# TOPOLOGICAL ENTROPY AND PERIODS OF SELF–MAPS ON COMPACT MANIFOLDS

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Abstract. Let $(\mathbb{M}, f)$ be a discrete dynamical system induced by a self–map $f$ defined on a smooth compact connected $n$–dimensional manifold $\mathbb{M}$. We provide sufficient conditions in terms of the Lefschetz zeta function in order that: (1) $f$ has positive topological entropy when $f$ is $\mathcal{C}^\infty$, and (2) $f$ has infinitely many periodic points when $f$ is $\mathcal{C}^1$ and $f(\mathbb{M}) \subseteq \mathrm{Int}(\mathbb{M})$. Moreover, for the particular manifolds $\mathbb{S}^n$, $\mathbb{S}^n \times \mathbb{S}^m$, $\mathbb{C}P^n$ and $\mathbb{H}P^n$ we improve the previous sufficient conditions.

## 1. Introduction and statement of the main results

Along this work we study discrete dynamical systems $(\mathbb{M}, f)$ induced by a continuous self–map $f$ where $\mathbb{M}$ is a smooth compact connected $n$–dimensional manifold. Frequently algebraic information of a given discrete dynamical system provides qualitative or quantitative results on their orbits. Here we use algebraic properties of the Lefschetz zeta function $\mathcal{Z}_f(f)$ associated to $f$, which is of the form $P(t)/Q(t)$ where $P(t)$ and $Q(t)$ are polynomials, to provide information on the positivity of the topological entropy, and on the infiniteness of the set of periodic points of the system.

Recall that a point $x \in \mathbb{M}$ is *periodic of period* $n$ if $f^n(x) = x$ and $f^k(x) \neq x$ for $k = 1, \ldots, n-1$. On the other hand, roughly speaking, the *topological entropy* of a system $h(f)$ is a non–negative real number (possibly infinite) which measures how much $f$ mixes up the phase space $\mathbb{M}$. When $h(f)$ is positive the dynamics of the system is said to be *complex* and the positivity of $h(f)$ is used as a measure of the so called *topological chaos*. See [1, 2, 12] for more details on the topological entropy.

In some settings the zero topological entropy is related with a concrete structure on the set of periods of the periodic points. For instance, for the systems of the form $([0,1], f)$ the notion $h(f) = 0$ is equivalent to all periodic points have periods powers of two, see [2, 3] for more details.

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Inspired in this apparently relation between the topological entropy and the periodic orbit structure and using as precedents the results of the papers [5, 10] and [11] we present our main results with applications to some concrete manifolds where we can compute the Lefschetz zeta function from their homological structure.

We shall provide results for general smooth compact connected manifolds, and for the particular manifolds $\mathbb{S}^n$ (the $n$–dimensional sphere), $\mathbb{S}^n \times \mathbb{S}^m$ (the product of the $n$–dimensional with the $m$–dimensional sphere), $\mathbb{C}P^n$ (the $n$–dimensional complex projective space) and $\mathbb{H}P^n$ (the $n$–dimensional quaternion projective space) we are able to determine easy sufficient conditions for having positive topological entropy and for having infinitely many periodic points in terms of the homology.

For a polynomial $H(t)$ we define $H^*(t)$ by

$$H(t) = (1-t)^\alpha (1+t)^\beta t^\gamma H^*(t),$$

where $\alpha$, $\beta$ and $\gamma$ are non-negative integers such that $1-t$, $1+t$ and $t$ do not divide $H^*(t)$. We also define $H^{**}(t)$ by

$$H(t) = (1-t)^\alpha (1+t)^\beta H^{**}(t),$$

where $\alpha$ and $\beta$ are non-negative integers such that $1-t$ and $1+t$ do not divide $H^{**}(t)$.

Our main results are the following three theorems, in them it appears the notion of Lefschetz zeta function $\mathcal{Z}_f(t)$ for a map $f$, for its definition see subsection 2.1.

