Facultad
de Ciencias
de la Empresa

Financial Mathematics: Fundamental Concepts

$$i_{TAE} = \left(1 + \frac{J(k)}{k}\right)^k - 1$$

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de Cartagena

edicionesUPCT

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Prologue

This manual, which is the fruit of our teaching experience, is a tool designed to serve as an aid and supplement for our students to facilitate understanding and development of the Mathematics of Financial Transactions. This work includes the basic subject matter required to successfully study Financial Mathematics. All chapters are accompanied by a sufficient number of solved exercises to enable students to get to grips with the subject matter. A total of 151.

The subject matter is divided into three parts:

Part one covers topics 1 to 5 and develops the main laws governing finance and their interrelationship: capitalisations, discounts, financial equivalence, etc., with 75 solved exercises.

The second part covers topics 6 to 8 and deals with financial income in detail and great depth. Emphasis is placed on the study of variable income, focusing on "income by annual blocks". This calculation technique enables the student to perform any income analysis. This part contains 58 solved exercises.

The third part is devoted to loans (topics 9 and 10), mainly analysing steady income loans, as well as the constitution transaction associated with the American amortization method. Regarding steady income loans, this part deals mainly with the French or progressive amortization method and the German or prepaid interest method. Part three also covers leasing transactions as an extension of the French method. This part contains 18 solved exercises.

We are aware that the content of this book is basic and that there are a lot of topics from the subject that it does not cover. However, we have written it for the purpose of providing support for our bilingual students in the subject of Financial Mathematics.

At the end of the manual there is an appendix which deals with the mathematical developments of the subject.

Cartagena, May 2017

1. Fundamental Concepts of Financial Mathematics

1.1. Fundamental concepts

1.1.1. Financial Phenomenon

A financial phenomenon is the exchange of economic goods in which time intervenes.

In every financial phenomenon two aspects should be taken into accoun: (A) the quantity, that is, the objective measurement of the asset exchanged, in monetary units, and (b), the moment in time to which the quantification refers.

1.1.2. Financial Capital

Financial Capital refers essentially to assets at the time of their maturity, generally coinciding with their availability, payment on demand or the payment of this asset.

All financial capital is to be defined by: (C, t).

1.1.3. Financial Space

Refers to the set of all the possible values of financial capital.

$$\mathrm{E} = \left[(\mathrm{C}, \mathrm{t}) / C \in R+,\ \mathrm{t} \in \mathrm{R}+ \right]$$

Financial capital will be generally represented through the following standard representations:

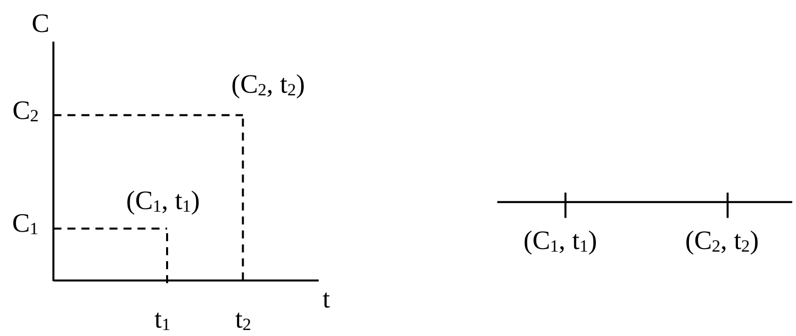


Figure 1.1

If to the financial space we provide an algebraic structure that allows us to assess simultaneously several financial capital movements, we can operate with these capital movements, compare, substitute or exchange them.

1.2. Financial equivalence. Properties of equivalence

1.2.1. Financial Equivalence

Any capital with an expiration date at any time "t" can be assessed at a reference time "p" that can be prior, simultaneous or subsequent to the former one.

The notation of this projection will be:

$$V = \text{Proy } p(C, t)$$

Projecting is simply assessing at a reference time.

If two different capital movements project themselves at a reference time "p", we can, with the obtained values, make a comparison so that they turn out to be same.

$$V_1 = \text{Proy } p(C_1, t_1) = \text{Proy } p(C_2, t_2) = V_2$$

In this case it is said that these capital movements are equivalent in p, representing this equivalence with the notation:

$$(C_1, t_1) \approx (C_2, t_2)$$

1.2.2. Properties of equivalence

1) Reflexive.

All financial capital is equivalent to itself.

$$(C_1, t_1) \approx (C_1, t_1)$$

2) Symmetrical.

If one capital is equivalent to another, this other is similar the first.

$$(C_1, t_1) \approx (C_2, t_2) \;\Rightarrow\; (C_2, t_2) \approx (C_1, t_1)$$

3) Transitive.

If one capital is equivalent to another and this other one is equivalent to a third one, the first and the third one are equivalent.

$$\left.\begin{array}{l}(C_1, t_1) \approx (C_2, t_2) \\ (C_2, t_2) \approx (C_3, t_3)\end{array}\right\} \;\Rightarrow\; (C_1, t_1) \approx (C_3, t_3)$$

1.2.3. Financial preference

It is said that capital (C_1, t_1) is preferable to another capital (C_2, t_2) if from the projection in "p" of both, we obtain the following results:

$$\text{Proy } p(C_1, t_1) > \text{Proy } p(C_2, t_2)$$

1.3. Financial laws. Properties

1.3.1. Financial Laws

A financial law is the formula or mathematical equation that is going to allow us to substitute a capital movement for another at a determined moment of time. Therefore, projections are realised according to two types of financial laws (or models):

- Capitalisation.
- Discount (Inverse transaction to capitalisation).

Within this classification, mathematical formulas can be exponential in character, in which case we will refer to them as compounds, or if they are linear functions, we will refer to them as simple.

<u>A) Capitalisation.</u>

In this type of financial law capital is projected to a subsequent moment, arriving at the projection by means of accrued interest.

Projection or valuation is realized at a subsequent moment.

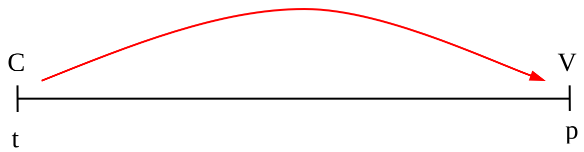


Figure 1.2

$$(C, t)\ C(t, p) = (V, p)$$

If to the capital (C, t) we apply the financial capitalisation law $C(t, p)$ it is transformed into (V, p) and to "V" is referred to as Capitalised Value of "C" in "p".

The capital that we arrive at after applying the financial law is known as “Final Capital" or "Total".

$$M = C_0 + I$$

<u>B) Discount.</u>

With this type of financial laws, capital is projected to a previous moment, arriving at the projection by means of disaggregation of interests.

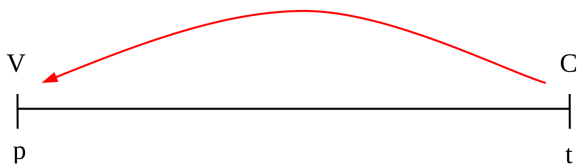


Figure 1.3

$$(C, t)\ D(t, p) = (V, p)$$

If to the capital (C, t) we apply the financial discount law $D(t, p)$, it is transformed into (V, p) and “V" is referred to as the Discounted Value of "C" in "p".

1.3.2. Properties of financial laws

1º) Financial capital movements, and also time, are always positive. Accordingly, any financial law is positive.

2ª) Any financial law applied to capital movements "C" has to be the same as the law applied, in identical terms, to a capital unit.

1.4. The principle of capital equivalence

Two capital movements are financially equivalent when the projections or valuations of both are equal at a specific moment.

1.4.1. Properties of equivalence

1ª) Reflexive: All capital is equivalent to itself.

2ª) Symmetrical: If a capital is equivalent to another, the latter is similar to the former.

3ª) Transitive: If one capital is equivalent to another and the latter at the same time is equivalent to a third one, the first and the third one are equivalent.

1.4.2. Sum of financial capital

The “capital sum of two capital units" refers to another capital unit "S" with expiration in "p", if we arrive at this capital unit through the sum of the projections or valuations of the added capital.

Let’s take the case of two capital units (C_1, t_1) and (C_2, t_2). Their projections in “p”, based on a financial law $F(t, p)$ are:

$$V_1 = C_1 \ F(t_1, p)$$
$$V_2 = C_2 \ F(t_2, p)$$

The financial sum will be obtained by adding: $S = V_1 + V_2$

1.5. Financial transaction. Classification

In the financial transaction, one or several capital units available at a specific time are replaced by another or several capital units available at another time. This is precisely its fundamental characteristic.

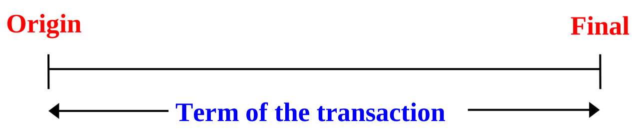


Figure 1.4

The person who **initiates** the financial transaction is known as the **creditor** and is the one who makes the **payment.**

The recipient of the financial transaction is known as the **debtor** and realizes the **compensation**.

Financial transactions can be classified according to a series of criteria:

1º) **According to its duration:**

- Short-term financial transactions.

- Long- term financial transactions.

The differentiating criterion is one year. Short-term transactions are those with a term of less than one year, whereas long-term transactions are those with a term of more than one year.

2º) **According to the temporal division of liabilities**:

- Simple transactions: Payment and compensation are formed by single capital (e.g. a fixed term deposit).

- Compound transactions: Payment and/or compensation are formed by more than one capital (example, a loan, a pension scheme).

3º) **According to financial law**:

- Capitalisation transactions: When the point of application is equal or subsequent to the last maturity.

- Discount transactions: When the point of application is equal or prior to the first maturity.

- Mixed transactions: When the point of application situated between the initiation and the end of the transaction.

4º) **According to the capital exchanged**:

- Transactions at a fixed rate and amount: All capital exchanged in the transaction is at a specific rate and for a specific amount.

- Random Operations: Some of the exchanged capital is random, that is to say, its availability or payables on demand depend on the occurrence of some event.

1.6. The financial factor

The financial factor is the mathematical expression that allows us to calculate the value of disposable capital at a particular moment, equivalent to other disposable capital at another moment, before or after. If the equivalent capital is after, the factor will be capitalisation and if it is prior it will be discount.

There are two capital units C_1 and C_2 available in n_1 and n_2, respectively. These capital units are equivalent based on a determined financial law, they are denominated generically $F(n,p)$. Since C_1 and C_2 are equivalent,

$$C_1 F(n_1, p) = C_2 F(n_2, p)$$

If we have defined the financial factor as the one that enables us to calculate the value of C_2, financially equivalent to C_1, the capitalisation factor will be that which verifies the following relation, being $L(n_1,p)$ the financial law of capitalisation for the capital C_1 and $L(n_2,p)$ that corresponding to C_2:

$$C_1 L(n_1, p) = C_2 L(n_2, p)$$

$$C_2 = C_1 \frac{L(n_1,p)}{L(n_2,p)} = C_1 \times C(n_1, n_2)$$

We call $C(n_1, n_2)$ on the financial factor of capitalisation, associated to the interval (n_1, n_2)

in which it is applied, and is the number by which one multiplies capital C_1, with expiration n_1 to obtain its equivalent value at the moment n_2.

If the financial law applied is the discount,

$$C_1 L^*(n_1, p) = C_2 L^*(n_2, p)$$

$$\frac{C_1}{C_2} = \frac{L^*(n_2, p)}{L^*(n_1, p)} = D(n_1, n_2) \Rightarrow \; C_1 = C_2 \times D(n_1, n_2)$$

The discount factor is given by the: $D(n_1, n_2)$

Properties of the financial factor:

1.- When the financial law applied is that of capitalisation, $C_2 > C_1$, capital grows according to the time elapsed owing to the interest accrued.

2.- The financial factor of capitalisation is always greater that 1, $C(n_1, n_2) > 1$. Consequently, the capitalisation factor shows the growth of a monetary unit in a period of time since the capital during a period of time produces interest.

3.- When we apply a financial law of discounts, $C_1 < C_2$, capital is diminished by the discount which entails the capital advance before its maturity.

4.- The financial discount factor is always less that 1, but more than zero, $0 < D(n_1, n_2) < 1$ It shows a decline from a monetary unit in a period of time when its maturity is anticipated.

5.- Both financial factors are related to: $C(n_1, n_2) = \dfrac{1}{D(n_1, n_2)}$

1.7. Mathematical reserve

Mathematical reserve shows us the existing financial balance in favour of an economic agent at a specific moment of the financial transaction.

There are three methods for calculating the mathematical reserve:

A) **Retrospective Method**: it is calculated taking into account the collections and payments from the start of the transaction up to the moment of the valuation.

B) **Prospective Method**: it is calculated taking into account the collections and payments pending from the moment of the valuation until the end of the transaction.

C) **Recurrent Method**: consists of calculating the reserve at a specific moment, based on the amount of the reserve at another moment.

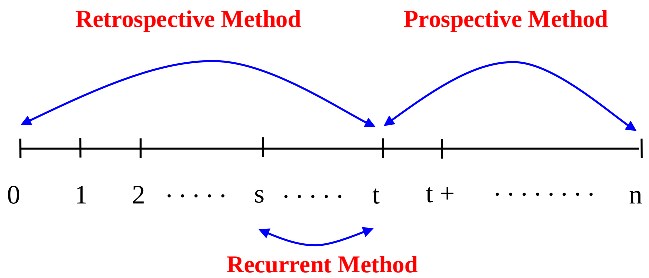


Figure 1.5

We refer to:

$V_{0t}(p)$: Value in "t" of the payment between 0 and t.

$V_{0t}(c)$: Value in "t" of the compensation between 0 and t.

$V_{tn}(p)$: Value in "t" of the payment between t and n.

$V_{tn}(c)$: Value in "t" of the compensation between t and n.

$V_t(p)$: Value of the payment at the moment "t".

$V_t(c)$: Value of the compensation at the moment "t".

R_t: Mathematical reserve in "t".

The values V_{0t} and V_{tn} are established according to the financial law agreed between the parties.

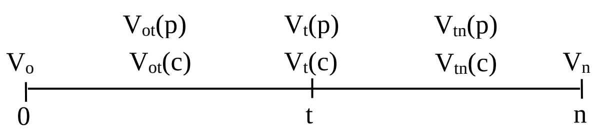


Figure 1.6

By the general principle of financial equivalence, at the moment "t" the following equivalence is produced:

$$V_t(p) = V_t(c) = V_{0t}(p) + V_{tn}(p) = V_{0t}(c) + V_{tn}(c)$$

We alter the equivalence members,

$$\underbrace{V_{0t}(p) - V_{0t}(c)}_{\substack{\text{Mathematical Reserve in "t"} \\ \text{Retrospective Method}}} = \underbrace{V_{tn}(c) - V_{tn}(p)}_{\substack{\text{Mathematical Reserve in "t"} \\ \text{Prospective Method}}}$$

- $R_t = V_{0t}(p) - V_{0t}(c)$: Reserve by the retrospective method.

- If $R_t > 0 \rightarrow V_{0t}(p) > V_{0t}(c)$, the financial balance or mathematical reserve is favourable to the creditor.

- If $R_t < 0 \rightarrow V_{0t}(p) < V_{0t}(c)$, the financial balance is favourable to the debtor.

- $R_t = V_{tn}(c) - V_{tn}(p)$: Reserve by the prospective method.

- If $R_t > 0 \rightarrow V_{tn}(c) > V_{tn}(p)$, the mathematical reserve is favourable to the debtor.

- If $R_t < 0 \quad \rightarrow V_{tn}(c) < V_{tn}(p)$, the mathematical reserve is favourable to the creditor.

The mathematical reserve by the Recurrent Method is based on knowing the reserve at a moment prior to "t", for example, moment "s",

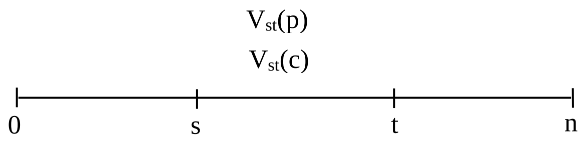


Figure 1.7

$$R_t = R_s + V_{st}(p) - V_{st}(c).$$

The study of the mathematical reserve becomes particularly important when a study of compound financial transactions is carried out.

Exercise 1.1. What capital is preferable and why: 1,000€ today or 1,100€ in the following year? Give reasons for your answer.

Exercise 1.2. Of the following financial transactions, show which would have a financial cost and which would be financially profitable:

- Lend money to a friend.
- Request and obtain a loan from a professional moneylender.
- Maintain a positive balance in a savings account.
- Request a home loan.
- Buy a car on instalments.

Exercise 1.3. Classify the following financial transactions from the point of view of the lender:

- A loan in monthly instalments over a two year period.
- Participant in a pension fund.
- Purchase of Public Treasury Bonds.

Exercise 1.4. Cite two examples in which the financial transaction produces interest.

Exercise 1.5. What is the discount? Cite two examples.

2. Financial Laws of Capitalisation

2.1. Simple and compound capitalisation

2.1.1. Introduction

Interest (capitalisation) can be: simple and compound. The interest is simple when the interest is not added to the capital, that is, each period of capitalisation is identical to the previous one since the capital is always the same. Simple interest is applied, generally, to short-term financial transactions.

Interest is compound when interest is accrued on the capital to produce new interest. Accordingly, interest in period "s" is greater than in period “s–1" due to the fact that in the calculation, interest for period “s–1" is accrued. Compound interest is applied, generally, to long-term financial transactions.

We will now consider a financial transaction where the investor invests C_0 monetary units of capital at moment "0". The term of the transaction is "n" periods and the interest rate remains constant throughout the entire transaction.

We refer to:

"n" as the term of the transaction.

$I_t =$ Total interest generated by the transaction.

"I" to the rate of interest or return of the transaction expressed on a per unit basis.

C_0 To the initial capital.

C_n to the total: $C_n = C_0 + I_t$

The "i" interest rate applies to C_0 during the "n" term of the transaction.

2.1.2. Calculation of the total or final capital in simple capitalisation

Based on the hypothesis that the capital imposes C_0 at the origin of the transaction, that is to say, at the moment 0, and as in the case of simple capitalisation interest is not accrued to the capital, at the end of a year the total will be:

$$C_1 = C_0 + C_0 i = C_0(1+i)$$

At the end of the second year the total will be equal to the previous one plus the interests pertaining to the first, that is,

$$C_2 = C_1 + C_0 i = C_0(1+i) + C_0 i = C_0 + C_0 i + C_0 i = C_0 + 2C_0 i = C_0(1+2i)$$

At the end of the third year:

$$C_3 = C_2 + C_0 i = C_0(1+2i) + C_0 i = C_0 + 2C_0 i + C_0 i = C_0 + 3C_0 i = C_0(1+3i)$$

At the end of the year “n–1"

$$C_{n-1} = C_{n-2} + C_0 i = C_0\left[1+(n-2)i\right] + C_0 i = C_0 + C_0(n-2)i + C_0 i =$$
$$= C_0 + (n-1)C_0 i = C_0\left[1+(n-1)i\right]$$

At the end of the year "n"

$$C_n = C_{n-1} + C_0 i = C_0\left[1+(n-1)i\right] + C_0 i = = C_0 + C_0(n-1)i + C_0 i = C_0 + nC_0 i = C_0(1+ni)$$

Therefore

$$C_n = C_0(1+ni)$$

The simple capitalisation is: $C(n_n, n_0) = (1+ni)$

Calling I_t the total interests produced:

$$I_t = C_n - C_0 = C_0(1+ni) - C_0 = C_0 + C_0 ni - C_0 = C_0 ni$$
$$I_t = C_0 ni$$

2.1.3. Calculation of the total or final capital end in compound capitalisation

C_1 corresponds to the value of the capital in year 1.

$$C_1 = C_0 + C_0 i = C_0(1+i)$$

That is, capital plus interest (remember that the interest rate is expressed on a per unit basis). Since capitalisation is compound, this capital again produces new interests and in this way the capital at the end of the second year will be:

$$C_2 = C_1 + C_1 i = C_0(1+i) + C_0(1+i)i = C_0(1+i)(1+i) = C_0(1+i)^2$$

At the end of the third year we will have:

$$C_3 = C_2 + C_2 i = C_0(1+i)^2 + C_0(1+i)^2 i = C_0(1+i)^2(1+i) = C_0(1+i)^3$$

In year "n–1" :

$$C_{n-1} = C_{n-2} + C_{n-2} i = C_0(1+i)^{n-2} + C_0(1+i)^{n-2} i = C_0(1+i)^{n-2}(1+i) = C_0(1+i)^{n-1}$$

In year "n" :

$$C_n = C_{n-1} + C_{n-1} i = C_0(1+i)^{n-1} + C_0(1+i)^{n-1} i = C_0(1+i)^{n-1}(1+i) = C_0(1+i)^n$$

Therefore,

$$C_n = C_0(1+i)^n$$

The general formula of the compound capitalisation that allows us to calculate the total or final capital based on the initial capital, knowing the interest rate and the term of the transaction.

The compounded capitalisation factor is $C(n_n, n_0) = (1+i)^n$

The total interest produced will be:

$$I_t = C_n - C_0 = C_0(1+i)^n - C_0 = C_0\left[(1+i)^n - 1\right]$$

Exercise 2.1. Someone wants to know which will be the total of a 9,000 € investment for a period of 10 years with an interest rate divided in two tranches: the first for 6 years at a simple 8% and the second for the remaining period, but at 10% simple interest rate. Perform the calculations in the following cases:

1.- The two tranches are applied so that the total amounts are not accrued.

2.- The two tranches are applied so that the second deposit is based on the total obtained in the first.

Solution.

1.- $C_n = C_0(1+n_1i_1 + n_2i_2) = 9{,}000[1+(6\times 0.08)+(4\times 0.1)] = 16{,}920$ €

2.- $C_n = C_0(1+n_1i_1)(1+n_2i_2) = 9{,}000(1+6\times 0.08)(1+4\times 0.1) = 18{,}648$ €

2.1.4. Graphic Representation

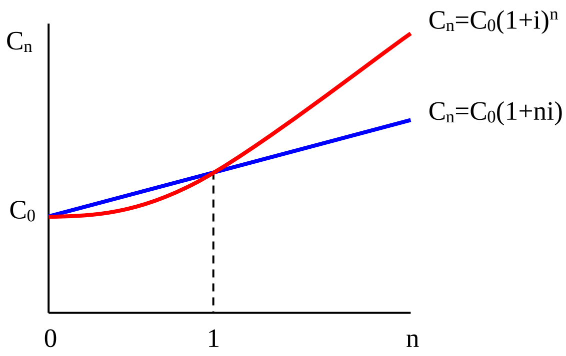


Figure 2.1

Calling:

C_nS the total in simple capitalisation.

C_nC the total in compounded capitalisation.

