorable, Bovdi in [3] showed that $(g - \alpha_g)(g^{-1} - \alpha_g)$ is a symmetric unit in the context of the classical involution. This idea was generalized by Jespers and Ruiz in [14] for arbitrary group involutions and by Broche and Polcino in [8] for oriented involutions. These generalization is resumed in the following lemma.

Lemma 5.6. *Let* G *be a torsion group with involution* φ *and non-trivial orientation homomorphism* σ *with kernel* N*. Let* R *be a* G*-favorable integral domain. Let* $g \in G$ *and* $\alpha_g \in R$ *such that* $(1 - \alpha_g^{|g|}) \in \mathcal{U}(R)$*. Define*

$$u = \begin{cases} (g - \alpha_g)(g^{\varphi} - \alpha_g) & \text{if } g \in N \\ (g + \alpha_g)(\alpha_g - g^{\varphi}) & \text{if } g \notin N \end{cases}$$

Then $u \in \mathcal{U}_\psi(RG)$.

The construction of these units is an essential step in the proof of the necessary condition of the following theorem.

Theorem 5.7. [8, 14] *Let* G *be a torsion group,* R *be a* G*-favourable integral domain and* ψ *an oriented involution. Then,* $\mathcal{U}_\psi(RG)$ *is commutative if and only if* $(RG)_\psi$ *is commutative (or, equivalently,* $\mathcal{U}_\psi(RG)$ *is a subgroup of* $\mathcal{U}(RG)$ *if and only if* $(RG)_\psi$ *is a subring of* RG*).*

Notice that if the orientation σ is trivial then ψ is the group involution φ extended R-linearly to the whole group ring RG.

Now we study the commutativity of the set of symmetric units when $*$ is the classical involution. For this involution Bovdi and Parmenter [5] proved the following result for integral group rings.

Theorem 5.8. [5] *If* $\mathcal{U}_*(\mathbb{Z}G)$ *is a subgroup in* $\mathcal{U}(\mathbb{Z}G)$*, then the set* $t(G)$ *of elements of* G *of finite order is a subgroup in* G*, every subgroup of* $t(G)$ *is normal in* G *and* $t(G)$ *is either abelian or a Hamiltonian 2-group. Conversely, suppose that the group* G *satisfies the above conditions and* $G/t(G)$ *is a right ordered group. Then* $\mathcal{U}_*(\mathbb{Z}G)$ *is a subgroup in* $\mathcal{U}(\mathbb{Z}G)$.

Notice that if G is not a torsion group then additional conditions are needed to obtain the converse in Theorem 5.8. This also happens in other contexts and the reason is that very little is known about the units in the torsion free part. Indeed the following appears as **Problem 10** in [27]:

If G is torsion free then all units $\mathcal{U}(RG)$ are trivial.

The benefit of working with integral group rings is that quite a number of strong results on the unit groups $\mathcal{U}(\mathbb{Z}G)$ are known. In particular, if u is a non-trivial bicylic unit then Marciniak and Sehgal [22] have shown that the group $\langle u, u^* \rangle$ is free. Clearly uu^*, $u^*u \in \mathcal{U}_*(\mathbb{Z}G)$ but they do not commute. This latter fact is an essential step in the proof of the previous theorem.

That the group $\langle u, u^* \rangle$ is a nonabelian free group for u a bicyclic unit is not always true. For instance when $*$ is the involution introduced by Novikov (4.1) Li in [19] characterized when $\langle u, u^* \rangle$ is a nonabelian free group.

The following theorem characterized when the set of symmetric units form a multiplicative group when $*$ is the classical involution and the characteristic of the ring R is different from 2. When the characteristic of R is 2 still remains an open problem.

Theorem 5.9. [7] *Let G be a nonabelian torsion group, let R be a commutative ring of characteristic 0 or $p \neq 2$ and $*$ the classical involution on G. Then, $\mathcal{U}_*(RG)$ form a multiplicative group if and only if G is a Hamiltonian 2-group.*

Notice that in the case in which $char(R) = 0$, Theorem 5.9 follows from Theorem 5.8.

As a consequence of the previous result and Theorem 2.3 we have the following corollary.

Corollary 5.10. *Let G be a torsion group, let R be a commutative ring with unity of prime characteristic $p \neq 2$ and $*$ be the classical involution on G. Then $\mathcal{U}_*(RG)$ is an abelian group if and only if $(RG)_*$ is a commutative ring.*

Notice that the previous result is not true when the group is torsion free. Indeed if G is a torsion free nilpotent group and R an infinite field, then the group algebra RG has only trivial units. Since G has no elements of order 2, the symmetric units commute (1 is the only symmetric unit), but the symmetric elements $(RG)_*$ do not necessarily commute.

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