**Theorem 1.** *Let* $(\mathbb{M}, f)$ *be a discrete dynamical system induced by a continuous self–map* $f$ *defined on a smooth compact connected* $n$*–dimensional manifold* $\mathbb{M}$*, and let* $\mathcal{Z}_f(t) = P(t)/Q(t)$ *be its Lefschetz zeta function.*

(a) *Assume that* $P^*(t)$ *or* $Q^*(t)$ *has odd degree. If* $f$ *is* $\mathcal{C}^\infty$*, then the topological entropy of* $f$ *is positive.*
(b) *Assume that* $P^*(t)$ *or* $Q^*(t)$ *has odd degree. If* $f$ *is a homeomorphism,* $\mathbb{M}$ *is without boundary and its dimension is* $\leq 3$*, then the topological entropy of* $f$ *is positive.*
(c) *Assume that* $P^{**}(t)$ *or* $Q^{**}(t)$ *has odd degree. If* $f$ *is* $\mathcal{C}^1$ *and* $f(\mathbb{M}) \subseteq \mathrm{Int}(\mathbb{M})$*, then* $f$ *has infinitely many periodic points.*

We note that statement (c) of Theorem 1 was proved in Theorems 3.1 of [5], but here for completeness we shall provide its proof.

For some special manifolds, as a consequence of their homological structure, we are able to provide more information first on their topological entropy and after on their periodic structure.

**Theorem 2.** *Let* $(\mathbb{M}, f)$ *be a discrete dynamical system induced by a* $\mathcal{C}^\infty$ *self–map* $f$ *defined on a smooth compact connected* $n$*–dimensional manifold* $\mathbb{M}$ *and* $\mathcal{Z}_f(t)$ *its Lefschetz zeta function. Then* $f$ *has positive topological entropy if one of the following conditions holds:*

(a) *if* $\mathbb{M} = \mathbb{S}^n$ *and* $D \notin \{-1, 0, 1\}$ *where* $\mathcal{Z}_f(t) = \dfrac{(1-Dt)^{(-1)^{n+1}}}{1-t}$*;*

(b) *if* $\mathbb{M} = \mathbb{S}^n \times \mathbb{S}^n$ *and* $D \notin \{-1, 0, 1\}$ *or the polynomial* $1-(a+d)t+(ad-bc)t^2$ *has a root different from the unity where* $\mathcal{Z}_f(t) = \dfrac{(1-(a+d)t+(ad-bc)t^2)^{(-1)^{n+1}}}{(1-t)(1-Dt)}$*;*

(c) *if* $\mathbb{M} = \mathbb{S}^n \times \mathbb{S}^m$ *and* $\{a, b, D\} \cap \{-1, 0, 1\} = \emptyset$ *where* $\mathcal{Z}_f(t) = \dfrac{(1-at)^{(-1)^{n+1}}(1-bt)^{(-1)^{m+1}}(1-Dt)^{(-1)^{n+m+1}}}{1-t}$*;*

(d) *if* $\mathbb{M}$ *is either* $\mathbb{C}P^n$ *or* $\mathbb{H}P^n$*, and the integer* $a \notin \{-1, 0, 1\}$ *where* $\mathcal{Z}_f(t) = \left(\prod_q (1-a^{q/l}t)\right)^{-1}$ *where* $q$ *runs over* $\{0, 2, 4, ..., 2n\}$ *with* $l = 2$ *when* $\mathbb{M} = \mathbb{C}P^n$*, and* $q$ *runs over* $\{0, 4, 8, ..., 4n\}$ *with* $l = 4$ *when* $\mathbb{M} = \mathbb{H}P^n$ *.*

Theorem 2 for the case of $\mathbb{M} = \mathbb{S}^n$ is well known, see [11] where Misiurewicz and Przytycki proved that if $f$ is $C^1$ on a smooth compact oriented manifold $\mathbb{M}$ of degree $D$, then $\log |D| \leq h(f)$.