When:

$$n=0 \quad \Rightarrow C_nS = C_nC$$
$$n=1 \quad \Rightarrow C_nS = C_nC$$
$$n>1 \quad \Rightarrow C_nS < C_nC$$
$$0<n<1 \quad \Rightarrow C_nS > C_nC$$

2.1.5. Analytic Justification

Based on the final capital and assuming that the initial capital is equal to a monetary unit, that is $C_0 = 1$,

$$C_n = C_0(1+i)^n \rightarrow C_n = (1+i)^n$$
$$C_n = C_0(1+ni) \rightarrow C_n = (1+ni)$$

We develop them $(1+i)^n$ by Newton's binomial,

$$(1+i)^n = 1+ni+\frac{n(n-1)}{2!}i^2+\frac{n(n-1)(n-2)}{3!}i^3+.......$$

Calling "x" as of the third member of the development,

$$\underbrace{(1+i)^n}_{\text{Compound cap. F}} = \underbrace{(1+ni)}_{\text{Simple cap. F}} + x$$

If we call C_nS the total in simple capitalisation and C_nC the total in compound capitalisation, we arrive at:

When:

$$n = 0 \quad \Rightarrow \quad C_nC = C_nS$$

$$n = 1 \quad \Rightarrow \quad C_nC = C_nS$$

$$0 < n < 1,\ (1+i)^n = (1+ni) - x \text{, then } C_nC < C_nS$$

$$n > 1,\ (1+i)^n = (1+ni) + x \text{, then } C_nC > C_nS$$

2.1.6. Interests for the commercial and calendar year in simple capitalisation

In commercial practice the year is divided into 12 months of 30 days each, so that the total days to be calculated for each year are 360; although occasionally a 365-day calendar year is used. Evidently, in accordance with whichever one is used, we would arrive at a different interest figure for each period.

We call:

I_c interest for the commercial year (k = 360).

I_n interest for the calendar year (k = 365).

On the other hand, we know that: $I_t = C_n - C_0 = C_0 ni$

$$I_c = C_0 n \frac{i}{360} \quad \Rightarrow \quad I_n = C_0 n \frac{i}{365}$$

We calculate the quotient between them,

$$\frac{I_c}{I_n} = C_0 n \frac{i}{360} : C_0 n \frac{i}{365} = \cancel{C_0} \times \cancel{n} \frac{\cancel{i}}{360} \times \frac{365}{\cancel{C_0} \times \cancel{n} \times \cancel{i}}$$

C_0, n and i are simplified and the quantities are divisible by 5, so that

$$\frac{I_c}{I_n} = \frac{73}{72}$$

We can establish a comparison I_c and I_n by difference:

$$I_c - I_n = \frac{C_0 \times n \times i}{360} - \frac{C_0 \times n \times i}{365} = C_0 \times n \times i \left[\frac{1}{360} - \frac{1}{365}\right] =$$

$$= C_0 \times n \times i \left[\frac{365 - 360}{360 \times 365}\right] = C_0 \times n \times i \left[\frac{5}{360 \times 365}\right] =$$

$$= C_0 \times n \times i \left[\frac{1}{360 \times 73}\right] = \frac{C_0 \times n \times i}{360} \frac{1}{73} = I_c \frac{1}{73} = \frac{I_c}{73}$$

Therefore,

$$I_c - I_n = \frac{I_c}{73}$$

We can also perform the following calculation

$$I_c - I_n = C_0 \times n \times i\left[\frac{5}{360 \times 365}\right] = C_0 \times n \times i\left[\frac{1}{72 \times 365}\right] = \frac{C_0 \times n \times i}{365}\frac{1}{72} = I_n \frac{1}{72} = \frac{I_n}{72}$$

We conclude that:

$$I_c - I_n = \frac{I_c}{72}$$

Exercise 2.2. Knowing that the interest on a certain loan is 86.50€ calculated on the basis of a commercial year. Calculate the interest for a calendar year.

Solution.

$$\frac{I_c}{I_n} = \frac{73}{72} \quad \Rightarrow \quad 72\, I_c = 73\, I_n$$

$$72 \times 86.50 = 73\, I_n$$

$$I_n = \frac{72 \times 86.50}{73} = 85.31€$$

Exercise 2.3. Knowing that the difference between the interest calculated by the commercial and calendar year is 205.48€. Calculate the interest for a calendar and a commercial year.

Solution.

$$I_c - I_n = \frac{I_c}{73} \quad \Rightarrow \quad 205.48 = \frac{I_c}{73} \quad \Rightarrow \quad I_c = 15{,}000\,€$$

$$I_c - I_n = \frac{I_n}{72} \quad \Rightarrow \quad 205.48 = \frac{I_n}{72} \quad \Rightarrow \quad I_n = 14{,}794.56€$$

2.1.7. Calculating the term of the transaction

A) Simple Capitalisation.

$$C_n = C_0(1 + ni) \quad \Rightarrow \quad (1 + ni) = \frac{C_n}{C_0}$$

$$n = \frac{\frac{C_n}{C_0} - 1}{i} = \frac{C_n - C_0}{C_0 i}$$

B) Compounded Capitalisation.

$$C_n = C_0(1+i)^n \Rightarrow \quad \mathrm{Lg}\, C_n = \mathrm{Lg}\, C_0 + n\, \mathrm{Lg}\,(1+i)$$

$$n = \frac{\mathrm{Lg}\, C_n - \mathrm{Lg}\, C_0}{\mathrm{Lg}\,(1+i)}$$

Exercise 2.4. A shopkeeper cannot pay cash for an invoice that amounted to 250 €, and has to pay 251.25€. What was the term of the loan if the 4.5% interest rate is added in simple capitalisation? (Consider the commercial year of 360 days).

Solution.

$$C_n = C_0(1 + ni) \quad \rightarrow \quad 251.25 = 250(1 + n \times 0.045)$$

$$n = \frac{\frac{251.25}{250} - 1}{0.045} = 0.1111\,\text{yerars} \times 360 = 40\,\text{days}$$

Exercise 2.5. Three capital units are placed at a simple interest rate. The first one at 5% annually for 4 years, the second at 4% annually for 3 years and 8 months and the third at 3% for 2 years and 16 days. In total, the interest accrued amounts to 310.25€. Knowing that the second capital is double the amount of the first one and the third one and three times the second one, calculate the amount of each capital as well as the interest accrued. (Consider the commercial year of 360 days).

Solution.

$$C_A \times 0.05 \times 4 + C_B \times 0.04 \times 3.6666 + C_C \times 0.03 \times 2.04444 = 310.25$$

$$C_B = 2C_A; \quad C_C = 3C_B$$

$$C_A = 360.19\,€; \quad C_B = 720.39\,€; \quad C_C = 2{,}161.18\,€$$

2.1.8. Calculation of the capital based on the previous one

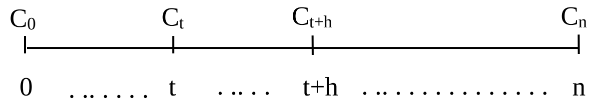


Figure 2.2

If we want to calculate the capital at moment "t + h" based on the capital in "t"

A) Simple Capitalisation.

$$C_{t+h} = C_t\left[1 + (t + h - t)i\right] = C_t(1 + hi)$$

B) Compound Capitalisation.

$$C_{t+h} = C_t(1+i)^{(t+h)-t} = C_t(1+i)^h$$

2.1.9. Interest rate calculation

A) Simple Capitalisation.

$$i = \frac{C_n - C_0}{C_0 n}$$

B) Compounded Capitalisation.

$$i = \sqrt[n]{\frac{C_n}{C_0}} - 1$$

By logarithms:

$$i = \text{Antilog}\left[\frac{\text{Lg}\,C_n - \text{Lg}\,C_0}{n}\right] - 1$$

Exercise 2.6. Calculate for how many months it is necessary for capital of €3,000 to have been deposited in a bank in order for the total interest accrued thereon to amount to €450 at an annual simple interest rate of 6%. *Solution: 30.*

Exercise 2.7. At what annual interest rate is it necessary to invest €12,000 in order to repay a debt of €15,500 which is due within 6 quarters? *Solution: 19.44%.*

Exercise 2.8. At what quarterly simple interest rate were €8,000 invested if the interest accrued thereon after 9 months amounted to €720? *Solution: 3%.*

Exercise 2.9. What amount will we obtain by investing €4,500 for 10 years in a financial institution that for the first 6 months offers us a bimonthly simple interest rate of 2%, for the following 4.5 years an annual compound interest rate of 4% and for the rest of the term a half-yearly simple interest rate of 5%? *Solution: €8,536.10*

2.2. Methods for calculating the total in compounded capitalisation with split time

A financial transaction can span over an exact number of years or an exact number of years plus a fraction. If we are dealing with an exact number of years the above is valid for now, but if dealing with an exact number of years plus a fraction, we can arrive at the solution by applying two methods: exponential and linear.

2.2.1. Exponential Method

The method involves the generalisation of the compound capitalisation formula, capitalising C_0 by the number of periods resulting from adding up the entire part plus the fraction.

We call "n" the exact number of years and "f" the fraction and "t" to the sum of both.

$$C_t = C_0(1+i)^t = C_0(1+i)^{n+f}$$

2.2.2. Linear Method

Its aim is to capitalise at a compound interest rate the entire part of periods and then capitalise the total obtained at at a simple interest rate by the fraction of the remaining time.

$$C_t = C_0(1+i)^n(1+if)$$

2.2.3. Difference between both methods

If we compare the expression of the total resulting from applying both methods:

Linear Method $\quad C_t = C_0(1+i)^n(1+i\times f)$

Exponential Method $\quad C_t = C_0(1+i)^{n+f} = C_0(1+i)^n(1+i)^f$

The difference between the two totals is based on the $(1+i)^f$ and $(1+if)$ factors and, therefore, since we know that "f" is less than the unit, we could say that the linear method delivers a greater amount than the one we would have obtained if we had applied the exponential method. To show this, we will develop the binomial power of the case where we applied the exponential method.

$$(1+i)^f = 1 + if + \frac{f(f-1)}{2!}i^2 + \frac{f(f-1)(f-2)}{3!}i^3 + ...$$

The development provides us with an alternating series of positive and negative terms. The third member of the series is negative, that is, $\frac{f(f-1)}{2!}i^2 < 0$, since $f < 1$, the fourth member is positive, etc. As a result, the series is decreasing, so that,

$$(1+i)^f = 1 + if - x \quad \Rightarrow \quad (1+i)^f < (1 + i \times f)$$

Graphically

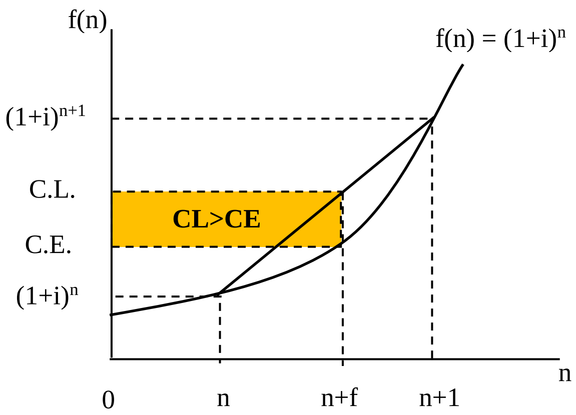


Figure 2.3

Exercise 2.10. 5,000€ in capital is invested in a transaction with a duration of 5 years, 3 months and 20 days. The compound interest rate applied to this transaction is 8%. Find the total by applying the exponential and linear methods (commercial year of 360 days).

Solution.

$$t = n + f \left\} \begin{aligned} n &= 5 \text{ anys} \\ f &= 3/12 + 20/360 = 0{,}305555 \end{aligned} \right.$$

Linear Method: $C_t = 5{,}000(1+0.08)^5(1+0.08 \times 0.30555) = 7{,}526.22$ €

Exponential Method: $C_t = 5{,}000(1+0.08)^{5.30555} = 7{,}521.45$ €

2.3. Calculating time using the linear method

If we apply the linear method, we could encounter a problem when calculating the term of the transaction as the unknown quantity is to be found in two places.

To resolve this issue, it is necessary to apply the exponential criterion to calculate the entire part of the duration and then apply the linear criterion after knowing the entire part of the time span so that we are able tocalculate the fractional part.

Exercise 2.11. Applying the linear method, calculate the period during which the 2,500€

capital should be deposited so that at a 5% compound interest rate, accrued interest totals 1,030 €.

Solution.

$$C_n = C_0 + I_t = 2{,}500 + 1{,}030 = 3{,}530 \text{ €}$$

1) $3{,}530 = 2{,}500(1+0.05)^n(1+0.05f)$

We assume that the second parenthesis is equal to the unit and we take logarithms.

$$\text{Log}\,3{,}530 = \text{Log}\,2{,}500 + n\,\text{Log}(1+0.05) \quad \rightarrow \quad n = 7.071243 = 7$$

2) We apply the linear method to the equation again, but this time, we know that "n" is equal to 7,

$$3{,}530 = 2{,}500(1+0.05)^7(1+0.05\times f)$$

$$f = 0.06964 = 25.07 \text{ days} \quad \Rightarrow \quad \text{Solution = 7 years and 25 days.}$$

Exercise 2.12. 6,000€ in capital was deposited at a 4% simple interest rate during a certain period of time, at the end of which, capital and interests are deposited at a 5% interest rate during a period that is one half year longer than the first period. If the new deposit generates 990€ in interest, how long was it deposited the first time?

Solution.

$$C_n = 6{,}000(1+0.04\,n) = 6{,}000 + 240\,n$$

$$\text{As } I_t = C_0 n i \rightarrow 990 = (6{,}000 + 240n)(n + \tfrac{1}{2})0.05$$

$$990 = 300n + 150 + 12n^2 + 6n \quad \rightarrow \quad 12n^2 + 306n - 840 = 0$$

$$n = \frac{-306 \pm \sqrt{306^2 + 4\times 12\times 840}}{2\times 12} = \left.\begin{matrix} 2.5 \text{ anys} \\ \text{Not Significant} \end{matrix}\right\}$$

Exercise 2.13. A person deposits 2,500€ in capital at a 3.8% compound interest rate during 9 years. At the end of each of the 8 first years, ¼ of the interest accrued is withdrawn. What capital will this person have on deposit at the conclusion of the ninth year?

Solution.

$$C_1 = C_0 + \tfrac{3}{4}C_0 i = C_0(1+\tfrac{3}{4}i)$$

$$C_2 = C_1 + \tfrac{3}{4}C_1 i = C_1(1+\tfrac{3}{4}i) = C_0(1+\tfrac{3}{4}i)^2$$

$$C_8 = C_7 + C_7\tfrac{3}{4}i = C_7(1+\tfrac{3}{4}i) = C_0(1+\tfrac{3}{4}i)^7(1+\tfrac{3}{4}i) = C_0(1+\tfrac{3}{4}i)^8$$

$$C_9 = C_8 + C_8 i = C_8(1+i) = C_0(1+\tfrac{3}{4}i)^8(1+i)$$

$$C_9 = 2{,}500(1+\tfrac{3}{4}0.038)^8(1+0.038) = 3{,}249.16 \text{ €}$$

Exercise 2.14. 3,000€ were deposited at compound interest for a certain period of time. If the deposit period had been for less than a year, the final capital would have been less than 660€. If the deposit period had been more than a year, the final capital would have been more than 690€. Calculate:

1º) The interest rate applied.

2º) The time "n", assuming the application of the linear method required four doubling the initial capital.

3º) Express the time lag if the exponential rather than the linear method had been applied.

Solution.

1)

$$C_n = 3{,}000(1+i)^n \quad (1)$$

$$C_n - 660 = 3{,}000(1+i)^{n-1} \quad (2)$$

$$C_n + 690 = 3{,}000(1+i)^{n+1} \quad (3)$$

We substitute (1) in (2)

$$3{,}000(1+i)^n - 660 = 3{,}000(1+i)^{n-1} \quad (4)$$

We substitute (1) in (3)

$$3{,}000(1+i)^n - 660 = 3{,}000(1+i)^{n-1} \quad (5)$$

In (4) we remove common factor $3{,}000(1+i)^{n-1}$

$$3{,}000(1+i)^{n-1}[(1+i)-1] = 660$$

In (5) we remove common factor $3{,}000(1+i)^n$

$$3{,}000(1+i)^n[1-(1+i)] = -690$$

We clear "i" in the two prior equations and we equalise:

$$\frac{660}{3{,}000(1+i)^{n-1}} = \frac{690}{3{,}000(1+i)^n}$$

$$\frac{3{,}000(1+i)^n}{3{,}000(1+i)^{n-1}} = \frac{690}{660} \to (1+i) = \frac{23}{22} = 1.04545; \quad i = 0.04545$$

2)

$$2C_0 = C_0(1+0.0454545)^n(1+0.0454545\,f)$$

$$n = 15.593244$$

$$2 = (1+0{,}0454545)^{15}(1+0{,}0454545\,f) \to$$
$$f = 0{,}58787374 = 7 \text{ months and 2 days}$$

Solution = 15 years, 7 months and 2 days.

3)

$$2C_0 = C_0(1+0.0454545)^n$$

Solution = 15 years, 7 months and 4 days.

Exercise 2.15. A person wins 50,000€ on the lottery and with this sum, makes the following transactions:

1. Deposits determined certain amount at a bank at an 8% annual compound interest rate and at the end of 5 years with the interest accrued, repays a 15,000€ loan from the same bank.

2. The person acquires a farm for 25,000€, selling it 5 years later at a price equivalent to the sum resulting from investing the total amount of the purchase at a 6% annual compound

interest rate at the end of the above-mentioned period.

3. The remainder of the available balance is put into a fixed deposit account for 5 years at 8% annual simple interest the first 3 years and at the same rate, but compounded annually, for the remainder of the period.

Calculate the amount of cash that he will have at the end of five years.

Solution.

Loan amount: $15{,}000 = C_0(1+0.08)^5 \quad \rightarrow \quad C_0 = 10{,}208.74$ €

Calculation of the fixed-term deposit:

Winnings.	50,000.00 €
Debt.	-10,208.74 €
Farm.	-25,000.00 €
Remainder (fixed- term deposit)	14,791.26 €

Value at the end of the 5 years:

Farm: $V_5 = 25{,}000\,(1+0.06)^5 = 33{,}455.63$ €

F.T. Deposit: $V_5 = 14{,}791.26\,(1+0.08\times 3)(1+0.08)^2 = 21{,}393.13$ €

Total value: 33,455.63 + 21,393.13 = 54,848.76€

Exercise 2.16. Calculate the term for a 6,000€ deposit at a compound interest rate of 9% in order to reach 8,650€. Perform the calculation: a) By applying the exponential method. b) By applying the linear method.

Solution. A) 4 years, 2 months and 28 days; b) 4 years, 2 months and 25 days.

Exercise 2.17. A person deposits in a financial entity a certain amount of capital at a 9% simple interest rate for four years, at the end of which he decides to transfer to another entity all the capital plus all the accrued interest, but this time, at 12%, earning 940€ in interest in three years. Calculate what capital was deposited in the first entity.

Solution. 1,919.93€

Exercise 2.18. A person won a cash prize of 20,000€ and decides to carry out the following financial transactions:

1º) Deposits in the financial entity "X" 25% of the capital at 9% compound interest for six years.

2º) Deposits in the financial entity "Y", at 11% compound interest, the capital necessary to repay the 10,000€ loan that expires in four years.

3º) The remainder of the prize is deposited in the entity "Z" which guarantees that the capital deposited will increase 3x in an 8-year period. Calculate:

A) Interests on the capital invested in the entity "X".

B) Capital deposited in the company "Y".

C) Capitalisation returns in the entity "Z" and interests accrued over five years for the deposit.

Solution. a) 3,385.50€; b) 6,587.31€; c) 14.72% and 8,303.43€

Exercise 2.19. Calculate how much interest would be accrued on a capital of €6,000 invested for 8 years in three financial institutions considering that they offer:

Financial institution "A": for the first 5 years an annual simple interest rate of 5% and for the remaining 3 years an annual compound interest rate of 4.5%.

Financial institution "B": for the first 4.5 years an annual compound interest rate of 6%, but with capitalisation performed according to the linear convention, and for the remainder of the term, the same interest rate but with capitalisation performed according to the linear convention.

Financial institution "C": for the first 175 days an annual simple interest rate of 5.5%, for the following 220 days an annual simple interest rate of 5% and for the remainder of the term an annual compound interest rate of 4% (calendar year). *Solution: A) €2,558.75; B) €3,567.14; C) €2,321.15*

3. Fractioned Capitalisation

3.1. Simple fractional and compound capitalisation

Until now we have considered annual interest, and we expressed the time in years. The question arises on what would occur if we divided the capitalisation period in semesters or quarters, that is, if we use the fraction corresponding to the effective annual rate.

In simple capitalisation the amounts are **proportional** because they keep between each other the same proportion as their times. Thus, we define the amount $i_k = \frac{i}{k}$ as a proportional amount. For example, the proportional monthly amount of a 12% simple interest rate would be 1%, since $i_{12} = \frac{i}{12} = \frac{0.12}{12} = 0.01$.

In this way, the total can be calculated through the following expression:

$$C_n = C_0 (1 + n \times k \times i_k)$$

For example, we calculate the total of a capital of 5,000€ in 3 years, at an 8% simple interest rate,

$$C_3 = 5{,}000(1 + 3 \times 0.08) = 6{,}200\,€$$

If now we calculate it with the corresponding proportional quarterly amount, we would obtain the same result,

$$C_3 = 5{,}000\left(1 + 3 \times 4 \times \frac{0.08}{4}\right) = 6{,}200\,€$$

We now apply the same reasoning used in the previous example, but at **compound capitalisation** and obtain,

$$C_3 = 5{,}000(1 + 0.08)^3 = 6{,}298.56\,€$$

With the corresponding proportional quarterly amount, we obtain:

$$C_3 = 5{,}000(1 + 0.02)^{12} = 6{,}341.20\,€$$

The result obtained (6,341.20€) on capitalising quarterly with a proportional amount is different than the one we obtained in the annual capitalisation (6,298.56€), this is due to the fact that in fractional compound capitalisation the proportional amount is not valid.

Analytically:

In compound capitalisation the total of a capital unit with a proportional amount of frequency k would be:

$$C_1 = \left(1+\frac{i}{k}\right)^k$$

We develop them $\left(1+\frac{i}{k}\right)^k$ by Newton's binomial,

$$\left(1+\frac{i}{k}\right)^k = 1+\frac{i}{k}k+\frac{k(k-1)}{2!}\left(\frac{i}{k}\right)^2+\frac{k(k-1)(k-2)}{3!}\left(\frac{i}{k}\right)^3+..........$$

Since $k>1$ from the third term of the series, they are all positive; consequently, its sum, which we will call x, will also be positive,

$$\left(1+\frac{i}{k}\right)^k = (1+i)+x$$

Therefore,

$$\left(1+\frac{i}{k}\right)^k > (1+i) \qquad (1)$$

This condition will also be met for n periods,

$$\left[\left(1+\frac{i}{k}\right)^k\right]^n > (1+i)^n$$

Since with the proportional amounts we have not achieved equivalence, we must seek an amount that operates in compound capitalisation that ensures the equivalence with the effective amount. In short, we must seek an amount (that we know is not proportional) that makes it possible to achieve equivalence in (1).

Therefore, we substitute $\frac{i}{k}$ for another amount that ensures equivalence, for example i_k and we call it "**Equivalence amount**".

$$(1+i_k)^k = (1+i)$$

3.2. Equivalence amounts

We define equivalence amounts as those which, applied to the same capital, during the same time, but referrring to different capitalisation periods, produce the same interests.

$$(1+i_k)^k = (1+i)$$

From the previous equation we can deduce,

$$i = (1+i_k)^k - 1$$

$$i_k = (1+i)^{1/k} - 1$$

Making it extensive to "n" year:

$$C_n = C_0(1+i)^n = C_0(1+i_k)^{nk}$$

In this equation we verify that the time $(n\times k)$ and the amount (i_k) refer to the same unit.

For example, if we apply a quarterly amount for 5 years:

$$C_5 = C_0(1+i_4)^{20}$$

We can also deduce the equivalence amount calculating the total of a capital unit to the equivalence amount i_k during the k fractions in which the unit period is divided.

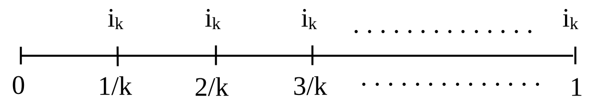


Figure 3.1

The capital units at the end of each k-th will be:

$$
\begin{aligned}
&C_{1/k} = 1+i_k \\
&C_{2/k} = C_{1/k} + C_{1/k}\, i_k = C_{1/k}(1+i_k) = (1+i_k)(1+i_k) = (1+i_k)^2 \\
&C_{3/k} = C_{2/k} + C_{2/k}\, i_k = C_{2/k}(1+i_k) = (1+i_k)^2(1+i_k) = (1+i_k)^3 \\
&\vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \vdots \\
&C_1 = C_{k-1/k} + C_{k-1/k}\, i_k = C_{k-1/k}(1+i_k) = (1+i_k)^{k-1}(1+i_k) = (1+i_k)^k
\end{aligned}
$$

Therefore, if we want "i" and" i_k " to be equivalent, we must verify, on the one hand, the equivalence between the amounts of a capital unit at "i" during a period, that is, (1+i), and, on the other hand, the equivalence of a capital unit at “ i_k ” during “k” fractions of the period corresponding to "i", that is, $(1+i_k)^k$.

Consequently,

$$(1+i) = (1+i_k)^k$$

Applying the example shown in the previous section (3.1),

$$(1+0.08) = (1+i_4)^4 \quad \rightarrow \quad i_4 = (1+0.08)^{1/4} - 1 = 0.019426546$$

$$C_3 = 5{,}000(1+0.019426546)^{12} = 6{,}298.56€$$

We verify how the annual amount of 8% is equivalent to the quarterly amount of 1.9426546% and, therefore, applied on the same capital (5,000€) over the same period (3 years) but capitalising differently (the first annually and the second quarterly) and that this produces the same result (6,298.56€).

Exercise 3.1. Calculate the equivalent monthly amount at 8% annually and verify that they are really equivalent by calculating the total of 1,750€ for previous amounts during 5 years.

Solution.

$$i_{12} = (1+0.08)^{1/12} - 1 = 0.00643403$$

$$C_{n1} = 1{,}750(1+0.08)^5 = 2{,}571.32\ €$$

$$C_{n2} = 1{,}750(1+0.0064340301)^{5\times 12} = 2{,}571.32\ €$$

• Comparing the proportional amount and equivalent amount $\left(i_k \approx i/k\right)$

We are going to demonstrate that the equivalent amount i_k is lower than the proportional amount i/k. To do this, we are going to carry out a binomial development in the expression of the equivalent amount.

$$i_k = (1+i)^{1/k} - 1$$

$$(1+i)^{1/k} = 1 + \frac{1}{k}i + \underbrace{\frac{\frac{1}{k}(\frac{1}{k}-1)}{2!}i^2 + \frac{\frac{1}{k}(\frac{1}{k}-1)(\frac{1}{k}-2)}{3!}i^3 + \ldots\ldots\ldots\ldots}_{\substack{\text{Alternating series of positive and negative terms } \left(\frac{1}{k}<1\right) \\ \text{Negative sum because of } \left(\frac{1}{k}-1\right)<0}}$$

$$(1+i)^{1/k} = (1+\frac{i}{k}) - x \rightarrow \quad (1+i)^{1/k} < (1+\frac{i}{k})$$

$$(1+i)^{1/k} - 1 < \frac{i}{k}$$

But $(1+i)^{1/k} - 1 = i_k$, therefore,

$$i_k < \frac{i}{k}$$

Exercise 3.2. Calculate the equivalent and proportional amounts of 9% annually with bimonthly capitalisation and check the relation existing between them.

Solution.

Proportional amount: $\quad i_k = \frac{i}{k} = \frac{0.09}{6} = 0.015$

Equivalent amount $\quad i_k = (1+i)^{1/k} - 1 = (1+0.09)^{1/6} - 1 = 0.01446$

Exercise 3.3. Someone wishes to invest 30,000€ that will not be needed for another two years. For this purpose two possible options are put forward and it is necessary to decide which is the most interesting.

The first option consists in buying for 30.000 € a bill of exchange for 36,750 €, with a maturity of two years.

The second option consists of lending those 30,000€ at a 5.25% half-yearly equivalent interest rate that should be settled at its maturity together with the accrued interest for those two years.

Indicate which of the two options is the most interesting.

Solution.

We can establish the comparison in two different ways:

a) By the cash amounts:

1) $36{,}750 = 30{,}000(1+i)^2 \quad \rightarrow \quad i = \sqrt{\frac{36{,}750}{30{,}000}} - 1 = 0106797$

2) $(1+i_k)^k = (1+i) \quad \rightarrow \quad i = (1+0.0525)^2 - 1 = 0.107756$

The best option is the second one since the resulting cash amount is greater.

b) By the totals:

1) $C_{n1} = 36{,}750$ €

2) $C_{n2} = 30{,}000(1+0.0525)^4 = 36{,}813.71$ €

The best option is the second one since the total is greater than that of the first.

Exercise 3.4. Calculate the amount obtained from an investment of €9,000 for 6 and a half years at a quarterly equivalent interest rate of 3.25%. Calculate also the amount based on the corresponding bimonthly equivalent interest rate. *Solution: €20,672.07*

3.3. E.A.R. (Effective Annual Rate) and the nominal amount. Relations of equivalence between them

In financial practice, and possibly due to the need to express the amount without a lot of decimals (which would be arrived at by using the equivalent amount), we work with a theoretical amount of cash other than the cash amount and the equivalent. We refer to the nominal amount, also known as cumulative nominal amount or capitalisable annual amount or convertible annual amount. We can define it as the annual amount proportional to the actual amount corresponding to a k frequency.

This nominal amount $J(k)$ is the result of multiplying the equivalent by the number of times that capitalisation takes place, or what is the same, the nominal value (as an external data) divided by the capitalisation frequency is equal to the equivalent for said frequency.

$$J(k) = ki_k$$

Clearing i_k,

$$i_k = \frac{J(k)}{k}$$

If we substitute this value in the equation that relate to the actual amount and the equivalent, that is, in $(1+i) = (1+i_k)^k$,

$$(1+i) = \left(1+\frac{J(k)}{k}\right)^k \qquad (2)$$

Clearing "i" :

$$TAE = i = \left(1+\frac{J(k)}{k}\right)^k - 1$$

To calculate $J(k)$ in terms of "i", in the equation (2) we raise both members to $\frac{1}{k}$:

$$(1+i)^{1/k} = \left[\left(1+\frac{J(k)}{k}\right)^k\right]^{1/k}$$

We operate,

$$(1+i)^{1/k} = \frac{k+J(k)}{k} \rightarrow \ k(1+i)^{1/k} - k = J(k)$$

We clear J (k),

$$J(k) = k\left[(1+i)^{1/k} - 1\right]$$

We observe that $i > J(k)$ for any k value and the difference becomes greater to the extent that it increases the fractioning. Indeed, we develop $\left(1+\frac{J(k)}{k}\right)^k$ by Newton's binomial,

$$\left(1+\frac{J(k)}{k}\right)^k = 1 + k\frac{J(k)}{k} + \underbrace{\frac{k(k-1)}{2!}\left(\frac{J(k)}{k}\right)^2 + \frac{k(k-1)(k-2)}{3!}\left(\frac{J(k)}{k}\right)^3 + \ldots..}_{\text{Since } k>1, \text{ after the third member } >0}$$

If we call "x" the sum of the development as from the third member,

$$(1+i) = [1+J(k)] + x$$

$$i = J(k) + x \quad \Rightarrow \quad TAE(i) > J(k)$$

Exercise 3.5. A bank pays a nominal 6% interest rate for its deposits, capitalisable on a half-yearly basis. According to these forecasts, an investor places the required capital so that, after 6 years, the total amount reaches 50,000€.

Two years after the initiation of the transaction the company changes the frequency of the capitalisation, passing said 6% to quarterly.

Four years after initiation of the transaction, the interest rate that is credited passes on to be 5.5% nominal value and the monthly capitalisation.

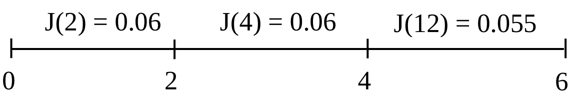
Calculate the amount deposited by the investor and the total that, in the new terms, the investor would obtain.

Solution.

Calculation of the amount deposited by the investor,

$$C_0\left(1+\frac{0.06}{2}\right)^{12} = 50{,}000 \quad \rightarrow \quad C_0 = 35{,}068.99 \text{ €}$$

The total obtained by the investor with the new terms.



$$C_6 = 35{,}068.994\left(1+\frac{0.06}{2}\right)^4\left(1+\frac{0.06}{4}\right)^8\left(1+\frac{0.055}{12}\right)^{24} = 49{,}620.80€$$

Exercise 3.6. A bank pays its depositors 9% nominal interest, capitalisable every six months. Another bank pays the same 9% but capitalisable quarterly. A third bank does it monthly. In which entity is it preferable to deposit the savings?

Solution.

Entity A) $\quad i = \left(1+\frac{0.09}{2}\right)^2 - 1 = 0.092025$

Entity B) $\quad i = \left(1+\dfrac{0.09}{4}\right)^4 - 1 = 0.093083$

Entity C) $\quad i = \left(1+\dfrac{0.09}{12}\right)^{12} - 1 = 0.093806$

The preferable entity is "C" since its cash amount is the greatest of the three.

3.4. Relation between the nominal amount and the frequency of capitalisation

We set the extreme values J (k) for $k = 0$, $k = \infty$ y $k = 1$. When $k \to 0$, the value J (k) is given by,

$$\lim_{k\to 0} J(k) = \lim_{k\to 0} k\left[(1+i)^{1/k} - 1\right] = \lim_{k\to 0} \frac{(1+i)^{1/k} - 1}{1/k} = \lim_{k\to 0} \frac{(1+i)^{1/k}\,\mathrm{Ln}(1+i)\,{-1}/{k^2}}{{-1}/{k^2}} = \infty$$

When $k \to \infty$, the value J (k) is, (see section 8.8), $\lim_{k\to\infty} J(k) = \mathrm{Ln}(1+i)$

When $k = 1$, the value J (k) is, $\lim_{k\to 1} J(k) = i$

as for the growth of the function, we calculate the first derived with respect to "k", calling the function "y",

$$y = k\left[(1+i)^{1/k} - 1\right]$$

$$y' = (1+i)^{1/k} - 1 + (1+i)^{1/k}\,\mathrm{Ln}(1+i)\,\frac{-1}{k^2}\,k = (1+i)^{1/k}\left[1 - \frac{\mathrm{Ln}(1+i)}{k}\right] - 1$$

When $k > 0$, $y' < 0$ and the function is decreasing. If we consider the same amount, then $J(2) > J(3) > J(4) > J(6) > J(12) > ......$

We calculate the second derivative,

$$y'' = -\frac{(1+i)^{1/k}\,\mathrm{Ln}(1+i)}{k^2} + \frac{(1+i)^{1/k}\left[\mathrm{Ln}(1+i)\right]^2}{k^3} + \frac{(1+i)^{1/k}\,\mathrm{Ln}(1+i)}{k^2} = \frac{(1+i)^{1/k}\left[\mathrm{Ln}(1+i)\right]^2}{k^3}$$

When $k > 0$, $y'' > 0$, therefore, the curve is concave. That is, the function has a decreasing concave nearing asymptotically the value of: $\mathrm{Ln}(1+i)$.

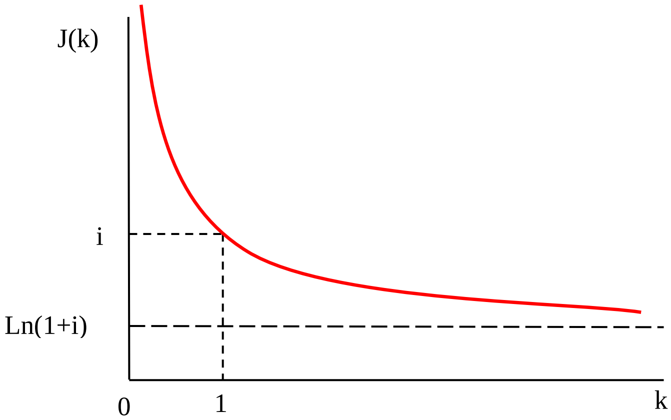


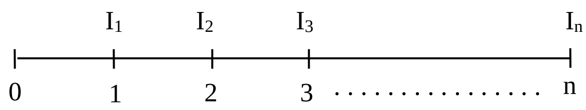
Figure 3.2

All in all, if the J (k) of the transaction is known, we want to calculate the E.A.R. (Equivalent Annual Rate), this rate will increase as the frequency (k) increases and, if the E.A.R of the transaction is known and we want to calculate the J (k), this will decrease as the frequency (k) increases. Therefore, to the extent that the number of capitalisations (k) increases, the E.A.R. increases and the nominal amount J (k) decreases.

3.5. Anticipated annual interest rate. Equivalence with E.A.R.

It may happen that interest is charged or paid in advance, therefore, accrual takes place in advance. If we take a financial transaction to "n" year, there will be some amounts at interest due differing from those at anticipated interest.

Interest due



Anticipated Interest

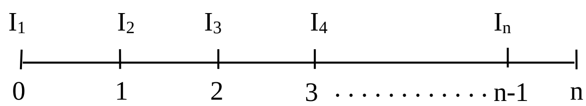


Figure 3.3

We call the Effective Annual Rate (E.A.R.) "i" and the Anticipated Effective Annual Rate (anticipated amount) “i_a”.

To better understand these concepts let us consider a 1€ one-year loan. At interest due, the lender delivers 1€ at the start of the transaction and after a year the lender should receive 1€ augmented by its corresponding interests (i).

Interest due

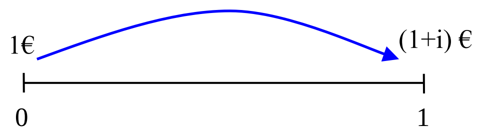


Figure 3.4

The borrower receives 1€ and returns (1+i) €

Anticipated Interest

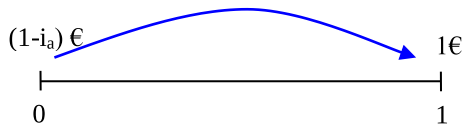


Figure 3.5

The borrower receives $(1-i_a)$ € and returns 1€

The total of the loan at anticipated interest is equal to the anticipated capital capitalised.

$$1=(1-i_a)(1+i)$$

Operating

$$i_a(1+i)=i \quad \Rightarrow \quad i_a=\frac{i}{(1+i)}$$

$$i_a=i-i\,i_a \quad \Rightarrow \quad i_a=i(1-i_a)$$

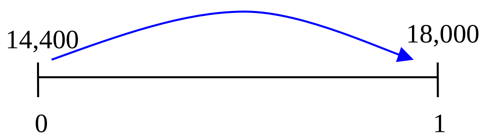
$$i=\frac{i_a}{1-i_a}$$

Exercise 3.7. A one-year 18,000 € loan is granted on which anticipated interests are paid today at a 20% rate. Calculate effective amount.

Solution.

$$i=\frac{i_a}{1-i_a}=\frac{0,2}{1-0,2}=0,25$$

Also,



$$14,400(1+i)=18,000 \quad \rightarrow \quad i=0.25$$

Exercise 3.8. A person has 75,000€ and decides to carry out the following transactions: lend 30,000 € for two years at an anticipated annual 10% interest rate, and lend the remainder of the money for 2 years but at 7% due. The two loans are a single reimbursement on expiration and periodical interest payments.

Calculate:

1º. Amounts effectively loaned in each of the transactions.

2º. Effective returns of each of the loan transactions.

3º. Returns on the entire transaction.

Solution.

1º) Loaned amounts.

A)

27,000	3,000	30,000
0	1	2

B)

48,000	3,360	3,360+48,000
0	1	2

The first loan amounts to 27,000 € and the second to 48,000€.

2º) Returns.

Returns on the first loan.

$$27{,}000(1+i)^2 = 3{,}000(1+i) + 30{,}000$$

We make $(1+i) = x$. We reflect,

$$x = \frac{3{,}000 \pm \sqrt{3{,}000^2 + 4 \times 27{,}000 \times 30{,}000}}{2 \times 27{,}000} = 1.1111$$

$$(1+i) = 1.1111 \quad \rightarrow \quad i = 0.1111$$

Returns on the second loan.

$$48{,}000(1+i)^2 = 3{,}360(1+i) + 51{,}360$$

We make $(1+i) = x$. We reflect,

$$x = \frac{3{,}360 \pm \sqrt{3{,}360^2 + 4 \times 48{,}000 \times 51{,}360}}{2 \times 48{,}000} = 1.07$$

$$(1+i) = 1.07 \quad \rightarrow \quad i = 0.07$$

The return is confirmed at 7%.

3º) Returns on the entire transaction.

$$75{,}000(1+i)^2 = 6{,}360(1+i) + 81{,}360$$

$$i = 0.0848$$

Exercise 3.9. A person who has 25,000€ in capital wishes to invest it for 3 years and goest to five financial entities which offer the following:

Entity "A": 9% simple per annum.

Entity "B": 9% compound per annum.

Entity “C”: 4.5% half-yearly equivalent.

Entity "D": 9% nominal capitalisable half-yearly.

Entity “E”: 0.8% proportional monthly.

Where would you be interested in depositing your money?

Solution.

In this case, the only way of comparing them is by calculating the total amounts,

“A” $C_n = 25{,}000(1+0.09\times 3) = 31{,}750$ €

“B” $C_n = 25{,}000(1+0.09)^3 = 32{,}375.725$ €

“C” $C_n = 25{,}000(1+0.045)^6 = 32{,}556.5031$ €

“D” $C_n = 25{,}000\left(1+\frac{0.09}{2}\right)^6 = 32{,}556.5031$ €

“E” $C_n = 25{,}000(1+0.008\times 3\times 12) = 32{,}200$ €

The best options are offered by entities "C" and "D".

Exercise 3.10. Calculate the APR resulting from the 5 transactions in the previous exercise. *Solution: A) 8.29%; B) 9%; C) 9.20%; D) 9.20%; E) 8.80%*

Exercise 3.11. An investor approaches two financial institutions with a view to investing his savings of €30,000 for a period of 6 years. The financial institutions offer the following:

Bank "P": for the first 2 years a four-monthly capitalisable nominal interest rate of 5%, for the next 18 months a bimonthly equivalent interest rate of 1.5% and for the remainder of the term a quarterly simple interest rate of 2%.

Bank "S": for the first 15 months a monthly nominal interest rate of 6%, for the next 21 months a quarterly equivalent interest rate of 3.5% and for the remainder of the term 6.5% APR.

Calculate: a) the amount that the investor would earn from the two banks, respectively, b) the APR from the two transactions. *Solution: a) P = €45,453.61, S = €49,687.07; b) P = 7.17%, S = 8.77%*

Exercise 3.12. Two investors deposit their capital for five years. The first one at 9% nominal capitalisable half-yearly and the second at 6% simple annual.

After this period has elapsed they withdraw the interest and again deposit the capital for another five-year period. This time, the first investor will deposit it at 6% simple interest and the second at 3% quarterly equivalent.

After 5 years, the first receives 3,000€ in total interests less than the second investor.

Calculate the total of the deposited capital with the amounts totalling 80,000€.

Solution

$$C_1 + C_2 = 80{,}000$$

$$I_1 = C_1(1+0.045)^{10} - C_1 + C_1(1+5\times 0.06) - C_1$$

$$I_2 = C_2(1+0.06\times 5) - C_2 + C_2(1+0.03)^{20} - C_2$$

$$I_1 + 3{,}000 = I_2$$

$$C_1 = 43{,}637.25\ €$$
$$C_2 = 36{,}362.74\ €$$

Exercise 3.13. Two capital units of equal amount are deposited in two financial entities. The first for 6 years and the second one for 8 years, both obtaining the same amount.

1º) Knowing that the first company capitalises at monthly returns of 0.75% and the second at a certain quarterly return. Calculate the value of this quarterly return.

2º) ¿What value should i_4 take so that the total obtained in the second entity is 30% more than the one obtained in the first entity?

Solution

1º) $\left.\begin{array}{l} C_{n1} = C_0(1+0.0075)^{72} \\ C_{n2} = C_0(1+i_4)^{32} \end{array}\right\} C_{n1} = C_{n2} \quad \rightarrow \quad C_0(1+0.0075)^{72} = C_0(1+i_4)^{32}$

$$i_4 = 0.01695$$

2º) $1{,}3\,(1+0.0075)^{72} = (1+i_4)^{32}$

$$i_4 = 0.02532$$

Exercise 3.14. Calculate the interest generated by a 7,000€ investment at a 5% half-yearly simple rate over a period of 15 months. Repeatthe exercise in the case that the amount is compounded.

Solution. 875€; 908.08€.

Exercise 3.15. Calculatethe total that results from investing 8,000€ for a period of 15 years if we know that the entity offers a monthly simple interest rate of 1% during the first years, a simple 2.5% quarterly interest during the following 7 years, and for the remaining term, a simple 5.5% half-yearly interest. Repeat the exercise in the case that the amounts are compounded.

Solution. 28,940.80€; 40,008.78€.

Exercise 3.16. A person who has 25,000€ cash wishes to invest for a period of 5 years and approaches four financial entities that offer the following:

Entity "A": 0.5% monthly equivalent.

Entity “B”: 9,000 € in interest for the entire period.

Entity “C”: 2.2% four-monthly equivalent.

Entity “D”: 12.5% half-yearly equivalent.

Calculate is the most advisable option for the investor:

A) According to E.A.R.

B) In accordance with the final totals.

Solution: a) A: 6.16%; B: 6.34%; C: 6.74%; D: 6.06%. b) A: 33,721.25 €; B: 34,000 €; C: 34,650.01 €; D: 33,559.95 €.