**Theorem 3.** *Let* $(\mathbb{M}, f)$ *be a discrete dynamical system induced by a* $\mathcal{C}^1$ *self–map* $f$ *defined on a smooth compact connected* $n$*–dimensional manifold* $\mathbb{M}$ *satisfying* $f(\mathbb{M}) \subseteq \mathrm{Int}(\mathbb{M})$*. Then* $f$ *has infinitely many periodic points if one of the conditions (a)–(d) of Theorem 2 holds.*

Similar results of Theorems 2 and 3 where proved in Theorems 3.3. and 3.4 of [5] for the $n$–dimensional torus.

The structure of the paper is the following. In section 2 we provide the results on the Lefschetz zeta function, on the cyclotomic polynomials and on the topological entropy that we need for proving in section 3 our Theorems 1, 2 and 3.

## 2. Preliminary results

2.1. **Lefschetz zeta function.** Given a discrete dynamical system $(\mathbb{M}, f)$ where $f$ is a continuous self–map defined on the compact $n$–dimensional manifold $\mathbb{M}$ the *Lefschetz number* is

$$L(f) = \sum_{k=0}^{n} (-1)^k \mathrm{trace}(f_{*k}),$$

where the induced homomorphism by $f$ on the $k$–th rational homology group of $\mathbb{M}$ is $f_{*k} : H_k(\mathbb{M}, \mathbb{Q}) \to H_k(\mathbb{M}, \mathbb{Q})$. We note that $H_k(\mathbb{M}, \mathbb{Q})$ is a finite dimensional vector space over $\mathbb{Q}$, and that $f_{*k}$ is a linear map given by a matrix with integer entries. The Lefschetz Fixed Point Theorem connects the fixed point theory with the algebraic topology via the following result.

**Theorem 4.** *Let* $(\mathbb{M}, f)$ *be a discrete dynamical system induced by a continuous self–map* $f$ *on compact manifold* $\mathbb{M}$ *and* $L(f)$ *be its Lefschetz number. If* $L(f) \neq 0$ *then* $f$ *has a fixed point.*

For a proof of Theorem 4 see [4].

Part of our interest in the present work is to provide information on the set of periodic points of $f$. To this objective we shall use the sequence of the Lefschetz numbers of all iterates of $f$ denoted by $\{L(f^m)\}_{m=0}^{\infty}$. We remark that the Lefschetz *zeta function* of $f$

$$\mathcal{Z}_f(t) = \exp\left(\sum_{m=1}^{\infty} \frac{L(f^m)}{m} t^m\right)$$

contains the information of all the sequence of the iterated Lefschetz numbers. Note that the function $\mathcal{Z}_f(t)$ can be computed also through

$$\mathcal{Z}_f(t) = \prod_{k=0}^{n} \det(I_{n_k} - t f_{*k})^{(-1)^{k+1}}, \tag{1}$$

where $n = \dim \mathbb{M}$, $n_k = \dim H_k(\mathbb{M}, \mathbb{Q})$, $I_{n_k}$ is the $n_k \times n_k$ identity matrix, and we take $\det(I_{n_k} - t f_{*k}) = 1$ if $n_k = 0$, for more details on the function $\mathcal{Z}_f(t)$ see [7].

From (1) the Lefschetz zeta function is a rational function and it contains the information of the infinite sequence of the iterated Lefschetz numbers. Note that this information is contained in two polynomials. Moreover, by [8, Theorem 6] we have the following property.

**Proposition 5.** *Let* $\mathbb{M}$ *be a smooth compact manifold and let* $(\mathbb{M}, f)$ *be a discrete dynamical system induced by a* $\mathcal{C}^1$ *self–map* $f$ *such that* $f(\mathbb{M}) \subseteq \mathrm{Int}(\mathbb{M})$*, and assume that* $f$ *has finitely many periodic points. Then* $\mathcal{Z}_f(t)$ *has a finite factorization in terms of the form* $(1 \pm t^r)^{\pm 1}$ *with* $r$ *a positive integer.*

2.2. **Cyclotomic polynomials.** The $n$–th cyclotomic polynomial is defined by

$$c_n(t) = \prod_{k} (w_k - t),$$

being $w_k = e^{2\pi ik/n}$ a primitive $n$–th root of unity and where $k$ runs over all the relative primes $\leq n$. See [9] for the properties of these polynomials.

For a positive integer $n$ the *Euler function* is $\varphi(n) = n \prod_{p|n, p \text{ prime}} \left(1 - \frac{1}{p}\right)$. It is known that the degree of the polynomial $c_n(t)$ is $\varphi(n)$. Note that $\varphi(n)$ is even for $n > 2$.

TABLE 1. The first thirty cyclotomic polynomials.

| | | |
|---|---|---|
| $c_1(t) = 1 - t$ | $c_2(t) = 1 + t$ | $c_3(t) = \dfrac{1 - t^3}{1 - t}$ |
| $c_4(t) = 1 + t^2$ | $c_5(t) = \dfrac{1 - t^5}{1 - t}$ | $c_6(t) = \dfrac{1 + t^3}{1 + t}$ |
| $c_7(t) = \dfrac{1 - t^7}{1 - t}$ | $c_8(t) = 1 + t^4$ | $c_9(t) = \dfrac{1 - t^9}{1 - t^3}$ |
| $c_{10}(t) = \dfrac{1 + t^5}{1 + t}$ | $c_{11}(t) = \dfrac{1 - t^{11}}{1 - t}$ | $c_{12}(t) = \dfrac{1 + t^6}{1 + t^2}$ |
| $c_{13}(t) = \dfrac{1 - t^{13}}{1 - t}$ | $c_{14}(t) = \dfrac{1 + t^7}{1 + t}$ | $c_{15}(t) = \dfrac{(1 - t^{15})(1 - t)}{(1 - t^3)(1 - t^5)}$ |
| $c_{16}(t) = 1 + t^8$ | $c_{17}(t) = \dfrac{1 - t^{17}}{1 - t}$ | $c_{18}(t) = \dfrac{1 + t^9}{1 + t^3}$ |
| $c_{19}(t) = \dfrac{1 - t^{19}}{1 - t}$ | $c_{20}(t) = \dfrac{1 + t^{10}}{1 + t^2}$ | $c_{21}(t) = \dfrac{(1 - t^{21})(1 - t)}{(1 - t^3)(1 - t^7)}$ |
| $c_{22}(t) = \dfrac{1 + t^{11}}{1 + t}$ | $c_{23}(t) = \dfrac{1 - t^{23}}{1 - t}$ | $c_{24}(t) = \dfrac{1 + t^{12}}{1 + t^4}$ |
| $c_{25}(t) = \dfrac{1 - t^{25}}{1 - t^5}$ | $c_{26}(t) = \dfrac{1 + t^{13}}{1 + t}$ | $c_{27}(t) = \dfrac{1 - t^{27}}{1 - t^9}$ |
| $c_{28}(t) = \dfrac{1 + t^{14}}{1 + t^2}$ | $c_{29}(t) = \dfrac{1 - t^{29}}{1 - t}$ | $c_{30}(t) = \dfrac{(1 + t^{15})(1 + t)}{(1 + t^3)(1 + t^5)}$ |

A proof of the next result can be found in [9].

**Proposition 6.** *Let $\xi$ be a primitive $n$–th root of the unity and $P(t)$ a polynomial with rational coefficients. If $P(\xi) = 0$ then $c_n(t)|P(t)$.*

**Lemma 7.** *If a polynomial has integer coefficients, constant term 1 and all of whose roots have modulus 1, then all of its roots are roots of unity.*

For a proof of Lemma 7 see [14].