Exercise 3.17. A certain financial entity offers us a set of blankets valued at 30,000 € for a fixed-term 18-month deposit. What amount will we have to deposit for a fixed term if we know that the entity pays a 5% simple interest when the fixed-term deposit is formalized. *Solution. 4,000€*

Exercise 3.18. Calculate the interest and APR that would be earned by investing €20,000 for a term of 8 years in the following financial institutions:

Bank "X": for the first 2 years a quarterly capitalisable nominal interest rate of 6%, for the following 18 months a monthly equivalent interest rate of 1% and for the remainder of the term a half-yearly simple interest rate of 2.5%.

Bank "Y": for the first 18 months a monthly capitalisable nominal interest rate of 8%, for the following 30 months the same interest rate but capitalisable bi-monthly and for the remainder of the term 4.5% APR.

Solution: X = €33,012.55, Y = €32,788.57; X = 6.46%, Y = 6.37%.

Exercise 3.19. Calculate the amount resulting from a fixed-term deposit of €8,000, taking into consideration that for the first 6 months the bank pays a quarterly equivalent interest rate of 2%, for the next 2.5 years a monthly nominal interest rate of 6% and for the following 2 years a bimonthly equivalent interest rate of 1.5%. It must be taken into account that 3 months after the transaction is initiated the investor contributes €1,000 to the fixed term and that after 2 years he withdraws €1,500. *Solution: €11,069.84*

Exercise 3.20. A saver places €10,000 in a deposit account which offers the following conditions: for the first 6 months after depositing the money there shall be an effective annual interest rate of 3%, for the next 6 months there shall be a quarterly nominal interest rate of 4% and for the last 6 months there shall be a bimonthly equivalent interest rate of 0.5%. How much money will there be in the account at the end of the term if the saver deposits €800 after 3 months and withdraws €1,200 after 9 months? *Solution: €10,113.20*

4. Capitalisation Operations

4.1. Conversion of a capital into another multiple of itself

A problem may arise occasionally when calculating the time that it takes for capital to become "x" times greater in compound capitalisation.

In simple capitalisation:

$$x\,C_0 = C_0(1+n\times i)$$

$$n=\frac{x-1}{i}; \qquad i=\frac{x-1}{n}$$

In compound capitalisation:

$$x\,C_0 = C_0(1+i)^t$$

We take logarithms,

$$\mathrm{Lg}\,x = t\,\mathrm{Lg}(1+i)$$

From here we can calculate, the time or the amount,

$$t=\frac{\mathrm{Lg}\,x}{\mathrm{Lg}(1+i)} \quad ; \quad i=\mathrm{Antilg}\left[\frac{\mathrm{Lg}\,x}{t}\right]-1$$

In conclusion, given an interest rate "i", the time required in order for capital to become "x" times greater is calculated separately than the amount of capital that is multiplied. Likewise, given a time period "t", the amount of capital that must be invested so that it becomes "x" times greater is calculated separately than the amount of capital that is multiplied.

Exercise 4.1. Calculate the time that it takes for 4,500 € in capital to increase threefold at an interest rate of 7% in compounded capitalisation.

Solution.

$$t=\frac{\mathrm{Lg}\,3}{\mathrm{Lg}\,(1+0.07)}=16{,}2357 \text{ years (16 years, 2 months and 25 days)}$$

4.2. Displacement of capital

Let us imagine a financial transaction in which, knowing the value of its capital at any t moment, we want to calculate the value of such capital in another previous or subsequent moment.

Assuming we have two capital units: C_1 due in t_1 and C_2 with expiration in t_2. In diagram,

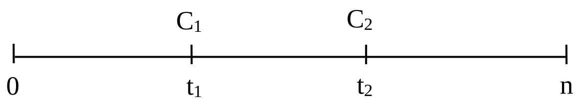


Figure 4.1

In order to be able to compare the capital units we have to value them at the same moment.

If the valuation is carried out in t_1:

C_1 it is already in t_1 y, therefore, its value will be Ct_1

$Ct_2 = C_2(1+i)^{-(t_2-t_1)}$

Now we can compare Ct_1 with Ct_2

If the valuation is carried out in t_2:

$Ct_1 = C_1(1+i)^{t_2-t_1}$

C_2 it is already in t_2 y, therefore, its value will be Ct_2

Now we can compare Ct_1 with Ct_2

4.3. Equivalent capital units

In compound capitalisation, two capital units are financially equivalent provided that their current values are equal. In the same way, they will be financially equivalent provided that the equivalence of its projections is verified at any moment "t" of the transaction, and also, when they have the same replacement capital.

In simple capitalisation it can be verified that the two capital units are financially equivalent when their current values in origin are equivalent. Nevertheless, the fact that the equality of its values can be verified at any moment "t", does not imply that the original values are equivalent and, consequently, it does not imply the financial equivalence.

This is not the case in compound capitalisation where financial equivalence is verified at any moment in which the capital units are valued.

4.3.1. Compound capitalisation

Let us assume that there are two capital units: C_1 and C_2 with expiration in t_1 and t_2, respectively (see figure 4.1). We calculate the value of both capital units at their origin:

$$C_{01} = C_1(1+i)^{-t_1}$$
$$C_{02} = C_2(1+i)^{-t_2}$$

By definition, in order for capital units C_1 and C_2 to be financially equivalent, their current value (or their replacement capital), have to be equal, that is, $C_{01} = C_{02}$. By replacing we have,

$$C_1(1+i)^{-t_1} = C_2(1+i)^{-t_2}$$

If we displace both capital units to the moment "t", after t_2, we will have to multiply by $(1+i)^t$,

$$C_1(1+i)^{t-t_1} = C_2(1+i)^{t-t_2}$$

By doing this, we can see that equality is verified at any moment "t".

4.3.2. Simple Capitalisation

Starting from the capital units of figure 4.1, calculate the value of both at their origin:

$$\begin{aligned} C_{01} &= C_1(1+t_1 i)^{-1} \qquad (1) \\ C_{02} &= C_2(1+t_2 i)^{-1} \end{aligned}$$

By definition, so that the capital units C_1 and C_2 are financially equivalent, their current value (or their replacement capital), have to be equal, that is, $C_{01} = C_{02}$. By replacing, we are left with:

$$C_1(1+t_1 i)^{-1} = C_2(1+t_2 i)^{-1}$$

Equivalent expressions. Nevertheless, capital equivalence at any moment "t" does not imply equivalence at their origin and therefore there is no equivalence. To show this we present the equality of C_1 and C_2 in "t"

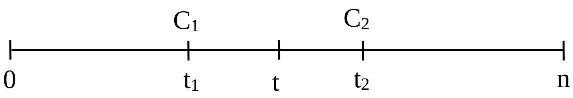


Figure 4.2

$$\left.\begin{aligned} C_{1t} &= C_1\left[1+(t-t_1)i\right] \\ C_{2t} &= C_2\left[1+(t_2-t)i\right]^{-1} \end{aligned}\right\} \quad C_{1t} = C_{2t}$$

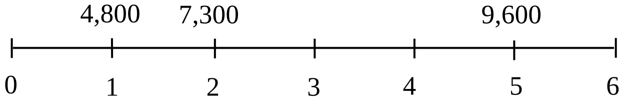
If we now value C_{1t} and C_{2t} at their origin,

$$\begin{aligned} C_{01} &= C_1\left[1+(t-t_1)i\right](1+t\,i)^{-1} \\ C_{02} &= C_2\left[1+(t_2-t)i\right]^{-1}(1+t\,i)^{-1} \end{aligned}$$

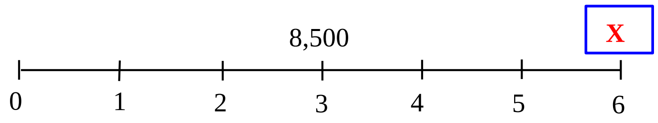
Expressions other than (1). Therefore, equivalence at a specific moment does not imply financial equivalence in simple capitalisation.

Exercise 4.2. A business offers its client two alternative financing plans for the acquisition of a machine. In each period, 10% of annual compound interest is included.

Plan “A”:



Plan "B"



Find the amount of the period number 6 of Plan "B" so that both are equivalent.

Solution.

We value the two plans at moment 0,

$$C_{A0} = 4{,}800\,(1+0.1)^{-1} + 7{,}300\,(1+0.1)^{-2} + 9{,}600\,(1+0.1)^{-5} = 16{,}357.53 \text{ €}$$

$$C_{B0} = 8{,}500\,(1+0.1)^{-3} + X\,(1+0.1)^{-6}$$

We equalize,

$$16{,}357.53 = 8{,}500\,(1+0.1)^{-3} + X\,(1+0.1)^{-6} \quad \rightarrow \quad X = 17{,}664.87 \text{ €}$$

We can also calculate equivalence taking the third year, for example, as the moment of the valuation.

$$4{,}800\,(1+0.1)^{2} + 7{,}300\,(1+0.1) + 9{,}600\,(1+0.1)^{-2} = 8{,}500 + X\,(1+0.1)^{-3}$$

$$X = 17{,}664.87 \text{ €}$$

If we perform the calculation with <u>simple capitalisation</u>, for example in the fourth year,

$$4{,}800\,(1+3\times 0.1) + 7{,}300\,(1+2\times 0.1) + 9{,}600\,(1+0.1)^{-1} =$$

$$= 8{,}500\,(1+0.1) + X\,(1+2\times 0.1)^{-1}$$

$$X = 17{,}252.72 \text{ €}$$

Now we calculate it at moment 0, checking the value obtained previously,

$$C_{A0} = 4{,}800\,(1+0.1)^{-1} + 7{,}300\,(1+2\times 0.1)^{-1} + 9{,}600\,(1+5\times 0.1)^{-1} = 16{,}846.96 \text{ €}$$

$$C_{B0} = 8{,}500\,(1+3\times 0.1)^{-1} + 17{,}252.72\,(1+6\times 0.1)^{-1} = 17{,}321.41 \text{ €}$$

As we can see, the results are different because simple capitalisation does not ensure financial equivalence at two different moments with the same capital units and the same interest rate. That is, simple capitalisation ensures us that the group of capital units are equivalent at a specific moment, for example, at the origin, but does not ensure their equivalence at a different moment.

Nevertheless, compound capitalisation ensures us the financial equivalence of a group of capital units at any moment in which these are valued.

This implies that compound capitalisation will always enable us to make the same financial choice from a group of equivalent capital units, regardless of the moment of its valuation, whereas this is not possible with simple capitalisation.

4.4. Single capital. Common expiration

4.4.1. Single Capital

$C_1, C_2, C_3, \ldots\ldots, C_k$ are capital units valued at the amount "i" of compound interest and expiring at moments $t_1, t_2, t_3, \ldots\ldots, t_k$, respectively. We call **Single capital** or equivalent capital capital "C" which is payable in "t" years and fulfills financial equivalence.

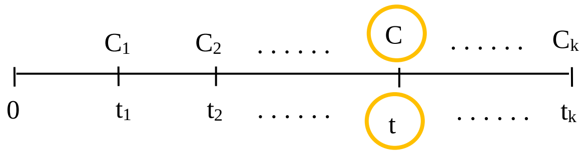


Figure 4.3

We can find equivalence at "t" or at the origin.

• Equivalence at "t":

$$C = C_1(1+i)^{t-t_1} + C_2(1+i)^{t-t_2} + \cdots\cdots + C_k(1+i)^{t-t_k}$$

• equivalence at the origin:

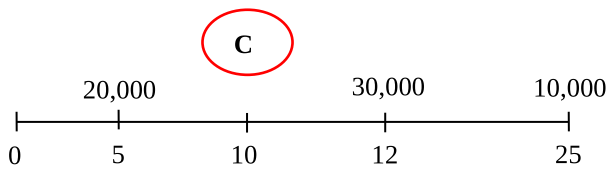
$$C(1+i)^{-t} = C_1(1+i)^{-t_1} + C_2(1+i)^{-t_2} + \cdots\cdots + C_k(1+i)^{-t_k}$$

$$C(1+i)^{-t} = \sum_{r=1}^{k} C_r(1+i)^{-t_r} \quad \Rightarrow \quad C = \frac{\sum_{r=1}^{k} C_r(1+i)^{-t_r}}{(1+i)^{-t}}$$

Exercise 4.3. A lender has granted three loans. One of 20,000€ due in 5 years, another of 30,000€ due in 12 years, and a third which is a reimbursement of 10,000€ in 25 years.

The person decides to enter into a transaction with fan entity pursuant to which it acquires these loans and is obliged to pay the lender the equivalent capital at 6% annual compound interest within 10 years. Find the equivalent capital.

Solution.



Valuation in the 10th year.

$$20{,}000\,(1+0.06)^{5} + 30{,}000\,(1+0.06)^{-2} + 10{,}000\,(1+0.06)^{-15} = C$$

$$C = 57{,}637.05\ €$$

Valuation at the origin:

$$20{,}000\,(1+0.06)^{-5} + 30{,}000\,(1+0.06)^{-12} + 10{,}000\,(1+0.06)^{-25} = C\,(1+0.06)^{-10}$$

$$C = 57{,}637.05\ €$$

Valuation in the 25th year:

$$20{,}000\,(1+0.06)^{20} + 30{,}000\,(1+0.06)^{13} + 10{,}000 = C\,(1+0.06)^{15}$$

$$C = 57{,}637.05\ €$$

4.4.2. Common Due Date

To understand what is the average due date we return to the definition of single capital. The "**common due date**" refers to the expiration of the single capital "C" equivalent to a group of capital units $C_1, C_2, C_3, \cdots C_n$.

In the equation of the single capital we operate to calculate "t",

$$C = \frac{\sum_{r=1}^{k} C_r (1+i)^{-t_r}}{(1+i)^{-t}} \quad \rightarrow \quad C(1+i)^{-t} = \sum_{r=1}^{k} C_r (1+i)^{-t_r}$$

We take logarithms,

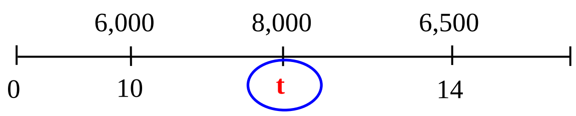
$$\text{Lg}\, C - t\, \text{Lg}\,(1+i) = \text{Lg} \sum_{r=1}^{k} C_r (1+i)^{-t_r}$$

$$t = \frac{\text{Lg}\, C - \text{Lg} \sum_{r=1}^{k} C_r (1+i)^{-t_r}}{\text{Lg}\,(1+i)}$$

Exercise 4.4. We want to replace two capital units at 8%. One of 6,000€, payable within 10 years and another of 6,500€, payable within14 years.

Calculate the expiration of a single capital unit of 8,000€.

Solution.



$$6{,}000\,(1+0.08)^{-10} + 6{,}500\,(1+0.08)^{-14} = 8{,}000\,(1+0.08)^{-t}$$

$$-t\,\text{Lg}\,(1+0.08) = \text{Lg}\, 0.624019712$$

$t = 6.1274$ years (6 years, 1 months and 16 days)

Exercise 4.5. A business has issued the following promissory notes for the repayment of a 15,000€ debt:

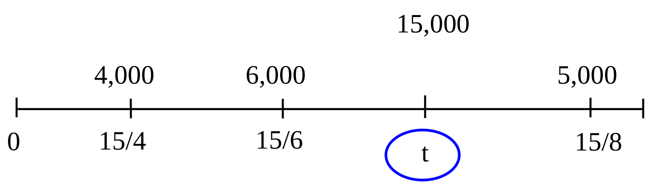
* Promissory note of 4,000€ due on April 15.

* Promissory note of 6,000€ due on June 15.

* Promissory note of 5,000€ due on August 15.

Calculate when the three promissory notes can be replaced by a single promissory note at a simple interest rate of 8% (financial law of simple capitalisation. Calendar year).

Solution.



$$5{,}000+6{,}000\left(1+61\frac{0.08}{365}\right)+4{,}000\left(1+122\frac{0.08}{365}\right)=15{,}000\left(1+t\frac{0.08}{365}\right)$$

$$t=56.93\approx 57 \text{ days (19 June)}$$

4.5. The average due date and the average amount

4.5.1. Average Due Date

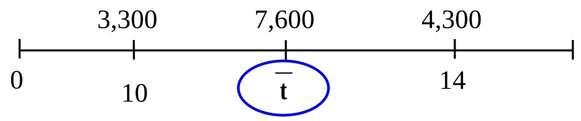
In a specific case in which the amount of single capital is the arithmetic sum of the replaced capital, the due date for this single capital is called **"average due date".**

In the equation "t" equation we make that $C=\sum_{r=1}^{n} C_r$. In this way we will arrive at the following average due date:

$$\overline{t}=\frac{\operatorname{Lg}\sum_{r=1}^{n} C_r-\operatorname{Lg}\sum_{r=1}^{k} C_r(1+i)^{-t_r}}{\operatorname{Lg}(1+i)}$$

Exercise 4.6. We want to replace two capital units at 5%. One for the amount of 3,300€ with due in 10 years and the other of for 4,300 € in 14 years. Calculate the average due date of the capital that replaces them.

Solution.



$$7{,}600\,(1+0.05)^{-t}=3{,}300\,(1+0.05)^{-10}+4{,}300\,(1+0.05)^{-14}$$

$$\overline{t}=\frac{\operatorname{Log}7{,}600-\operatorname{Log}4{,}197.70}{\operatorname{Log}1.05}=12.16659 \text{ years (12 years and 2 months)}$$

4.5.2. Average Amount

We define as average amount of several capital units as the amount which applied to a group of capital over a certain period generates the same interest as the several rates applied to the same capitals during the same period.

A) Simple capitalisation

Let $C_1, C_2, C_3, \ldots\ldots\ldots, C_n$ represent the totals of the capital units $c_1, c_2, c_3, \ldots\ldots\ldots, c_n$ deposited during $n_1, n_2, n_3, \ldots\ldots\ldots, n_n$ years at the $i_1, i_2, \ldots\ldots\ldots, i_n$ simple interest rates, respectively, and let $\overline{i}$ represent the average rate.

$$C_1=c_1(1+n_1 i_1)$$
$$C_2=c_2(1+n_2 i_2)$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$C_n=c_n(1+n_n i_n)$$

By definition,

$$c_1(1+n_1i_1)+c_2(1+n_2i_2)......+c_n(1+n_ni_n)=c_1(1+n_1\bar{i})+c_2(1+n_2\bar{i})......+c_n(1+n_n\bar{i})$$

Operating,

$$c_1+c_1n_1i_1+c_2+c_2n_2i_2+....+c_n+c_nn_ni_n=c_1+c_1n_1\bar{i}+c_2+c_2n_2\bar{i}+....+c_n+c_nn_n\bar{i}$$

$$c_1+c_1n_1i_1+c_2+c_2n_2i_2+....+c_n+c_nn_ni_n=c_1+c_1n_1\bar{i}+c_2+c_2n_2\bar{i}+....+c_n+c_nn_n\bar{i}$$

$$\sum_{r=1}^{n}c_rn_ri_r=\bar{i}\sum_{r=1}^{n}c_rn_r \quad\rightarrow\quad \bar{i}=\frac{\sum_{r=1}^{n}c_rn_ri_r}{\sum_{r=1}^{n}c_rn_r}$$

Weighted arithmetic average equation.

If all the capital units and periods are equal, that is, $c_1=c_2=c_3=........=c_n=c$ and the term is also identical for all capital units, $n_1=n_2=n_3=........=n_n=n$.

$$\bar{i}=\frac{c_1n_1i_1+c_2n_2i_2+........+c_nn_ni_n}{cn+cn+........+cn}=\frac{i_1+i_2+........+i_n}{N}\quad\frac{\sum_{r=1}^{n}i_r}{N}$$

Simple arithmetic average equation.

Exercise 4.7. Calculate the average rate for 3 deposits each for the amount of 5,000 € over a 6-year period: the first at 6%, the second at 8%, and the third at 13%.

Solution.

$$\bar{i}=\frac{0.06+0.08+0.13}{3}=0.09$$

Exercise 4.8. Calculate the average amount of three deposits: 10,000€ for 3 years, 30,000€ for 6 years, and 50,000€ for 2 years, at the simple interest rate of 3%, 6% and 7%, respectively.

Solution.

$$\bar{i}=\frac{10{,}000\times3\times0.03+30{,}000\times6\times0.06+50{,}000\times2\times0.07}{10{,}000\times3+30{,}000\times6+50{,}000\times2}=0.06032$$

Tthe total interest at the average rate will be:

$$\bar{I}=0.06032258\ [C_1n_1+C_2n_2+C_3n_3]=18{,}700\ \text{€}$$

$$I_1=10{,}000\times3\times0.03=900\ \text{€}$$

$$I_2=30{,}000\times6\times0.06=10{,}800\ \text{€}$$

$$I_3=50{,}000\times2\times0.07=7{,}000\ \text{€}$$

$$I_1+I_2+I_3=900+10{,}800+7{,}000=18{,}700\ \text{€}$$

B) Compound capitalisation

Let $c_1,c_2,c_3,.........,c_n$ represent the capital units deposited during "n" years at the $i_1,i_2,.........,i_n$ compound interest rates, respectively, and let $C_1,C_2,C_3,.........,C_n$ represent the totals of the "n" capital units and $\overline{i}$ the average rate,

$$C_1 = c_1\left[(1+i_1)^n\right]$$
$$C_2 = c_2\left[(1+i_2)^n\right]$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$C_n = c_n\left[(1+i_n)^n\right]$$

We apply average rate definition,

$$c_1\left[(1+i_1)^n\right]+c_2\left[(1+i_2)^n\right]+\ldots\ldots..+c_n\left[(1+i_n)^n\right]=$$
$$=c_1\left[\left(1+\overline{i}\right)^n\right]+c_2\left[\left(1+\overline{i}\right)^n\right]+\ldots\ldots..+c_n\left[\left(1+\overline{i}\right)^n\right]$$

$$\sum_{r=1}^{n} c_r(1+i_r)^n = \left(1+\bar{i}\right)^n \sum_{r=1}^{n} c_r$$

$$\left(1+\overline{i}\right)^n = \frac{\sum_{r=1}^{n} C_r(1+i_r)^n}{\sum_{r=1}^{n} C_r}$$

If we call “x” the second member of the previous equivalnece and we take logarithms,

$$\overline{i} = \text{Antilog}\left[\frac{\text{Logx}}{n}\right]-1$$

Exercise 4.9. Calculate the average amount at a compound interest rate of the following 5-year investments.

A: 50,000€ at 8%.

B: 25,000€ at 7.5%.

C: 64,000€ at 9%

Solution.

$$C_{nA} = 50{,}000(1+0{,}08)^5 = 73{,}466.40€$$
$$C_{nB} = 25{,}000(1+0.075)^5 = 35{,}890.73€$$
$$C_{nC} = 64{,}000(1+0.09)^5 = 98{,}471.93€$$
$$c_A + c_B + c_C = 139{,}000 \quad \rightarrow \quad C_{nA} + C_{nB} + C_{nC} = 207{,}829.06$$
$$139{,}000\left(1+\bar{i}\right)^5 = 207{,}829.06$$
$$\overline{i} = 0.08377$$

Exercise 4.10. Calculate the single investment to be carried out in compound capitalisation in substitution of the capital units whose amounts and valuation amounts shown below:

1.- 3,000€ at 8%.