2.3. **Topological entropy.** As we showed in subsection 2.1, given a discrete dynamical system $(\mathbb{M}, f)$ with $f$ a continuous self–map defined on a compact compact $n$–dimensional manifold $\mathbb{M}$, the map $f$ induces an action on the homology groups of $\mathbb{M}$, which we denote $f_{*k} : H_k(\mathbb{M}, \mathbb{Q}) \to H_k(\mathbb{M}, \mathbb{Q})$, for $k \in \{0, 1, \ldots, m\}$. The *spectral radii* of these maps are denoted $\mathrm{sp}(f_{*k})$, and they are equal to the largest modulus of all the eigenvalues of the linear map $f_{*k}$. The *spectral radius of* $f_*$ is

$$\mathrm{sp}(f_*) = \max_{k=0,\ldots,m} \mathrm{sp}(f_{*k}).$$

**Theorem 8.** *If $f$ is $C^\infty$ on a smooth compact manifold $\mathbb{M}$, then $\log(\mathrm{sp}(f_*)) \leq h(f)$.*

Theorem 8 is due to Yomdin [15].

**Theorem 9.** *If $f$ is a homeomorphism and $\mathbb{M}$ is a compact manifold without boundary with dimension $\leq 3$, then $\log(\mathrm{sp}(f_{*1})) \leq h(f)$.*

Theorem 9 is the Corollary 1 of Manning [10].

## 3. Proof of the results

We need the following results for proving our theorems.

**Lemma 10.** *Let $(\mathbb{M}, f)$ be a discrete dynamical system induced by a $\mathcal{C}^\infty$ self–map $f$ defined on a smooth compact connected $n$–dimensional manifold $\mathbb{M}$. If the topological entropy of $f$ is zero, then all the eigenvalues of the induced homomorphisms $f_{*k}$'s are zero or root of unity.*

*Proof.* By Theorem 8 we know $\log(\mathrm{sp}(f_*)) \leq h(f)$. Note that $\mathrm{sp}(f_*) \geq 1$ because $f_{*0} = (1)$ since $\mathbb{M}$ is connected. Therefore, due to the inequality $\log(\mathrm{sp}(f_*)) \leq h(f)$, we have that $\mathrm{sp}(f_*) = 1$ otherwise $h(f)$ would be positive. Hence all the eigenvalues of the $f_{*k}$'s are zero or have modulus 1.

Let $f_{*k}$ be one of the induced homomorphisms having an eigenvalue of modulus 1. Its characteristic polynomial will be of the form $t^m p(t)$ where $m$ is a non–positive integer and $p(t)$ is a polynomial with integer coefficients. Let $p_0$ be the constant term of the polynomial $p(t)$. Clearly $p_0$ is the product of all eigenvalues different from zero of the mentioned characteristic polynomial. So, since all these eigenvalues have modulus 1, $p_0$ is an integer of modulus 1, i.e., $p_0$ is 1 or $-1$. By Lemma 7 all the roots of the polynomial $p(t)$ are roots of unity finishing the proof. □

**Lemma 11.** *Let* $(\mathbb{M}, f)$ *be a discrete dynamical system induced by a* $\mathcal{C}^1$ *self–map* $f$ *defined on a smooth compact connected* $n$*–dimensional manifold* $\mathbb{M}$*. Assume that* $f(\mathbb{M}) \subseteq \mathrm{Int}(\mathbb{M})$*. If* $f$ *has finitely many periodic points, then all the eigenvalues of the induced homomorphisms* $f_{*k}$*'s are zero or root of unity.*

*Proof.* Since by Proposition 5 the Lefschetz zeta function (1) has a finite factorization in terms of the form $(1 \pm t^r)^{\pm 1}$ with $r$ a positive integer, it follows that all the eigenvalues of the $f_{*k}$'s are roots of unity. This completes the proof. $\square$

*Proof of Theorem 1.* >From the definitions of the polynomial $H^*$ and of the Lefschetz zeta function we have

$$\mathcal{Z}_f(t) = \frac{P(t)}{Q(t)} = (1-t)^a (1+t)^b t^c \frac{P^*(t)}{Q^*(t)}$$

where $a, b$ and $c$ are non–negative integers.

Assume now that the topological entropy $h(f) = 0$. Then by Lemma 10 all the eigenvalues of the induced homomorphisms $f_{*k}$'s are zero or root of unity. Therefore, by (1) all the roots of the polynomials $P^*(t)$ and $Q^*(t)$ are roots of the unity different from $\pm 1$. Hence, by Proposition 6 the polynomials $P^*(t)$ and $Q^*(t)$ are product of cyclotomic polynomials different from $c_1(t)$ and $c_2(t)$ (see Table 1). Consequently $P^*(t)$ and $Q^*(t)$ have even degree because all the cyclotomic polynomials which appear in them have even degree since the Euler function $\varphi(n)$ for $n > 2$ only takes even values. But this is a contradiction with the assumption that $P^*(t)$ or $Q^*(t)$ has odd degree. This completes the proof of statement (a).

The proof of statement (b) is the same than the proof of statement (a) but uses the result of Lemma 10 under the assumption of statement (b), and substituting in the proof of Lemma 10 the Theorem 8 by the Theorem 9.

Under the assumptions of statement (c), by Lemma 11 all the eigenvalues of the induced homomorphisms $f_{*k}$'s are zero or root of unity. From the definition of the polynomial $H^{**}$ and of the Lefschetz zeta function we have

$$\mathcal{Z}_f(t) = \frac{P(t)}{Q(t)} = (1-t)^a (1+t)^b \frac{P^{**}(t)}{Q^{**}(t)}$$

where $a$ and $b$ are non–negative integers. By Proposition 5 all the roots of the polynomials $P^{**}(t)$ and $Q^{**}(t)$ are roots of unity different from $\pm 1$. Therefore, the rest of the proof of statement (c) follows as in the last part of the proof of statement (a). This completes the proof of the theorem. $\square$

*Proof of Theorem 2.* For $n \geq 1$ let $f : \mathbb{S}^n \to \mathbb{S}^n$ be a $\mathcal{C}^\infty$ map. The homological groups of $\mathbb{S}^n$ over $\mathbb{Q}$ and the induced linear maps are of the form

$$H_q(\mathbb{S}^n, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } q \in \{0, n\}, \\ 0 & \text{otherwise,} \end{cases}$$

where $f_{*0} = (1)$, $f_{*i} = (0)$ for $i = 1, ..., n-1$ and $f_{*n} = (D)$ where $D$ is the degree of the map $f$, see for more details [6].

From (1) we have that

$$\mathcal{Z}_f(t) = \frac{(1 - Dt)^{(-1)^{n+1}}}{1 - t}.$$

Hence if $D \notin \{-1, 0, 1\}$, by Lemma 10 and (1) we have $h(f) > 0$. This proves the theorem under condition (a).

Let $f : \mathbb{S}^n \times \mathbb{S}^n \to \mathbb{S}^n \times \mathbb{S}^n$ be a $\mathcal{C}^\infty$ map. The homological groups of $\mathbb{S}^n \times \mathbb{S}^n$ over $\mathbb{Q}$ are

$$H_q(\mathbb{S}^n \times \mathbb{S}^n, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } q \in \{0, 2n\}, \\ \mathbb{Q} \oplus \mathbb{Q} & \text{if } q = n, \\ 0 & \text{otherwise.} \end{cases}$$

The induced linear maps are $f_{*0} = (1)$, $f_{*n} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{Z}$, $f_{*2n} = (D)$ where $D$ is the degree of the map $f$, and $f_{*i} = (0)$ for $i \in \{1, ..., 2n\}$, $i \neq 1, n, 2n$ (see for more details [6]). From (1) the Lefschetz zeta function of $f$ is

$$\mathcal{Z}_f(t) = \frac{(1 - (a+d)t + (ad - bc)t^2)^{(-1)^{n+1}}}{(1 - t)(1 - Dt)}.$$

Assume now that the topological entropy $h(f) = 0$. Then by Lemma 10 and (1) all the eigenvalues of the induced homomorphisms $f_{*k}$'s are zero or root of unity which is a contradiction with the condition (b). So the theorem is proved under condition (b).