2.- 2,500€ at 9%.

3.- 1,300€ at 10%.

4.- 1,900€ at 12%.

The single amount is at 8% and the term for all capital units is 5 years.

Solution.

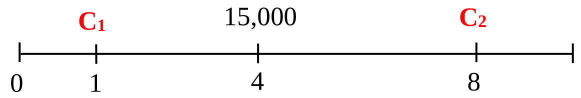
$$C\,(1+0.08)^5 = 3{,}000(1+0.08)^5 + 2{,}500(1+0.09)^5 + 1{,}300(1+0.1)^5 + 1{,}900(1+0.12)^5$$

$$C = 9{,}321.71\ €$$

Exercise 4.11. A person that has to pay 15,000€ within 4 years proposes to the lender the replacement of this payment by another two payments due in 1 year and in 8 years.

If the interest rate of the transaction is 8% and the sum of the replacement capital units are equal to the capital unit replaced, calculate the amounts.

Solution.



$$C_1 + C_2 = 15{,}000$$

$$15{,}000\,(1+0.08)^{-4} = C_1(1+0.08)^{-1} + C_2(1+0.08)^{-8}$$

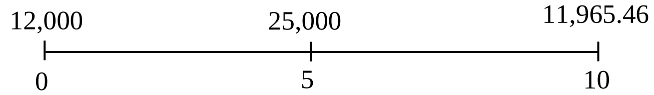
$$15{,}000\,(1+0.08)^{-4} = (15{,}000 - C_2)(1+0.08)^{-1} + C_2(1+0.08)^{-8}$$

$$C_1 = 7{,}575.14\ €$$

$$C_2 = 7{,}424.85\ €$$

Exercise 4.12. A person has to pay 25,000€ in 5 years and to settle the debt proposes to the creditor a payment of 12,000€ now and antoher payment of 11,965.46€ in 10 years Calculate at what amount the transaction was valued.

Solution.



$$25{,}000\,(1+i)^{-5} = 12{,}000 + 11{,}965.46(1+i)^{-10}$$

We make $x = (1+i)^{-5}$

$$11{,}965.46\,x^2 - 25{,}000\,x + 12{,}000 = 0$$

$$x = \left.\frac{25{,}000 \pm \sqrt{25{,}000^2 - 4 \times 11{,}965.467 \times 12{,}000}}{2 \times 11{,}965.467}\right\} = \begin{matrix} 1.342089 \\ 0.7472576 \end{matrix}$$

$$(1+i)^{-5} = 0.7472576 \quad \Rightarrow \quad i = 0.06$$

Exercise 4.13. Mr. "A" will pay Mr. "B" 7,500€ within a year and 5,000€ within 6 years. Mr. "B" undertakes to return the loan in the following way: 2,000€ in 2 years, 5,000€ in 4 years and another payment in 8 years At a rate of 8%, calculate:

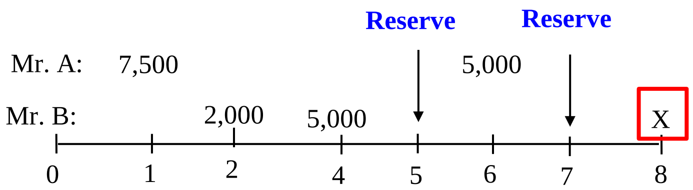
A) Value of the last payment.

B) Mathematical reserve by the prospective method in the 5th year.

C) Reserve by the retrospective method in the 5th year.

D) Reserve in the 7th year, using the reserve value of the 5th year.

Solution.



A)

$$7{,}500(1+0.08)^{-1}+5{,}000(1+0.08)^{-6}=$$

$$=2{,}000(1+0.08)^{-2}+5{,}000(1+0.08)^{-4}+X(1+0.08)^{-8}$$

$$X = 8{,}709.48\ €$$

B)

$$\text{Mr. A.:}\ V_5=5{,}000\,(1+0.08)^{-1}=4{,}629.62\,€$$

$$\text{Mr. B.:}\ V_5=8{,}709.48\,(1+0.08)^{-3}=6{,}913.87\,€$$

$$R_5=V_5(B)-V_5(A)=6{,}913.87-4{,}629.62=2{,}284.25\,€.$$

C)

$$\text{Mr. A.:}\ V_5=7{,}500\,(1+0.08)^{4}=10{,}203.66\,€$$

$$\text{Mr. B.:}\ V_5=2{,}000\,(1+0.08)^{3}+5{,}000\,(1+0.08)=7{,}919.42\,€$$

$$R_5=V_5(A)-V_5(B)=10{,}203.66-7{,}919.42=2{,}284.25\,€.$$

D)

$$R_7=R_5\,(1+0{,}08)^{2}=2{,}284.25\,(1+0.08)^{2}+5{,}000(1+0.08)=8{,}064.33\ €$$

Exercise 4.14. A person wishes to buy a home worth 200,000€ and approaches a financial entity to finance the purchase. After studying the transaction, the entity agrees to the financial loan request, but under the following terms and conditions:

On 1 January of x3 the financial entity lends the requested amount and, at the same time, the borrower undertakes to repay it every 1st of January according to the following plan:

x4: 25,000 €.

x5: 30,000 €.

x6: 40,000 €.

x7: 55,000 €.

x8: 70,000 €.

x10: Pending quantity to be determined.

If the valuation rate of compound interest is 10%, calculate:

A) Quantity to be repaid in x10.

B) Mathematical reserve by theretrospective and prospective methods after making the payment for year x6.

C) Mathematical Reserve by the recurrent method in the year x8, previously calculating the reserve in x5, by applying the retrospective method (assuming that the calculation of the reserves is performed after payments or collections in relation to the respective year).

Solution. A) 80,670.09 €; B) 162,950 €; C) 66,669.49 €.

Exercise 4.15. An investment for 3 years, 5 months and 20 days is made. Of the couple of options shown below, which is more cost-effective?

A) Simple or compound capitalisation.

B) Considering compound capitalisation: linear method or exponential method.

C) An E.A.R. of 6% or a nominal monthly rate of 6%.

D) An effective 6% rate or a nominal half-yearly rate of 6%.

E) A nominal 4% monthly rate or a nominal 4% half-yearly rate.

F) A 9% at the due date or an anticipated 9% rate.

Solution. A) Compound; B) Linear compound; C) Nominal amount; D)Nominal amount; E) Nominal monthly valuey; F) Anticipated amount.

Exercise 4.16. An entity has to pay the following promissory notes: 6,000 € within 3 years; 5,000 € within 5 years; 9,000 € within 7 years and 10,000 € within 9 years. If the interest is calculated at 12% annually, under the financial law of compound capitalisation, calculate when the four promissory notes can be replaced by a single one (rounding of the result).

Solution. 6 years and 27 days.

Exercise 4.17. Our company has to purchase a machine and the supplier offers us two equivalent payment methods with 10% interest:

Payment method 1: A payment two years after delivery of the piece of machinery of €6,000 and another payment of €6,050 after 4 years.

Payment method 2: A payment of €6,100 a year after purchase and another of €6,080 5 years after the first instalment.

You are asked to: assess the two offers at the time of purchase of the machine and after 10 years in terms of simple and compound interest rates and analyse the differences with regard to the investment decision based on the capitalisation rates used to perform the calculations.

Solution:
SC)

$$\left[\left(V(0)_1 = 9{,}321.43\right) < \left(V(0)_2 = 9{,}598.78\right)\right] \rightarrow (2) > (1)$$

$$\left[\left(V(10)_1 = 20{,}480\right) > \left(V(10)_2 = 18{,}880\right)\right] \rightarrow (2) < (1)$$

CC)

$$\left[\left(V(0)_1 = 9{,}090.90\right) < \left(V(0)_2 = 9{,}320.65\right)\right] \rightarrow (2) > (1)$$

$$\left[\left(V(10)_1 = 23{,}579.47\right) < \left(V(10)_2 = 24{,}175.38\right)\right] \rightarrow (2) > (1)$$

5. The Financial Discount Transaction

5.1. The commercial discount transaction

Discount refers to the amount deducted from capital by anticipating the money, that is, by collecting it before its expiration. Thus, a business possessing a batch of bills of exchange for future collection, can go to a bank to have them discounted, that is, that the bank delivers the cash pertaining to the bills of exchange after charging a fee and discount related expenses.

The discount can be: simple and compound, which are also classified as a commercial and rational or mathematic discount.

The commercial discount represents the cost, in monetary units, of the transaction for the business; while the rational discount represents the profit, in monetary units, for the bank.

5.2. Simple update

5.2.1. Simple Rational Discount

The financial law for simple rational discount is reciprocal to the law of simple capitalisation. The rational discount is calculated on the basis of the cash amount.

We refer to:

"n" as the term of the transaction.

"I" as the interest rate or return of the transaction expressed on a per unit basis.

C_0 as cash.

C_n as the nominal.

Since the rational discount is applied on the basis of the cash amount.

$$D_r = C_0\, n\, i \qquad (1)$$

We can also obtain the discount by subtracting the nominal and the cash amount, that is,

$$D_r = C_n - C_0 \qquad (2)$$

In (2) we reflect the cash

$$C_0 = C_n - D_r \qquad (3)$$

We substitute (3) in (1),

$$D_r = (C_n - D_r)\, n\, i$$

Operating,

$$D_r = C_n \frac{n\,i}{1+n\,i}$$

We calculate the cash amount by the difference,

$$C_0 = C_n - D_r$$

$$C_0 = C_n - C_n \frac{n\,i}{1+n\,i} = C_n\left(1 - \frac{n\,i}{1+n\,i}\right) = C_n\left(\frac{1}{1+n\,i}\right) = C_n(1+n\,i)^{-1}$$

The simple rational discount factor is reciprocal to the simple capitalisation factor and is expresses as follows: $(1+n\,i)^{-1}$

5.2.2. Simple Commercial Discount

The financial law for simple commercial discount is that in which the discounts of a period are proportional to the period of the discount under consideration. In commercial discount, the discount rate "d" is used instead of the effective annual rate.

By definition, the commercial discount applies to the nominal value, therefore,

$$D_c = C_n\,n\,d$$

Cash, depending on the nominal, will be given by,

$$C_0 = C_n - D_c = C_n - C_n\,n\,d$$

$$C_0 = C_n(1 - n\,d)$$

The financial factor of the simple commercial discount will be given by $(1 - n\,d)$

Exercise 5.1. Calculate the rational and commercial or mathematics discount of a bill of exchange with a nominal value of 12,000€ at a 6% simple discount during 60 days (Commercial year of 360 days).

Solution.

$$D_c = C_n\,n\,d = 12{,}000 \times 60 \times \frac{0{,}06}{360} = 120\ €$$

$$C_0 = C_n - D_c = 12{,}000 - 120 = 11{,}880\ €$$

To calculate the rational discount we should assume that $i = d$,

$$D_r = C_n \frac{n\,i}{1+n\,i} = 12{,}000 \frac{60\frac{0.06}{360}}{1+60\frac{0.06}{360}} = 118.811\ €$$

$$C_0 = C_n - D_r = 12{,}000 - 118.811 = 11{,}881.18\ €$$

5.2.3. The equivalent interest rate is the same as the discount rate

We seek the relation between "i" and "d" so that the simple rational discounts and the simple commercial discount are the same, that is, the results are equivalent when we apply both laws. Graphically,

Figure 5.1

A disposable Euro at the "n" moment, on discounting the"d" amount becomes $(1-d\times n)$, if we now capitalise this cash value up to "n" to the "i" amount the total will be,

$$(1-d\times n)(1+n\times i)$$

So that "d" and "i" are equivalent, the total obtained has to be equal to the nominal capital that has been split, obtaining, in this way, the initial result.

$$(1-d\times n)(1+n\times i)=1$$

operating,

$$i=\frac{d}{1-n\times d}; \qquad d=\frac{i}{1+n\times i}$$

If n = 1,

$$i=\frac{d}{1-d}; \quad d=\frac{i}{1+i}$$

As can be observed, i > d

We can also reach this result if we equalize the equation of both discounts.

$$\left.\begin{array}{r} D_r=C_n\dfrac{n\,i}{1+n\times i} \\ D_c=C_n\;n\times d \end{array}\right\} \quad D_c=D_r \quad \Rightarrow \quad C_n\;n\,d=C_n\frac{n\times i}{1+n\times i}$$

$$C_n\;n\,d+C_n\;n\times d\times n\times i=C_n\;n\times i \tag{4}$$

$$C_n\;n\times d=i\,(C_n\;n-C_n\;n\times d\times n)$$

$$i=\frac{C_n\;n\times d}{C_n\;n(1-d\times n)} \quad \Rightarrow \quad i=\frac{d}{1-n\times d}$$

Similarly, operating in (4),

$$d(C_n\;n+C_n\;n\times n\times i)=C_n\;n\times i$$

$$d=\frac{C_n\;n\times i}{C_n\;n(1+n\times i)} \quad \Rightarrow \quad d=\frac{i}{1+n\times i}$$

As shown above, “d” and “i” are equivalent and not the same. But if $d=i$ this would be an error that would increase to the extent that "n" and "i" would be greater. The error is quantified as follows:

Calling F_1 the rational discount factor, F_2 the commercial discount factor and E the error,

$$E=F_1-F_2=\frac{1}{1+n\,i}-(1-n\,i)=\frac{1-1-n\times i+n\times i+n^2i^2}{1+n\times i}=\frac{n^2\times i^2}{1+n\times i}>0$$

A positive amount that increases at the same proportion as time and the rate.

Exercise 5.2. The commercial discount of a draft amounts to 1,000€ and the rational discount is 952.38€. Calculate the nominal of the draft based on that d=i.

Solution.

$$D_c = C_n \ n \times i \quad \rightarrow \quad 1.000 = C_n \ n \times i \qquad (5)$$

$$D_r = \frac{C_n \ n \times i}{1 + n \times i} \quad \rightarrow \quad 952{,}38 = \frac{C_n \ n \times i}{1 + n \times i} \qquad (6)$$

We substitute (5) in (6),

$$952.38 = \frac{1{,}000}{1 + n \times i} \quad \rightarrow \quad (1 + n \times i) = 1.05 \quad \rightarrow \quad n \times i = 0.05$$

We again substitute in (5),

$$1{,}000 = C_n \ 0.05 \quad \rightarrow \quad C_n = 20{,}000 \text{ €}.$$

Exercise 5.3. On 22 August, a businessman presents a draft of 10,000€ nominal value, expiring on 20 November, for collection. Assuming that the settlmeent payment takes place upon presentation of the draft, that the discount is 16%, and the collection fee 0.4%, calculate the net payable amount (Calendar year of 365 days).

Solution.

We count 90 days calculating the months by the calendar year (year of 365 days) for this purpose we divide by 365.

$D_c = C_n \ n \times d = 10{,}000 \times 90 \times \frac{0.16}{365} =$	394.52 €
Comisión = 0.4% sobre 10,000 =	40.00 €
Total =	**434.52 €**

Net to be paid = 10,000 – 434.52 = 9,565.48€

5.3. Compound update

5.3.1. Compound Rational Discount

The rational discount is applied on the basis of the cash amount and is defined as its interest rate for the time remaining until its expiration.

The rational discount is the difference between the end capital or nominal (C_n) and the initial capital or cash (C_0), therefore the discount will be determined by the following equation:

$$D_r = C_n - C_0 = C_n - C_n (1+i)^{-n} = C_n \left[1 - (1+i)^{-n}\right]$$

The cash value would be:

$$C_0 = C_n (1+i)^{-n}$$

The financial factor of the compound rational discount is determined by $(1+i)^{-n}$. There is complete identity between the compound update with interest amounts and compound capitalisation.

5.3.2. Compound Commercial Discount

The commercial discount is calculated on the nominal and is defined as the nominal interest from the cash payment until the due date. The commercial discount operates with "d" discount amounts.

If the nominal value C_n we discount a period at the "d" discount amount, we obtain the cash of C_{n-1}:

$$C_{n-1} = C_n - C_n d = C_n(1-d)$$

In the period “n-2" the cash will be the result to subtracting from the nominal of the period “n-1" its corresponding discount:

$$C_{n-2} = C_{n-1} - C_{n-1}d = C_{n-1}(1-d) = C_n(1-d)(1-d) = C_n(1-d)^2$$

In period "n-3" :

$$C_{n-3} = C_{n-2} - C_{n-2}d = C_{n-2}(1-d) = C_n(1-d)^2(1-d) = C_n(1-d)^3$$

In period "1" :

$$C_1 = C_2 - C_2 d = C_2(1-d) = C_n(1-d)^{n-2}(1-d) = C_n(1-d)^{n-1}$$

In period "0":

$$C_0 = C_1 - C_1 d = C_1(1-d) = C_n(1-d)^{n-1}(1-d) = C_n(1-d)^n$$

Therefore:

$$C_0 = C_n(1-d)^n$$

Being $(1-d)^n$ the compound discount financial factor.

The compound commercial discount will determined by the difference between the nominal and the cash value,

$$D_c = C_n - C_0$$

$$D_c = C_n - C_n(1-d)^n = C_n\left[1-(1-d)^n\right]$$

5.3.3. Interest rate equivalent to the discount rate

To find the relation between "i" and "d" we equalize the equations of the rational and commercial discounts,

$$\left.\begin{aligned} D_r &= C_n\left[1-(1+i)^{-n}\right] \\ D_c &= C_n\left[1-(1-d)^n\right] \end{aligned}\right\} \quad D_c = D_r \quad \Rightarrow \quad (1+i)^{-n} = (1-d)^n$$

We extract the umpteenth root to both members of the equality,

$$\sqrt[n]{\frac{1}{(1+i)^n}} = \sqrt[n]{(1-d)^n} \quad \Rightarrow \quad \frac{1}{(1+i)} = (1-d)$$

$$i = \frac{d}{1-d} \quad \Rightarrow \quad d = \frac{i}{1+i}$$

5.4. Comparing the simple rational discount and the composed rational discount

The comparison is going to be limited to the effective values of the two discounts.
We refer to:
C_{0S} as the effective value resulting from the application of the simple rational discount.
C_{0C} as the effective value resulting from applying the compound rational discount.
DRS as the simple rational discount.
DRC as the compound rational discount.

$$C_{0S} = C_n - \frac{C_n n \times i}{1 + n \times i} = C_n \frac{1}{1 + n \times i} = C_n (1 + n \times i)^{-1}$$

$$C_{0C} = C_n - C_n \left[1 - (1 + i)^{-n}\right] = C_n (1 + i)^{-n}$$

Therefore, the comparison is circumscribed to:

$$(1 + n \times i)^{-1} \approx (1 + i)^{-n}$$

There could be four different situations when we compare the two discounts:
1ª. When n = 0:

$$C_{0S} = C_{0C} \quad \Rightarrow \quad DRS = DRC$$

2ª. When n = 1:

$$\left.\begin{array}{l} C_{0S} = C_n (1 + i)^{-1} \\ C_{0C} = C_n (1 + i)^{-1} \end{array}\right\} \quad \Rightarrow \quad DRS = DRC$$

3ª. When $0 < n < 1$:
We develop them $(1 + i)^n$ by Newton's binomial,

$$(1 + i)^n = 1 + n\, i + \underbrace{\frac{n(n-1)}{2!} i^2 + \frac{n(n-1)(n-2)}{3!} i^3 + \ldots\ldots}_{\text{Since } n<1 \text{, the sum is a negative quantity}}$$

Calling "x" as from the third member of the development,

$$(1 + i)^n = (1 + n \times i) - x \quad \Rightarrow \quad (1 + i)^n < (1 + n \times i)$$

Its reciprocals will verify that:

$$\frac{1}{(1 + i)^n} > \frac{1}{(1 + n \times i)}$$

Therefore,

$$C_{0S} < C_{0C}$$

and consequently,

$$DRS > DRC$$

4ª. When n> 1:

$$(1 + i)^n = (1 + n \times i) + x \quad \Rightarrow \quad (1 + i)^n > (1 + n \times i)$$

Its reciprocal will verify that:

$$\frac{1}{(1+i)^n} < \frac{1}{(1+n\times i)} \quad \Rightarrow \quad C_{0S} > C_{0C}$$

and consequently,

$$DRS < DRC$$

Exercise 5.4. A 15,000 € nominal bill of exchange with an expiration date in 3 months is submitted for discounting at three different entities that offer:

- Entity "X" : a 12% simple commercial discount
- Entity "Y" : a 12% simple mathematical discount.
- Entity “Z”: a 12% compound mathematical discount.

What company offers a greater discount?

Solution.

Entity “X”:

$$C_0 = C_n\,(1-n\times d) = 15{,}000\left(1-\frac{3}{12}0.12\right) = 14{,}550\,€$$

$$D_C = C_n - C_0 = 15{,}000 - 14{,}550 = 450\,€$$

Entity "Y":

$$C_0 = C_n\,(1+n\,i)^{-1} = 15{,}000\left(1+\frac{3}{12}0.12\right)^{-1} = 14{,}563.10\,€$$

$$D_C = C_n - C_0 = 15{,}000 - 14{,}563.1068 = 436.89\,€$$

Entity"Z":

$$C_0 = C_n\,(1+i)^{-n} = 15{,}000\,(1+0.12)^{-3/12} = 14{,}580.98\,€$$

$$D_C = C_n - C_0 = 15{,}000 - 14{,}580.98 = 419.01\,€$$

$$X > Y > Z$$

As we can see, when $0 < n < 1$, DRS (436.89) > DRC (419.01).

Exercise 5.5. The simple commercial discount of 15,000 € nominal bill of exchange amounts to 375€, but if the expiration takes place 30 days before and the type of discount was a 1% greater than that previously used, the discount would be 25 € less. What is the discount rate and the expiration date? (Consider the commercial year of 360 days).

Solution.

$$D_c = C_n\; n\times d \quad \to \quad 15{,}000\frac{n\times d}{360} = 375\ € \tag{7}$$

$$15{,}000\left(\frac{n-30}{360}\right)(d+0.01) = 350 \tag{8}$$

From (7) we deduct

$$n\times d = 9$$

Substituting in (8) we arrive at:

$$n\times d + 0.01n - 30\,d - 0.3 = 8.4$$

$$9 + 0.01\frac{9}{d} - 30\,d - 0.3 = 8.4$$

$$30\,d^2 - 0{,}3\,d - 0.09 = 0$$

$$d = \frac{0.3 \pm \sqrt{0.3^2 + 4 \times 30 \times 0.09}}{2 \times 30} = 0.06$$

$$d = 0.06$$

$$n = 150 \text{ days}$$

5.5. Equivalence of capital units and the discount

To study the capital equivalence we are going to apply the simple and compound commercial discount financial laws.

On the basis of the following capital units and their respective maturities, we are going to calculate firstly, the single capital "C" equivalent to all of them and, secondly, the common due date.