For $1 \leq n < m$, let $f : \mathbb{S}^n \times \mathbb{S}^m \to \mathbb{S}^n \times \mathbb{S}^m$ be a $\mathcal{C}^\infty$ map. The homological groups of $\mathbb{S}^n \times \mathbb{S}^m$ over $\mathbb{Q}$ and the induced linear maps are

$$H_q(\mathbb{S}^n \times \mathbb{S}^m, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } q \in \{0, n, m, n+m\}, \\ 0 & \text{otherwise,} \end{cases}$$

where $f_{*0} = (1)$, $f_{*n} = (a)$, $f_{*m} = (b)$ with $a, b \in \mathbb{Z}$, $f_{*n+m} = (D)$ where $D$ is the degree of the map $f$ and $f_{*i} = (0)$ for $i \in \{0, ..., n+m\}$,

$i \neq 0, n, m, n+m$ (see for more details [6]). From (1) the Lefschetz zeta function of $f$ is

$$\mathcal{Z}_f(t) = \frac{(1-at)^{(-1)^{n+1}}(1-bt)^{(-1)^{m+1}}(1-Dt)^{(-1)^{n+m+1}}}{1-t}.$$

Under condition (c) we have $\{a, b, D\} \cap \{-1, 0, 1\} = \emptyset$, so the function $\mathcal{Z}_f(t)$ has some root which is not root of unity. Then by Lemma 10 and (1) we have $h(f) > 0$. Hence the theorem is proved under condition (c).

For $n \geq 1$ let $f : \mathbb{C}P^n \to \mathbb{C}P^n$ be a $\mathcal{C}^\infty$ map. The homological groups of $\mathbb{C}P^n$ over $\mathbb{Q}$ and the induced linear maps are

$$H_q(\mathbb{C}P^n, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } q \in \{0, 2, 4, ..., 2n\}, \\ 0 & \text{otherwise}, \end{cases}$$

where $f_{*0} = (1)$, $f_{*q} = (a^{\frac{q}{2}})$ where $a \in \mathbb{Z}$ and $q \in \{0, 2, 4, ..., 2n\}$, $f_{*q} = (0)$ otherwise (see for more details [13, Corollary 5.28]).

For $n \geq 1$ let $f : \mathbb{H}P^n \to \mathbb{H}P^n$ be a $\mathcal{C}^\infty$ map. The homological groups of $\mathbb{H}P^n$ over $\mathbb{Q}$ and the induced linear maps are

$$H_q(\mathbb{H}P^n, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } q \in \{0, 4, 8, ..., 4n\}, \\ 0 & \text{otherwise}, \end{cases}$$

where $f_{*0} = (1)$, $f_{*q} = (a^{\frac{q}{4}})$ where $a \in \mathbb{Z}$ and $q \in \{0, 4, 8, ..., 4n\}$, $f_{*q} = (0)$ otherwise (see for more details [13, Corollary 5.33]).

For these last two spaces from (1) we obtain that

$$\mathcal{Z}_f(t) = \left( \prod_q (1 - a^{q/l} t) \right)^{-1}$$

where $q$ runs over $\{0, 2, 4, ..., 2n\}$ with $l = 2$ when $\mathbb{M} = \mathbb{C}P^n$ and $q$ runs over $\{0, 4, 8, ..., 4n\}$ with $l = 4$ when $\mathbb{M} = \mathbb{H}P^n$. Under condition (d), since the integer $a \notin \{-1, 0, 1\}$ the function $\mathcal{Z}_f(t)$ has a root which is not root of unity, and by Lemma 10 and (1) we have $h(f) > 0$. Therefore, the theorem follows under condition (d). $\square$

*Proof of Theorem 3.* By Lemma 11, if $f$ has finitely many periodic points then all the eigenvalues of the homomorphims $f_{*k}$'s are roots of the unity. Consequently, the conditions (a)-(d) from Theorem 2 are not satisfied. Hence, $f$ has infinitely many periodic points. $\square$

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