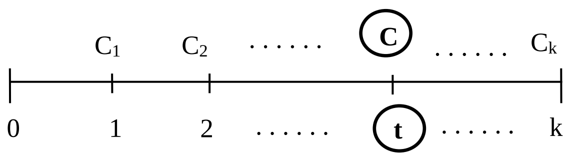


Figure 5.2

$$C(1-d)^t = C_1(1-d) + C_2(1-d)^2 + \cdots\cdots + C_k(1-d)^k$$

$$C(1-d)^t = \sum_{r=1}^{n} C_r(1-d)^r$$

From this equation we can deduct the single capital "C" or the common expiration "t" :

$$C = \frac{\sum_{r=1}^{n} C_r(1-d)^r}{(1-d)^t}$$

The common due date is calculated by taking logarithms in the equation of the single capital.

If, instead of the commercial compound discount, we the simple commercial discount.

$$C(1-td) = C_1(1-d) + C_2(1-2d) + \cdots\cdots + C_n(1-nd)$$

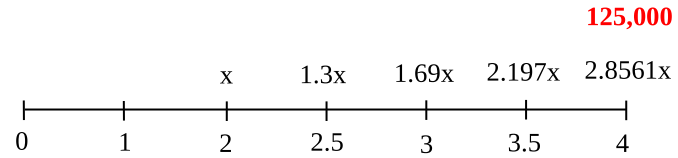
$$C(1-td) = \sum_{r=1}^{n} C_r(1-rd)$$

$$C = \frac{\sum_{r=1}^{n} C_r(1-r \times d)}{(1-t \times d)}; \qquad t = \frac{C - \sum_{r=1}^{n} C_r(1-r \times d)}{Cd}$$

Exercise 5.6. If they have to pay 125,000 € within 4 years and it is agreed with the creditor to substitute this payment with another 5 payments under the following terms: the first in two years and the rest every half-year that follows so that each one is 30% greater the previous.

Calculate the payment to be made in 2 years if the compound discount is 9%.

Solution.



$$125{,}000(1-0.09)^{4} = x(1-0.09)^{2} + 1.3x(1-0.09)^{2,5} + 1.69x(1-0.09)^{3} +$$
$$+ 2.197x(1-0.09)^{3,5} + 2.8561x(1-0.09)^{4}$$
$$x = 12{,}858.15\ €$$

Exercise 5.7. Calculate the average due date of the following commercial drafts, applying an annual compound mathematical discount of 6%. Resolve it also with the commercial discount that results from its equivalence.

Commercial draft list:

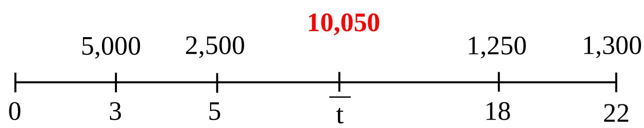
5,000€ in 3 years.

2,500€ in 5 years.

1,250€ in 18 years.

1,300€ in 22 years.

Solution.



A)

$$10{,}050\,(1+0.06)^{-\bar{t}} = 5{,}000\,(1+0.06)^{-3} + 2{,}500\,(1+0.06)^{-5} +$$
$$+ 1{,}250\,(1+0.06)^{-18} + 1{,}300\,(1+0.06)^{-22}$$
$$\bar{t} = 6.5411 \text{ years}$$

B)

$$d = \frac{i}{1+i} = \frac{0.06}{1+0.06} = 0.0566037$$

$$10{,}050\,(1-0.056603)^{\bar{t}} = 5{,}000\,(1-0.056603)^{3} + 2{,}.500\,(1-0.056603)^{5} +$$
$$+ 1{,}250\,(1-0.056603)^{18} + 1{,}300\,(1-0.056603)^{22}$$
$$\bar{t} = 6.5411 \text{ years (6 years, 6 months and 15 days)}$$

5.6. Bank fees and commissions. Default, return and protest of bills of exchange. Re-exchange bill

Bank commissions are amounts of money that the bank collects from its clients for specific services rendered. There are several types of commissions:

A) Commissions for services such as for the use of credit cards and ATMs, for transferring money to other accounts or to other entities, for overdrafts, for account maintenance, for collection of cheques, etc.

B) Commission for specific products such as, for example, for managing pension schemes, for management of investment funds, for buying and selling securities, for administration and custody of securities, etc.

C) Commission for loans and credit facilities. For example, commission for opening credits facilities and loans, expenses for studying loan requests, commission for anticipated cancellation, etc.

The norms that regulate the banking commissions in Spain are the Ministerial Order of December 12, 1989 on interest rates, commission, performance and information norms for clients and adversiting of credit entities, and Circular 8/1990 dated 7 September of the Bank of Spain on transparency in the transactions of credit entities and protection of clients, with its subsequent amendments.

The protest of a bill of exchange is a notarial deed that should be drafted within five days after the expiration date so as to certify that it has not been paid or accepted. Once the protest is made, the drawee has two days to pay the amount before a notary public or to xplain the reasons considered relevant.

In practice, when the bill of exchange is a direct debit, the protest is used by the Clearing House (extra-notariaI protest).

The **Re-exchange bill** is an extrajudicial means for claiming payment of an outstanding bill of exchange that its holder has in his possession. The value of the re-exchange bill comprises the value of the outstanding bill of exchange, the protest charges and the re-exchange charges (article 62 LC). The bank commissions for the return of the bill of exchange, usually a percentage of the nominal value, should also be taken into account.

The Bank of Spain considers that the **commissions for return of checks and bills of exchange are not appropriate** if the financial entity has not informed the clients previously that, in the event of the holder's failure to pay, a commission will be charged for the return.

Exercise 5.8. A person receives from a bank a return notice of 1,500€ nominal value that expired on 24 June. This person withdrew the bill of exchange on 10 July. The balance of the current account on the expiration date of the bill was zero.

Keeping in mind that the protest costs of the protested bills of exchange is 65€; that the bank charged 20% interest for the overdraft (consider a commercial year of 360 days); that the return commission is 1.5% of the nominal value; that the protest commission amounts to 40€ and the postal charges are 4€, calculate the amount that this person would have paid on that date.

Solution.

Nominal	1,500.00€
Return Commission (1.5% s/ 1,500)	22.50€
Protest commission	40.00€
Protest charges	65.00€
Postal charges	4,00v
Sum	1,631.50€
Overdraft interest:	
$I_T = 1{,}631.50\dfrac{0.2 \times 16}{360} =$	14.50€
Total	1,646.00€

Exercise 5.9. A draft of 8,000€ with expiration on June 20 is returned due to non-payment. The return commission amounts to 1.5% of the nominal and the mailing expenses amount to 3 €.

The act of protest presented accordingly has generated the following expenses which the financial entity has debited in the client's account: notary public, 75€ and protest commission, 30€.

In order to collect the amount owed, the drawer resorts to a re-exchange bill. Assume that the settlement date was June 25. Calculate the nominal value of the new bill of exchange, if its expiration date is August 25 and the bank charges 32 € for the new draft. The discount applied is 9% and the commission for collection is 1‰. (Consider a commercial year of 360 days).

Solution.

A) Calculating the amount owed to the bank.

Draft:

Nominal	8,000€
Return commission (1.5% s/ 8,000)	120€
Postal charges	3€
Total	8,123€

Protest:

Notary public	75€
Protest commission	30€
Total	105€

Total = 8,123 + 105 = 8,228 €

B) Calculation of the amount of the re-exchange bill.

The effective sum of the bill of exchange = 8,228 + 32 = 8,260€

$$8{,}260 = C_n - C_n nd - C_n G$$

$$8{,}260 = C_n - C_n \frac{0.09 \times 60}{360} - C_n \frac{1}{1{,}000}$$

$$C_n = 8{,}394.30 \text{ €}$$

Exercise 5.10. On 20 January 20 a company deposits the following draft remittance in a financial entity:

Nº Draft	Nominal	Due date
1	1,000€	15/03
2	1,500€	18/04
3	800€	20/05
4	3,000€	17/02

The entity applies a commercial discount of 7% for maturities of up to 90 days and 8% for greater maturities. The transaction fee is 2‰ of the nominal values and the expenses total 2€ per draft (consider a commercial year of 360 days). Calculate:

A) The net amount credited and discount for the remittance.

B) E.A.R. of the transaction.

Solution.

A)

Nº Draft	Nom.	Days	Todd	Dis.	Com.	Charge s.	Liquid
1	1,000€	55	7%	10.69	2.00	2.00	985.31€
2	1,500€	88	7%	25.66	3.00	2.00	1,469.34€
3	800€	120	8%	21.33	1.60	2.00	775.07€
4	3,000€	27	7%	15.75	6.00	2.00	2.976,25v
Total	6,300€			73.43	12.60	8.00	6,205.97€

B)

$$\text{Vto.Medio} = \frac{1{,}000 \times 55 + 1{,}500 \times 88 + 800 \times 120 + 3{,}000 \times 27}{6{,}300} = 57.77 \text{ days}$$

$$6{,}205.97 = 6{,}300(1+i)^{-57.77/365}$$

$$\text{E.A.R.} = i = 0.09965$$

Exercise 5.11. A business that has a commercial draft of 25,000€ drawn on 20 March with an expiration date on 16 July decides to present it for discount 28 March. The financial entity applies a commercial discount of 6% for transactions of up to 90 days and 8% for transactions exceeding this term. The negotiation commission is f 1.5 per thousand and there is an additional set charge of 25€ for every draft negotiated.

On July 16 a non-payment is produced and the bank debits the following charges to its client: return commission, 1%; fixed charges, 10€. Likewise, the protest of the draft is produced generating the following charges: Notary public fees, 300€; protest commission, 1.3% of the nominal value.

The entity and its client reach an agreement on 24 July to circulate a re-exchange bill with expiration date 29 September. Nevertheless, this new draft is negotiated in another financial entity at a commercial discount of 6 % for transactions of 60 days and 25% more for previous ones for longer term transactions, as well as a transaction fee of 0.5%. It is also known that the new draft amount to 40€. (Commercial year of 360 days).

Calculate:

A) The discount and net amount received from the first discount.

B) Value of there-exchange bill.

Solution. A) 600; 24,337.50; B) 26,482.23.

Exercise 5.12. A business has in its portfolio a trade bill of 27,000€ date of payment order 15 October and maturity date 20 January. Nevertheless, confronted with a liquidity problem it decides to present it for discount on 28 October. The financial entity applies a commercial discount of 7%, with a commission of 2‰ (minimum 25 €) and fixed charges of 20 €. Assuming that entity credits the remittance on 31 October, calculate the amount of the discount and the net amount received. (Consider a commercial year of 360 days).

Solution. 420 (Discount includes the commission and charges: 494 €); 26,506.

Exercise 5.13. A bill with a nominal value of €6,250 whose payment order date is 02/12/14, and which matures on 01/04/15, is submitted to the bank on 07/12/14 to be discounted at a simple commercial paper discount rate of 16%, with the bank applying to this transaction a collection fee of 4% of the nominal amount of the bill submitted. On 08/12/14, the bank proceeds to deliver the cash from the discounted bill.

Upon maturity of the discounted bill, the bank submits it for collection as it is unpaid and, consequently, 15 days after the failure to pay is registered, it proceeds to create a 30-day redraft or replacement against the company, applying to the bill a simple commercial paper discount rate of 17%, bill creation expenses of €350 and a collection fee of 5% on the nominal value of the redraft. Calculate:

a) The cash amount paid to the company for the discounted bill.

b) The nominal value of the redraft.

Solution: a) €5,911.11; b) €7,052.53

Exercise 5.14. A person that has 150,000€ decides to enter into a series of financial transactions starting on January 1, of year x1.

1º) Makes a fixed-term deposit of 10,000€ at 8.5% per annum.

2º) Acquires an apartment with the following payment terms and conditions:

- Deferral interest: 6%
- Draft nº 1: 20,000€, due 01/01/x1.
- Draft nº 2: 40,000€, due 30/06/x2.
- Draft nº 3: 30,000€, due 01/01/x6.

3º) Invests 25,000€ in securities that yield an equivalent monthly amount at 0.75% interest rate.

4º) The remainder of the money is loaned to a friend under the following terms and conditions:

- Annual interest payments.
- Lump-sum reimbursement on maturity.
- Anticipated Interest of 7.5%.

- Repayment terms: 5 years.

The interest payments, when collected, are automatically deposited in the savings account taht pays 2% interest in arrears.

Calculate the loan amount and the total disposable sum at the end of the fifth year.

Solution.

a) Calculate the amount of the loan:

Initial disposable capital	150,000.00€
Fixed-term deposit	(10.000,00) €
Bills of exchange for the purchase of the apartment:	
$V(0)=20{,}000+40{,}000(1+0.06)^{-1.5}+30{,}000(1+0.06)^{-5}=$	
	(79.070,04) €
Securities portfolio	(25.000,00) €
Sum	35,929.96 €

Nominal value of the loan:

$$x(1-i_a)=35{,}929.96 \quad \Rightarrow \quad x(1-0.075)=35{,}929.96$$

$$x = 38{,}843.20 \text{ €}$$

Interests to be received per annum will be:

$$I_T=38{,}843.20\times 0.075=2{,}913.24 \text{ €}$$

Total disposable of the loan at the end of the fifth year:

$$V(5)=2{,}913.24\left[(1+0.02)^4+(1+0.02)^3+(1+0.02)^2+(1+0.02)\right]+$$
$$+38{,}843.20=51{,}090.57 \text{ €}$$

b) Total disposable sum at the end of the fifth year.

Loan	51,090.57 €
Fixed-term deposit: $V(5)=10{,}000(1+0.085)^5=$	15,036.56 €
Portfolio: $V(5)=25{,}000(1+0.0075)^{60}=$	39,142.02 €
Total	105,269.17 €

6. Financial Income. General Valuation of Constant Annual Income

6.1. Concept of income

We understand as **income** every succession of capital that is part of the same financial transaction, disposable or payable on demand, in equidistant moments of time. The **source of income** (that which generates the income) can be a capital unit, rental income from a farm or town property, the payments made to settle a debt (loan), the amounts that make up the capital, the payments to be made in the purchase of a home, etc.

We call $C_1, C_2, \ldots\ldots\ldots, C_n$ the different capital units that generate the source of income in the periods $t_1, t_2, \ldots\ldots\ldots, t_n$, respectively.

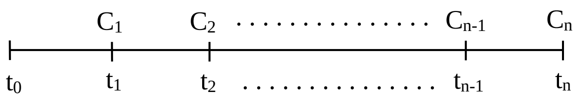


Figure 6.1

We will call the economic horizon the time defined by the interval $[t_0, t_n]$, that is, the interval in which the income is generated, and each period will be called <u>the income period</u> $[t_n, t_{n-1}]$. We will call the number of periods between t_0 y t_n the <u>income duration.</u>

t_0 will be the <u>income source</u>

t_n will be the <u>final income.</u>

We will call each of the capital units that make up the income the <u>income term</u>. When the income term is annual, we will denominate it **annuity**. When the terms are valued in t_0 we will have the <u>current value</u> or initial value and when they are valued in t_n we will have the <u>final value</u> or capital value.

We can define the current value of a capital unit as the value that a sum of money would today is expected to have in the future, knowing the interest rate at which we could invest if we had it at our disposal now.

Likewise, we can define the final value of a capital unit as the future value of a sum of money that is invested today at a specified interest rate.

6.2. General income classification

Income can be classified according to a series of criteria:

1º) **According to the nature of the terms that compose it:**

- Constant Income: income the terms of which are equal.

- Variable Income: when its terms vary according to a specific criterion. Within these we can find: variable income in arithmetic progression and variable income in geometric progression.

2º) **According to the periodicity of its maturity:**

- Annual Income: all its terms have annual periodicity.

- Split income or of periodicity of less than a year.

- Income with a periodicity of more than one year.

3º) **According to the moment of the valuation**:

- Immediate income: when the income is valued between the first and the last term defined by the economic horizon.

- Deferred income: when the income is valued before the expiration of the first term. The period between this moment and the expiration of the first term is called "**deferment**". For example, calculate today the value of an income that will be collected within 5 years.

- Anticipated income: when the income is valued after the last term, that is, after t_n. The time that separates both terms is called "**anticipation**". For example, calculate the value within 10 years of income that will be collected over the next 6 years.

4º) **According to availability of the terms**:

- Pre-payable Income: that in which the first term is cashed in at the beginning of the period. If we consider the period corresponding to a calendar year, income would be cashed in the first day of January.

- Post-payable income: that in which the first term is cashed in at the end of the period. If we consider the period corresponding to a calendar year, income would be cashed in on 31 December.

5º) **According to the number of terms that compose the income**:

- Tempory income: income with a finite number of periods or terms.

- Perpetual income: income with an infinite number of periods or terms, or the number of these terms is not known beforehand.

6º) **According to the valuation method**:

- Simple income: valued in simple capitalisation.

- Compound income: valued in compound capitalisation.

6.3. Financial value of income. Post-payability and pre-payability

6.3.1. Temporary, variable, immediate and post-payable income

We are going to calculate the current and final value of this income, as well as the relation between them.

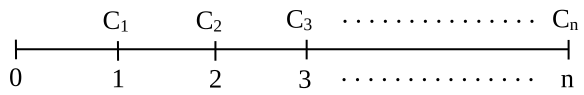


Figure 6.2

$$V(0)=C_1(1+i)^{-1}+C_2(1+i)^{-2}+C_3(1+i)^{-3}+\cdots\cdots+C_n(1+i)^{-n}=\sum_{r=1}^{n}C_r(1+i)^{-r}$$

$$V(n)=C_n+C_{n-1}(1+i)+C_{n-2}(1+i)^2+\cdots\cdots+C_1(1+i)^{n-1}=\sum_{r=n}^{1}C_r(1+i)^{n-r}$$

In compound capitalisation it is verified that:

$$V(n)=V(0)\,(1+i)^n$$

$$V(0)=V(n)\,(1+i)^{-n}$$

6.3.2. Temporary, variable, immediate and pre-payable

We calculate the current and final value of this income, as well as the relation between them.

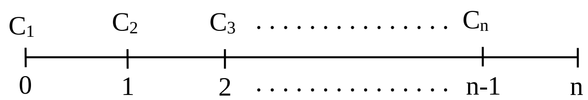


Figure 6.3

$$\ddot{V}(0)=C_1+C_2(1+i)^{-1}+C_3(1+i)^{-2}+\cdots\cdots+C_n(1+i)^{-(n-1)}=\sum_{r=1}^{n}C_r(1+i)^{-(r-1)}$$

$$\ddot{V}(n)=C_n(1+i)+C_{n-1}(1+i)^2+C_{n-2}(1+i)^3+\cdots\cdots+C_1(1+i)^n=\sum_{r=n}^{1}C_r(1+i)^{n-r+1}$$

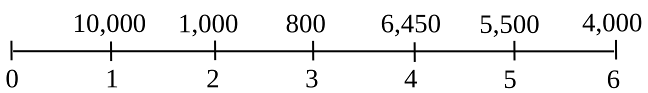
In compound capitalisation it is verified that:

$$\ddot{V}(0)=V(0)\,(1+i)$$

$$\ddot{V}(n)=\ddot{V}(0)\,(1+i)^n$$

$$\ddot{V}(n)=V(0)\,(1+i)\,(1+i)^n$$

Exercise 6.1. Calculate the current and final value of an income at 10% annually, knowing that it includes the following six terms:



Solution.

$$V(0) = 10{,}000(1+0.1)^{-1} + 1{,}000(1+0.1)^{-2} + 800(1+0.1)^{-3} + 6{,}450(1+0.1)^{-4} +$$
$$+ 5{,}500(1+0.1)^{-5} + 4{,}000(1+0.1)^{-6} = 20{,}596.807\text{ €}$$

$$V(6) = 10{,}000(1+0.1)^{5} + 1{,}000(1+0.1)^{4} + 800(1+0.1)^{3} + 6{,}450(1+0.1)^{2} +$$
$$+ 5{,}500(1+0.1) + 4{,}000 = 36{,}488.5\text{ €}$$

$$V(6) = V(0)(1+0.1)^{6} = 20{,}596.807(1+0.1)^{6} = 36{,}488.5\text{ €}$$

Exercise 6.2. It is known that a person is going to earn the following income over the next five years:

Year 1: 2,000€
Year 2: 1,500€
Year 3: 3,000€
Year 4: 6,000€
Year 5: 4,600€

Calculate the current and final value of income at an annual compound rate of interest of 6% in the event that income:

A) Is paid at the end of each period, that is, it is post-payable.

B) Is paid at the start of each period, that is, it is pre-payable.

Solution.

A)

$$V(0) = 2{,}000(1+0.06)^{-1} + 1{,}500(1+0.06)^{-2} + 3{,}000(1+0.06)^{-3} + 6{,}000(1+0.06)^{-4} +$$
$$+ 4{,}600(1+0.06)^{-5} = 13{,}930.59\text{ €}$$

$$V(5) = 2{,}000(1+0.06)^{4} + 1{,}500(1+0.06)^{3} + 3{,}000(1+0.06)^{2} +$$
$$+ 6{,}000(1+0.06) + 4{,}600 = 18{,}642.27\text{ €}$$

Also,

$$V(5) = V(0)(1+0.06)^{5} = 13{,}930.59(1+0.06)^{5} = 18{,}642.27\text{ €}$$

$$V(0) = V(5)(1+0.06)^{-5} = 18{,}642.27(1+0.06)^{-5} = 13{,}930.59\text{ €}$$

B)

$$\ddot{V}(0) = 2{,}000 + 1{,}500(1+0.06)^{-1} + 3{,}000(1+0.06)^{-2} + 6{,}000(1+0.06)^{-3} +$$
$$+ 4{,}600(1+0.06)^{-4} = 14{,}766.43\text{ €}$$

$$\ddot{V}(5) = 2{,}000(1+0.06)^{5} + 1{,}500(1+0.06)^{4} + 3{,}000(1+0.06)^{3} +$$
$$+ 6{,}000(1+0.06)^{2} + 4{,}600(1+0.06) = 19{,}760.81\text{ €}$$

Also,

$$\ddot{V}(5) = \ddot{V}(0)(1+0.06)^{5} = 14{,}766.43(1+0.06)^{5} = 19{,}760.81\text{ €}$$

$$\ddot{V}(0) = \ddot{V}(5)(1+0.06)^{-5} = 19{,}760.81(1+0.06)^{-5} = 14{,}766.43\text{ €}$$

6.4. Calculation of the current and the final value of immediate income

6.4.1. Current value of a constant, immediate and post-payable income.

Since it is a constant income, its diagram is as follows:

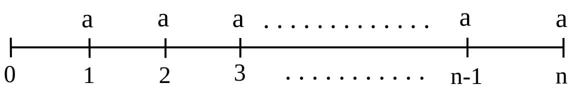


Figure 6.4

Based on the principle of financial equivalence, we calculate the value of all the terms at moment "0", multiplying by the corresponding discount factors.

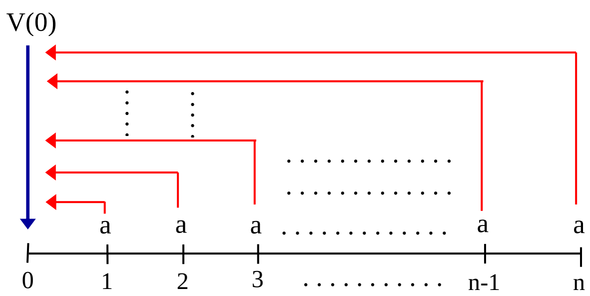


Figure 6.5

$$V(0) = a(1+i)^{-1} + a(1+i)^{-2} + \dots\dots\dots + a(1+i)^{-(n-1)} + a(1+i)^{-n} =$$
$$= a\left[(1+i)^{-1} + (1+i)^{-2} + \dots\dots\dots + (1+i)^{-(n-1)} + (1+i)^{-n}\right]$$

The expression of the square bracket is a decreasing geometric progression of reason $(1+i)^{-1}$. The general term of the sum is $S = \dfrac{x_1 - x_n r}{1-r}$.

$$V(0) = a\left[\frac{(1+i)^{-1} - (1+i)^{-n}(1+i)^{-1}}{1-(1+i)^{-1}}\right]$$

We take the common factor $(1+i)^{-1}$ in the numerator and denominator:

$$V(0) = a\frac{(1+i)^{-1}}{(1+i)^{-1}}\left[\frac{1-(1+i)^{-n}}{(1+i)-1}\right] = a\left[\frac{1-(1+i)^{-n}}{i}\right]$$

If the expression $\dfrac{1-(1+i)^{-n}}{i}$ we designate it as $a_{\overline{n}|i}$

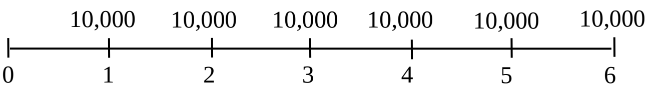
$$V(0) = a\, a_{\overline{n}|i}$$

If it is a unit income:

$$V(0) = a_{\overline{n}|i}$$

Exercise 6.3. Calculate the current value of the 6 constant payments of 10,000€ each, at a compound interest rate of 8%, that a company has to pay for a machine,

Solution.



$$V(0) = a\, a_{\overline{n|}\,i} = 10{,}000\, a_{\overline{6|}\,0.08} = 10{,}000 \times 4.622879664 = 46{,}228.79\ €$$

6.4.2. Current value of an immediate, constant and pre-payable income

The pre-payable income is that in which the effectiveness of the terms occurs at the start of each income period or interval.

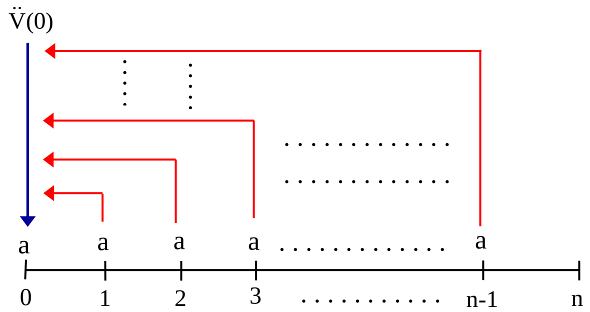


Figure 6.6

$$\ddot{V}(0) = a + a(1+i)^{-1} + a(1+i)^{-2} + \dots\dots + a(1+i)^{-(n-1)} =$$

$$= a\left[\underbrace{1 + (1+i)^{-1} + (1+i)^{-2} + \dots\dots + (1+i)^{-(n-1)}}_{\text{Prog.Geométrica decreciente de razón } (1+i)^{-1}}\right]$$

$$\ddot{V}(0) = a\left[\frac{1-(1+i)^{-(n-1)}(1+i)^{-1}}{1-(1+i)^{-1}}\right] = a\left[\frac{1-(1+i)^{-n}}{1-(1+i)^{-1}}\right]$$

We remove common $(1+i)^{-1}$ factor in the denominator:

$$\ddot{V}(0) = a\left[\frac{1-(1+i)^{-n}}{(1+i)^{-1}[(1+i)-1]}\right] = a(1+i)\left[\frac{1-(1+i)^{-n}}{i}\right] = a(1+i)\, a_{\overline{n|}\,i}$$

Calling $\ddot{a}_{\overline{n|}\,i}$ the current value of the pre-payable income,

$$\ddot{V}(0) = a\, \ddot{a}_{\overline{n|}\,i}$$

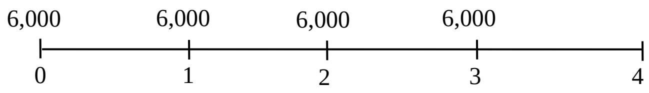
If unit income:

$$\ddot{a}_{\overline{n|}\,i} = (1+i)\, a_{\overline{n|}\,i}$$

The value of $\ddot{a}_{\overline{n|}\,i}$ is not tabulated and, therefore, in order t be able to calculate it, we must find the value of $a_{\overline{n|}\,i}$ and multiply it by (1+i).

Exercise 6.4. Calculate the price in cash of a machine it we know that 6.000€ has to be paid at the beginning of the next four years. These payments include interest at a 9% compound interest rate.

Solution.



$$\ddot{V}(0) = a\,\ddot{a}_{\overline{n}|\,i} = a\,(1+i)\,a_{\overline{n}|\,i} = 6{,}000(1+0.09)\,a_{\overline{4}|0.09} = 21{,}187.768\ €$$

6.4.3. Final value of constant, immediate and post-payable income

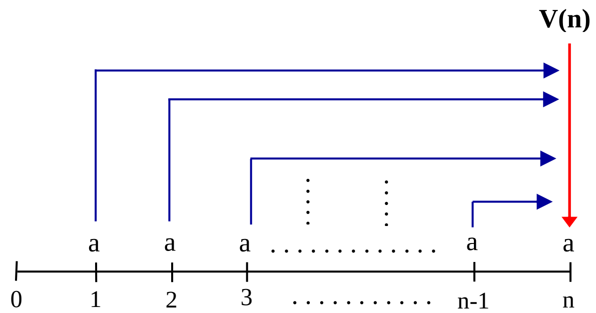


Figure 6.7

We multiply the income terms by the corresponding capitalisation factors, starting with the term whose maturity is produced in "n" to finish with the term whose maturity is produced at the end of the first period,

$$V(n) = a + a(1+i) + a(1+i)^2 + \dots\dots\dots + a(1+i)^{n-1} =$$
$$= a\left[1 + (1+i) + (1+i)^2 + \dots\dots\dots + (1+i)^{n-1}\right]$$

The expression of the square bracket is the growing geometric progression of reason (1+i), whose general term is $S = \dfrac{x_n r - x_1}{r - 1}$. We substitute,

$$V(n) = a\left[\frac{(1+i)^{n-1}(1+i) - 1}{(1+i) - 1}\right] = a\left[\frac{(1+i)^n - 1}{i}\right]$$

Calling $s_{\overline{n}|i}$ la the expression $\dfrac{(1+i)^n - 1}{i}$,

$$V(n) = a\,s_{\overline{n}|i}$$

If it is unit income

$$V(n) = s_{\overline{n}|i}$$

Exercise 6.5. A person deposits in the bank 10.000 euros annually 20 years before retirement. What amount will be received upon retirement if the bank pays 4.5% annual compound interest?

Solution.

$$V(n) = a\, s_{\overline{n}|i} = 10.000\, s_{\overline{20}|0,045} = 10,000 \times 31.37142277 = 313,714.22$$

6.4.4. Final value of constant, immediate and pre-payable income

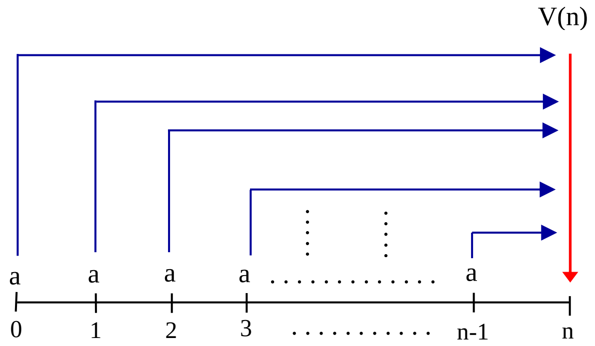


Figure 6.8

$$\ddot{V}(n) = a(1+i) + a(1+i)^2 + \dots\dots.. + a(1+i)^n =$$

$$= a\left[(1+i) + (1+i)^2 + \dots\dots.. + (1+i)^n\right] = a\left[\frac{(1+i)^n(1+i) - (1+i)}{(1+i) - 1}\right] =$$

$$= a(1+i)\left[\frac{(1+i)^n - 1}{i}\right] = a(1+i)\, s_{\overline{n}|i}$$

$$\ddot{V}(n) = a\, \ddot{s}_{\overline{n}|i} = a\,(1+i)\, s_{\overline{n}|i}$$

If it is unit income

$$\ddot{s}_{\overline{n}|i} = (1+i)\, s_{\overline{n}|i}$$

Exercise 6.6. A 40-year-old person deposits €6,000 in a financial institution each year in order to obtain a certain amount of capital upon their retirement. If the financial institution pays them 3% interest per annum: Calculate how much capital they will have at retirement age (67 years) in the following cases:

a) The amounts are deposited in the financial institution at the end of each year.

b) The amounts are deposited in the financial institution at the beginning of each year.

Solution.

a)

$$V(65) = a\, s_{\overline{n}|i} = a s_{\overline{n}|i} = 6,000 s_{\overline{27}|0.03} = 244,257.80$$

b)

$$\ddot{V}(65) = a\, \ddot{s}_{\overline{n}|i} = a\,(1+i) s_{\overline{n}|i} = 6,000(1+0.03) s_{\overline{27}|0.03} = 251,585.54\ €$$

6.4.5. Relation between current value and final value

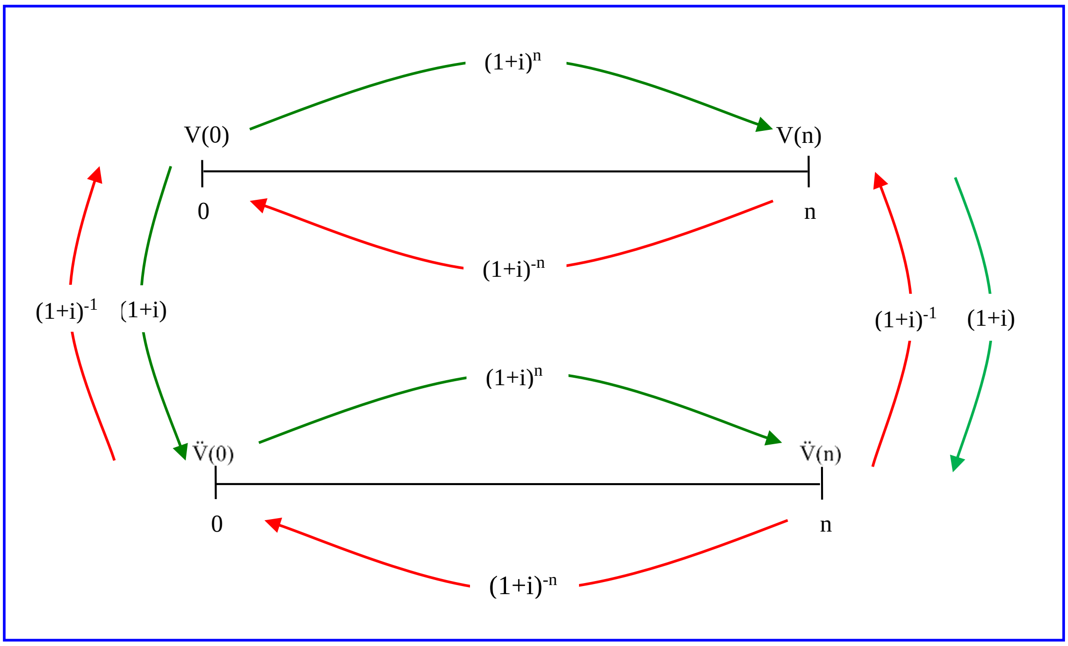


Figure 6.9

With respect to post-payable income

$$V(n) = V(0)(1+i)^n \quad \Rightarrow \quad s_{\overline{n}|i} = a_{\overline{n}|i}(1+i)^n$$

$$V(0) = V(n)(1+i)^{-n} \quad \Rightarrow \quad a_{\overline{n}|i} = s_{\overline{n}|i}(1+i)^{-n}$$

If we multiply $a_{\overline{n}|i}$ by $(1+i)^n$

$$a_{\overline{n}|i}(1+i)^n = \frac{1-(1+i)^{-n}}{i}(1+i)^n = \frac{(1+i)^n - (1+i)^{-n}(1+i)^n}{i} =$$

$$= \frac{(1+i)^n - 1}{i} = s_{\overline{n}|i}$$

With respect to the pre-payable

$$\ddot{V}(n) = \ddot{V}(0)(1+i)^n \quad \Rightarrow \quad \ddot{s}_{\overline{n}|i} = \ddot{a}_{\overline{n}|i}(1+i)^n$$

$$\ddot{V}(0) = \ddot{V}(n)(1+i)^{-n} \quad \Rightarrow \quad \ddot{a}_{\overline{n}|i} = \ddot{s}_{\overline{n}|i}(1+i)^{-n}$$

If we multiply the current value of the pre-payable income by $(1+i)^n$

$$\ddot{a}_{\overline{n}|i}(1+i)^n = \left[(1+i)\frac{1-(1+i)^{-n}}{i}\right](1+i)^n = \frac{(1+i)^{n+1} - (1+i)}{i} =$$

$$= (1+i)\left[\frac{(1+i)^n - 1}{i}\right] = (1+i)\, s_{\overline{n}|i} = \ddot{s}_{\overline{n}|i}$$

Exercise 6.7. Calculate the current and final value of the 1,500€ annual income which is to be paid during 8 years, in the cases where such income is: a) post-payable and b) pre-payable.

Solution. a) V(0) = 9,314.69; V(n) = 14,846.20; b) V(0) = 9.873.57; V(n) = 15,736.97

6.5. Deferred income

Income is deferred when its valuation is performed at a date prior to the date of availability or payability of the first term. The period that separates this moment and the maturity of the first term is called deferment.

The diagram of the financial transaction is as follows:

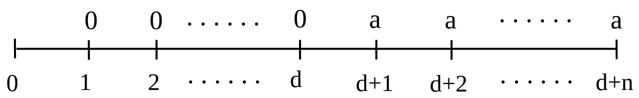


Figure 6.10

6.5.1. Current value of constant, post-payable and deferred income

We calculate the current value of income corresponding to the periods when income is generated. Once income is updated to "d", we multiply it by its corresponding update factor for the periods between "0" and "d", so that:

$$V(0) = a\, a_{\overline{n}|i}(1+i)^{-d}$$

If it is a unit income:

$$d/a_{\overline{n}|i} = a_{\overline{n}|i}(1+i)^{-d}$$

The total value can also be calculated in another way:

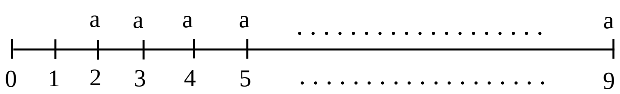
$$d/a_{\overline{n}|i} = a_{\overline{n}|i}(1+i)^{-d} = \left[\frac{1-(1+i)^{-n}}{i}\right](1+i)^{-d} = \frac{(1+i)^{-d} - (1+i)^{-(n+d)}}{i} =$$

We add and subtract 1 to the numerator,

$$= \frac{(1+i)^{-d} - (1+i)^{-(n+d)} + 1 - 1}{i} = \frac{1-(1+i)^{-(n+d)}}{i} - \frac{1-(1+i)^{-d}}{i} = a_{\overline{n+d}|i} - a_{\overline{d}|i}$$

We will consider that income is deferred post-payable when there is more than one period from the origin until the first income term (greater than the frequency of income) and the deferment period will be considered as the time exceeding the frequency of income.

For example, income of 8 terms which will begin to be paid at the end of the second year (or the beginning of the third)



In this case, income has an annual frequency and two periods have elapsed from the origin until income starts to be paid; however, the deferment will be for one period since time exceeds the income frequency.

$$V(0) = a\, a_{\overline{8}|i}(1+i)^{-1}$$

6.5.2. Current value of constant, post-payable and deferred income

The calculation of the final value it is not affected by the income deferment,

$$V(n) = d/S_{\overline{n}|i} = a\, S_{\overline{n}|i}$$

If it is a unit income:

$$d/S_{\overline{n}|i} = S_{\overline{n}|i}$$

We can check it by capitalising the current value up to the period "$d+n$",

$$d/S_{\overline{n}|i} = d/a_{\overline{n}|i}\,(1+i)^{d+n} = a_{\overline{n}|i}\,(1+i)^{-d}(1+i)^{d+n} = a_{\overline{n}|i}\,(1+i)^{n} = S_{\overline{n}|i}$$

6.5.3. Current value of constant, pre-payable and deferred income

The diagram of the financial transaction is as follows:

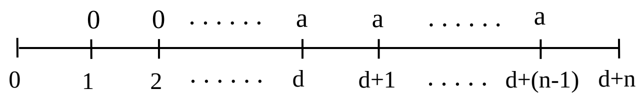


Figure 6.11

$$\ddot{V}(0) = a\, \ddot{a}_{\overline{n}|i}(1+i)^{-d} = a\, a_{\overline{n}|i}(1+i)(1+i)^{-d}$$

If it is a unit income:

$$d/\ddot{a}_{\overline{n}|i} = \ddot{a}_{\overline{n}|i}(1+i)^{-d}$$

6.5.4. Final value of constant, post-payable and deferred income

$$\ddot{V}(n) = a\, d/\ddot{S}_{\overline{n}|i} = a\, \ddot{a}_{\overline{n}|i}\,(1+i)^{-d}(1+i)^{d+n} =$$

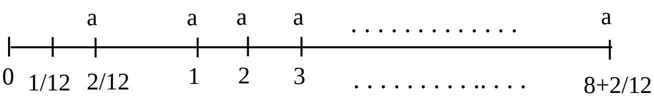
$$= a\, a_{\overline{n}|i}\,(1+i)(1+i)^{-d}(1+i)^{d+n} = a\, a_{\overline{n}|i}\,(1+i)(1+i)^{n} = a\, \ddot{S}_{\overline{n}|i}$$

If it is a unit income:

$$d/\ddot{S}_{\overline{n}|i} = \ddot{S}_{\overline{n}|i}$$

We will consider that income is deferred pre-payable when there is less than one period corresponding to the income frequency from the origin until the first income term elapses, and the deferment period will be considered the period from the origin ntil the the time that elapses from the origin until the first term is effective.

For example, income with 8 annual terms which will start to be paid in two months.



In this case the frequency of income is annual and two months have elapsed since the origin until income starts to paid (less than a year which is the income frequency), therefore the income pre-payable in 2/12 and deferred 2 months.

$$V(0) = a\, a_{\overline{8|}i}(1+i)(1+i)^{-2/12}$$

6.6. Anticipated income.

Income is anticipated when its valuation is performed at a date subsequent to the date of availability or payability on demand of the last term of its accrual. The period between the last term and the moment of valuation is called anticipation.

6.6.1. Current value of constant, post-payable and anticipated income

The income diagram is:

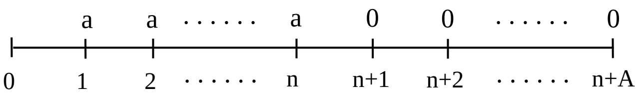


Figure 6.12

Anticipation of the income does not affect the calculation of the current value,

$$V(0) = a\, a_{\overline{n|}i}$$

If it is unit income

$$A/a_{\overline{n|}i} = a_{\overline{n|}i}$$

6.6.2. Final value of constant, post-payable and anticipated income

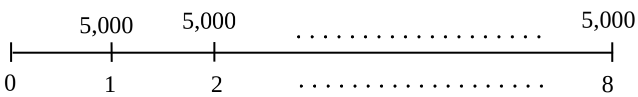
$$V(n) = a\, s_{\overline{n|}i}\,(1+i)^{A}$$

If it is unit income

$$A/s_{\overline{n|}i} = s_{\overline{n|}i}\,(1+i)^{A}$$

Exercise 6.8. Calculate the final and current value of an 8 term income of 5,000 € each if it is valued at 6%, in the cases where income is: a) post-payable and b) pre-payable.

Solution.



A)

$$V(0) = a\, a_{\overline{n}|i} = 5{,}000\, a_{\overline{8}|0.06} = 31{,}048.97\ €$$

$$V(n) = a\, s_{\overline{n}|i} = 5{,}000\, s_{\overline{8}|0.06} = 49{,}487.34\ €$$

Also:

$$V(n) = V(0)(1+i)^n = 31{,}048.97\,(1+0.06)^8 = 49{,}487.34\ €$$

B)

$$\ddot{V}(0) = a\, \ddot{a}_{\overline{n}|i} = 5{,}000\, \ddot{a}_{\overline{8}|0.06} = 5{,}000\, a_{\overline{8}|0.06}(1+0.06) = 32{,}911.9\ €$$

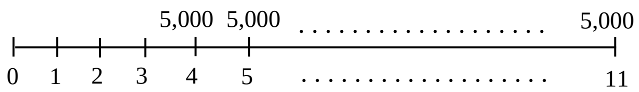
$$\ddot{V}(n) = a\, \ddot{s}_{\overline{8}|0.06} = 5{,}000\, \ddot{s}_{\overline{8}|0.06} = 5{,}000\, s_{\overline{8}|0.06}(1+0.06) = 52{,}456.58\ €$$

Also:

$$\ddot{V}(n) = \ddot{V}(0)(1+i)^n = V(0)(1+i)(1+i)^n = 31{,}048.97\,(1+0.06)(1+0.06)^8 = 52{,}456.58\ €$$

Exercise 6.9. Calculate the current and final value of 5000 € income with a duration of 8 years if we know that it will begin to be paid at the start of the 5th year and it is valued at 6%.

Solution.



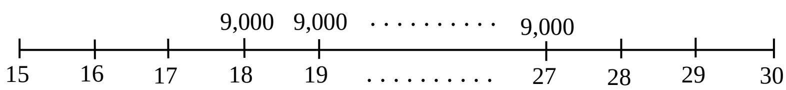
$$V(0) = 5{,}000\, a_{\overline{8}|0.06}(1+0.06)^{-3} = 26{,}069.313\ €$$

$$V(n) = 5{,}000\, s_{\overline{8}|0.06} = 49{,}487.34\ €$$

$$V(n) = V(0)(1+i)^n = 26{,}069.313\,(1+0.06)^{11} = 49{,}487.34\ €$$

Exercise 6.10. A 15 year old inherits an income of 9,000€ per annum during 10 years. The individual, however, will not be eligible for this income until he turns 18 and will not be able to start collecting it until he turns 30. Calculate, at a valuation rate of 5%: A) the current value of this income, B) the value of the first income payment, and C) the value at the first withdrawal.

Solution.



A)

$$V(15) = 9{,}000\, a_{\overline{10}|0.05}(1+0.05)^{-2} = 63{,}034.57\ €$$

B)

$$V(18) = 9{,}000\ \ddot{a}_{\overline{10}|0.05} = 72{,}970.39\ €$$

Also,

$$V(18) = V(15)(1+0.05)^3 = 72{,}970.39\ €$$

C)

$$V(30) = 9{,}000\ \ddot{s}_{\overline{10}|0.05}(1+0.05)^3 = 131{,}044.34\ €$$

Also,

$$V(30) = V(15)(1+0.05)^{15} = 63{,}034.57\ (1+0.05)^{15} = 131{,}044.34\ €$$

Exercise 6.11. Calculate the cash value of a property taking into account that a total of 12 payments of €800 are to be made, at an interest rate of 5%, if the first payment is made:

a) At the end of the 3rd year after the property is purchased.

b) At the beginning of the 5th year.

c) 7 months after purchase.

d) 14 days after purchase.

Solution: a) €6,431.38; b) €6,125.12; c) €7,236.22; d) €7,431.21

6.7. Perpetual income

Perpetual income refers to income with an unlimited number of periods or more than 100 annual terms, or which are not known beforehand. For its valuation we use the mathematical limit principle. A clear example of perpetual income is a pension plan life annuity after retirement.

The income diagram is:

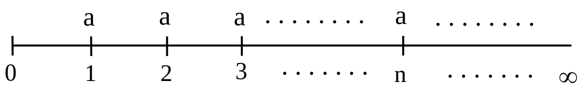


Figure 6.13

6.7.1. Current value of constant, immediate, perpetual and post-payable income

We update the different terms of the income until "0",

$$V(0) = a(1+i)^{-1} + a(1+i)^{-2} + \dots\dots\dots + a(1+i)^{-n} + \dots\dots\dots =$$
$$= a\left[(1+i)^{-1} + (1+i)^{-2} + \dots\dots\dots + (1+i)^{-n} + \dots\dots\dots\right]$$

The expression of the square bracket is the sum of a decreasing geometrical progression and unlimited geometrical progression of reason $(1+i)^{-1}$ and whose general term is: $S = \frac{x_1}{1-r}$,

$$V(0) = a\left[\frac{(1+i)^{-1}}{1-(1+i)^{-1}}\right] = a\frac{(1+i)^{-1}}{(1+i)^{-1}}\left[\frac{1}{(1+i)-1}\right] = a\left[\frac{1}{i}\right] = a\ a_{\overline{\infty}|i}$$

If it is unit income

$$a_{\overline{\infty}|i} = \frac{1}{i}$$

Similarly,

$$V(0) = \lim_{n\to\infty} a_{\overline{n}|i} = \lim_{n\to\infty} \frac{1-(1+i)^{-n}}{i} = \frac{1}{i}$$

6.7.2. Current value of constant, immediate, perpetual and pre-payable income

$$\ddot{V}(0) = a + a(1+i)^{-1} + a(1+i)^{-2} + \dots\dots\dots + a(1+i)^{-n} + \dots\dots\dots =$$
$$= a\left[1 + (1+i)^{-1} + (1+i)^{-2} + \dots\dots\dots + (1+i)^{-n} + \dots\dots\dots\right] =$$
$$= a\left[\frac{1}{1-(1+i)^{-1}}\right] = a\left[\frac{1}{(1+i)^{-1}\left[(1+i)-1\right]}\right] = a(1+i)\frac{1}{i} = a\,\ddot{a}_{\overline{\infty}|i}$$

If it is unit income

$$\ddot{a}_{\overline{\infty}|i} = a_{\overline{\infty}|i}(1+i) = \frac{1}{i}(1+i)$$

Similarly,

$$\ddot{V}(0) = \lim_{n\to\infty}(1+i)\,a_{\overline{\infty}|i} = \lim_{n\to\infty}(1+i)\frac{1-(1+i)^{-n}}{i} = (1+i)\frac{1}{i}$$

6.7.3. Current value of constant, deferred, perpetual and post-payable income

$$V(0) = a\ d/a_{\overline{\infty}|i} = a\,a_{\overline{\infty}|i}(1+i)^{-d} = a\frac{1}{i}(1+i)^{-d}$$

6.7.4. Current value of constant, deferred, perpetual and pre-payable income

$$V(0) = a\ d/\ddot{a}_{\overline{\infty}|i} = a\,\ddot{a}_{\overline{\infty}|i}(1+i)^{-d} = a\,a_{\overline{\infty}|i}(1+i)^{-d}(1+i) = a\,\frac{1}{i}(1+i)^{-d}(1+i)$$

Exercise 6.12. Several persons that vie for the purchase of a flat have offered:

Mr. “A”: 123,000 € in cash.

Mr. “B”: 40,000 € in cash and 15,000 € every year until for a total of 10 payments.

Mr. “C”: 16,000 € annually during 15 years, making the first payment when signing the contract.

Mr. “D”: 242.000 € within 6 years.

Which is the most interesting offer if the interest rate is 12% annually?

Solution.

A) V(0) = 123,000 €.

B) $V(0) = 40{,}000 + 15{,}000\,a_{\overline{10}|0.12} = 124{,}753.34$ €

C) $V(0) = 16{,}000\,\ddot{a}_{\overline{15}|0.12} = 122{,}050.69$ €

D) $V(0) = 242{,}000\,(1+0.12)^{-6} = 122{,}604.73€$

The most interesting offer is “B”. Classified in the following order: B>A>D>C.

Exercise 6.13. Calculate the current value of income with a constant annual amount of 4,800 € if the interest rate is 7.5%, assuming the following:

A) That income is paid during 10 periods, with the first payment coinciding with the beginning of the 7th year.

B) That the income is perpetual, deferred 5 years and that its terms are effective at the start of the first year.

Solution.

A) $V(0) = 4{,}800\, a_{\overline{10}|0.075}(1+0.075)^{-5} = 22{,}949.92\ €$

B) $V(0) = 4{,}800\, a_{\overline{\infty}|0.075}(1+0.075)(1+0.075)^{-5} = 47{,}923.23\ €$

6.8. Time calculation in immediate post-payable income

To calculate the number of income accrual periods we can resort to logarithmic calculation. If we start from, for example, from the current value,

$$V(0) = a\, a_{\overline{n}|i} = a\frac{1-(1+i)^{-n}}{i}\ ;\ V(0)\, i = a - a(1+i)^{-n}\ \ (1+i)^{-n} = \left(1-\frac{V(0)i}{a}\right)$$

We use logarithms to calculate “n”,

$$-n\,Lg\,(1+i) = Lg\left(1-\frac{V(0)\,i}{a}\right) \quad \Rightarrow \quad n = -\frac{Lg\left(1-\frac{V(0)i}{a}\right)}{Lg\,(1+i)}$$

If it is unit capital,

$$n = -\frac{Lg(1-V(0)i)}{Lg\,(1+i)}$$

It could happen, however, that the result of “n” is not an exact number of periods, that is, that the periods comprise an entire plus a fractional part,

$$n = p + f$$

In this way there will be a final amount that will not reach the value of an annuity and which we will call “a_f”,

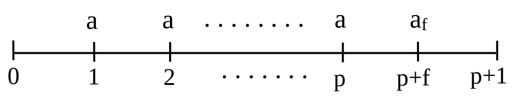


Figure 6.14

$$V(0) = a\, a_{\overline{p|}i} + a_f (1+i)^{-(p+f)}$$

The most common case in this type of problem is the amortization of a loan that makes it necessary for "n" to represent the exact number of the periods. The problem can be solved in two different ways:

"OPTION A": (It is based on carrying a_f up to moment "p"

If we force "n" to be an exact number of periods, the last annuity will be an amount other than "a", that is, it will be an "a+x" amount.

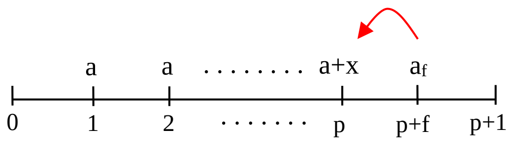


Figure 6.15

$$V(0) = a\, a_{\overline{p|}i} + x (1+i)^{-p}$$

Equalizing and substituting V(10) by the one calculated previously,

$$a\, a_{\overline{p|}i} + a_f (1+i)^{-(p+f)} = a\, a_{\overline{p|}i} + x (1+i)^{-p}$$

$$x = a_f (1+i)^{-f}$$

"OPTION B": (It is based on carrying a_f up to moment "p+1"

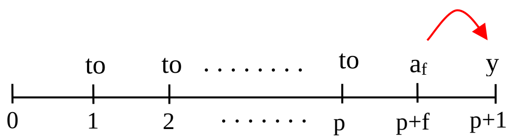


Figure 6.16

$$V(0) = a\, a_{\overline{p|}i} + y (1+i)^{-(p+1)}$$

Equalizing and substituting V(0) by its value,

$$a\, a_{\overline{p|}i} + a_f (1+i)^{-(p+f)} = a\, a_{\overline{p|}i} + y (1+i)^{-(p+1)}$$

$$y = a_f (1+i)^{1-f}$$

Exercise 6.14. A loan granted now will be cancelled with 11.000 € payments at the end of each year. If the valuation rate is 10%, calculate:

1st .- What will be the number of annual payments if the amount granted in the loan is 76.000 €?

2nd .- If time is not the exact figure in years, what will be the annual payment corresponding to that fraction of the year?

3rd .- Transform the solution to comply with the requisite that the number of years is exact (options A and B).

Solution.

1st .-

$$76{,}000 = 11{,}000\, a_{\overline{n|}0.1} \quad \Rightarrow \quad a_{\overline{n|}0.1} = 6.90909$$

$$\frac{1-(1+0.1)^{-n}}{0.1} = 6.90909 \quad \Rightarrow \quad n = 12.31893\text{years}$$

2nd.-

$$V(0) = a\, a_{\overline{p|}i} + a_f\,(1+i)^{-(p+f)}$$

$$76{,}000 = 11{,}000\, a_{\overline{12|}0.1} + a_f\,(1+0.1)^{-12.318934}$$

$$a_f = 3{,}395.085\,€$$

3rd.- Option "A" (12 years).

$$x = a_f\,(1+i)^{-f} = 3{,}395.085(1+0.1)^{-0.318934} = 3{,}293.43\,€$$

The last loan instalment would amount to: $11{,}000 + 3{,}293.435 = 14{,}293.43$ €

We can also obtain this result in another way:

$$76{,}000 = 11{,}000\, a_{\overline{11|}0.1} + (11{,}000 + x)(1+0.1)^{-12}$$

$$x = 3{,}293.43\ €$$

Option "B" (13 years).

$$y = a_f\,(1+i)^{1-f} = 3{,}395.085(1+0.1)^{1-0.318934} = 3{,}622.77\,€$$

The last instalment would amount to: 3,622.778€

We can also obtain this result operating the following manner:

$$76{,}000 = 11{,}000\, a_{\overline{12|}0.1} + x(1+0.1)^{-13}$$

$$x = 3{,}622.77\ €$$

Exercise 6.15. Calculate how many annual instalments of €6,000 must be made in order to repay a loan of €80,000 at an interest rate of 5%. If the number of periods is not an exact amount, force the solution until you find it.

Solution: First option: 21 payments of €6,000 and another after 22 years of 8,989.56 €; second option: 22 payments of 6,000 € and another after 23 years of 3,139.04 €.

Exercise 6.16. We need to have 36.000€ within 8 years and the following equivalent financial alternatives are considered:

A) Make 8 pre-payable annual deposits of a constant amount at 8%.

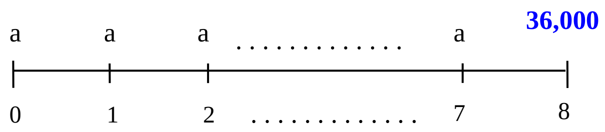
B) Make certain deposits during the 6 first years and the deposit twice the amount of the previous deposit during the remaining years, also at 8%.

C) Deposit at the beginning of each year a specific "C" amount during the 8 years, but at a rate of 7% during the first 5 years and 9% for the remainder.

Calculate the amount of the deposits to be made in each of the repayment methods.

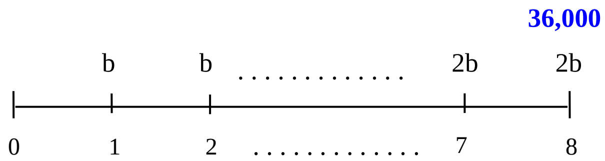
Solution.

A)



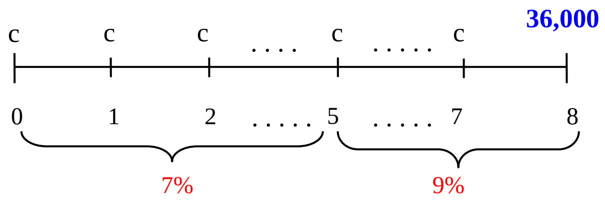
$$36{,}000 = a\,\ddot{s}_{\overline{8}|0.08} \quad\Rightarrow\quad a = 3{,}133.82\,€$$

B)



$$36{,}000 = b\,s_{\overline{6}|0.08}(1+0.08)^2 + 2b\,s_{\overline{2}|0.08} \quad\Rightarrow\quad b = 2{,}830.93\,€$$

C)



$$36{,}000 = C\,\ddot{s}_{\overline{5}|0.07}(1+0.09)^3 + C\,\ddot{s}_{\overline{3}|0.09} \quad\Rightarrow\quad C = 3{,}119.09\,€$$

Exercise 6.17. Solve the previous exercise using current values rather than final values. *Solution: A) 3,133.82 €; B) 2,830.93 €; C) 3,119.09 €*

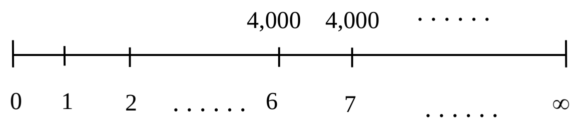
Exercise 6.18. In the following cases, calculate the current value of the yield of of a country property that generates income of 4,000€ annually, if the payment is made at the end of the sixth year:

A) The rate is 6%.

B) The rates are are: 5.5% for the first ten years, 6% for the following six and 7% for the remainder.

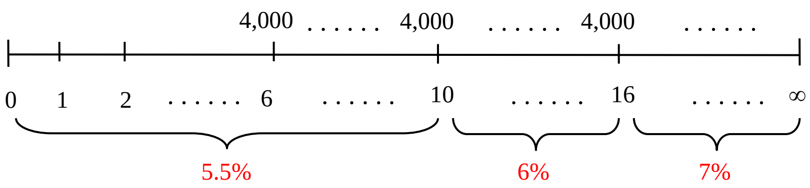
Solution.

A)



$$V(0) = 4{,}000\,a_{\overline{\infty}|0.06}(1+0.06)^{-5} = 49{,}817.21\,€$$

B)



$$V(0) = 4{,}000\, a_{\overline{5}|0.055}(1+0.055)^{-5} + 4{,}000\, a_{\overline{6}|0.06}(1+0.055)^{-10} +$$
$$+\,4{,}000\, a_{\overline{\infty}|0.07}(1+0.06)^{-6}(1+0.055)^{-10} = 48{,}167.54\,€$$

7. Constant Fractional Income and with Periodicity of More Than One Year

7.1. Fractional, immediate, post-payable income

Fractional income is income in which the periodicity of its terms is less than a year, that is, it becomes effective every K-th part of the year (for example, every month, semester, etc).

Let us assume a monetary unit that is split in "k" parts each k-th of year:

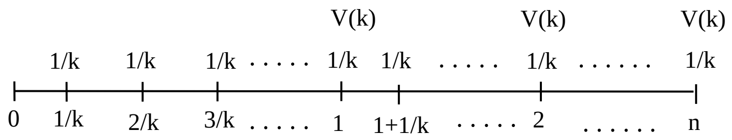


Figure 7.1

Based on the diagram of figure 7.1 for the period (0.1), we calculate the value acquired V(k) in a year by the "k" fractional payments at "i" rates, that obviously will be a greater value that the sum of the k-ths in which we have divided the monetary unit (the sumof which is equal to one):

$$V(k) = \frac{1}{k} + \frac{1}{k}(1+i)^{1/k} + \frac{1}{k}(1+i)^{2/k} + \ldots\ldots\ldots + \frac{1}{k}(1+i)^{k-1/k} =$$

$$= \frac{1}{k}\left[1 + (1+i)^{1/k} + (1+i)^{2/k} + \ldots\ldots\ldots + (1+i)^{k-1/k}\right]$$

The expression of the square bracket is the growing geometric progression of reason (1+i), whose general term is $(1+i)^{1/k}$. $S = \frac{x_n r - x_1}{r-1}$,

$$V(k) = \frac{1}{k}\left[\frac{(1+i)^{k-1/k}(1+i)^{1/k} - 1}{(1+i)^{1/k} - 1}\right] = \frac{1}{k}\left[\frac{i}{(1+i)^{1/k} - 1}\right] = \frac{i}{k\left[(1+i)^{1/k} - 1\right]} = \frac{i}{J(k)}$$

Therefore, $\frac{i}{J(k)}$ is equivalent to the total which would be obtained from capitalising the 1/k parts in which the monetary unit has been divided during a year, at the "i" rate. For example, if $\frac{0.05}{J(12)} = 1.02271479$, this means that 1 € divided into12 months and invested at 5% annually will amount to 1.02271479 € by the end of the year.

Once the V(k) corresponding to every year has been calculated, the diagram of figure 7.1

will have been converted in the following:

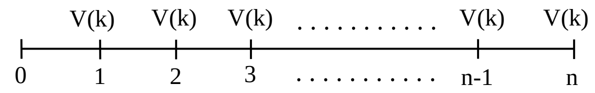


Figure 7.2

7.1.1. Current value

The current value of the income represented in figure 7.2 corresponds to the current value of a post-payable income of "n" periods, at "i" and term V(k):

$$a^{(k)}_{\overline{n}|i} = V(k)\, a_{\overline{n}|i} = \frac{i}{J(k)}\, a_{\overline{n}|i}$$

If the income amounts to “a” monetary units and is divided in "k" times, then,

$$V(0) = a\,k\, a^{(k)}_{\overline{n}|i} = a\,k \frac{i}{J(k)}\, a_{\overline{n}|i}$$

The difference between annual and fractional income for a capital unit will be defined as,

$$\text{Dif.} = a^{(k)}_{\overline{n}|i} - a_{\overline{n}|i} = a_{\overline{n}|i} \frac{i}{J(k)} - a_{\overline{n}|i} = a_{\overline{n}|i}\left[\frac{i}{J(k)} - 1\right]$$

Given that i > J(k) (see chapter 3), the difference $\left[\frac{i}{J(k)} - 1\right]$ is called “ frequency capital gain” of a fractional income with respect to income with an annual frequency.

7.1.2. Final value

Based on figure 7.2,

$$s^{(k)}_{\overline{n}|i} = V(k)\, s_{\overline{n}|i} = \frac{i}{J(k)}\, s_{\overline{n}|i}$$

If t income amounts to “a” monetary units and is divided "k" times,

$$V(n) = a\,k\, s^{(k)}_{\overline{n}|i} = a\,k \frac{i}{J(k)}\, s_{\overline{n}|i}$$

7.2. Fractional, immediate, pre-payable income

In this case the diagram corresponding to a period is as follows:

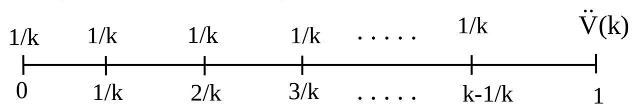


Figure 7.3

$$\ddot{V}(k) = \frac{1}{k}(1+i)^{1/k} + \frac{1}{k}(1+i)^{2/k} + \dots\dots\dots + \frac{1}{k}(1+i) = = \frac{1}{k}\left[(1+i)^{1/k} + (1+i)^{2/k} + \dots\dots\dots + (1+i)\right] =$$

$$= \frac{1}{k}\left[\frac{(1+i)(1+i)^{1/k} - (1+i)^{1/k}}{(1+i)^{1/k} - 1}\right] = \frac{1}{k}(1+i)^{1/k}\left[\frac{(1+i)-1}{(1+i)^{1/k} - 1}\right] =$$

$$= (1+i)^{1/k}\frac{i}{k\left[(1+i)^{1/k} - 1\right]} = (1+i)^{1/k}\frac{i}{J(k)}$$

Once the V(k) corresponding to every year has been calculated, the diagram of figure 7.3 will be converted into the following:

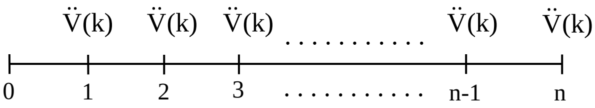


Figure 7.4

7.2.1. Current value

The current value of the income represented in figure 7.4 corresponds to the post-payable income of "n" periods, at "i" and term V(k):

$$\ddot{a}^{(k)}_{\overline{n|}\,i} = \ddot{V}(k)\, a_{\overline{n|}\,i} = (1+i)^{1/k}\frac{i}{J(k)}\, a_{\overline{n|}\,i}$$

If the income amounts to “a” monetary units and is divided "k" times,

$$\ddot{V}(0) = a\,k\,\ddot{a}^{(k)}_{\overline{n|}\,i} = a\,k\frac{i}{J(k)}\, a_{\overline{n|}\,i}\,(1+i)^{1/k}$$

7.2.2. Final value

$$\ddot{s}^{(k)}_{\overline{n|}\,i} = V(k)\,\ddot{s}_{\overline{n|}\,i} = \frac{i}{J(k)}\,\ddot{s}_{\overline{n|}\,i}$$

If the income amounts to “a” monetary units and is divided "k" times,

$$\ddot{V}(n) = a\,k\,\ddot{s}^{(k)}_{\overline{n|}\,i} = a\,k\frac{i}{J(k)}\, s_{\overline{n|}\,i}\,(1+i)^{1/k}$$

7.3. Deferred fractional income

7.3.1. Post-payable income. Current value

$$d/a^{(k)}_{\overline{n|}\,i} = a_{\overline{n|}\,i}\,\frac{i}{J(k)}(1+i)^{-d}$$

If the income amounts to “a” monetary units and is divided "k" times,

$$d/V(0) = a\,k\, a_{\overline{n|}\,i}\,\frac{i}{J(k)}(1+i)^{-d}$$

7.3.2. Post-payable income. Current value

We go directly to the case where the income amounts to “a” monetary units and is divided i “k” times.

$$d/\ddot{V}(0) = a\,k\,a_{\overline{n}|i}\,\frac{i}{J(k)}(1+i)^{-d}(1+i)^{1/k}$$

7.4. Anticipated fractional income

7.4.1. Post-payable income. Final value

$$A/S^{(k)}_{\overline{n}|i} = S_{\overline{n}|i}\,\frac{i}{J(k)}(1+i)^{A}$$

If the income amounts to “a” monetary units and is divided "k" times,

$$A/V(n) = a\,k\,S_{\overline{n}|i}\,\frac{i}{J(k)}(1+i)^{A}$$

7.4.2. Pre-payable income. Final value

If the income amounts to “a” monetary units and is divided "k" times,

$$A/\ddot{V}(n) = a\,k\,S_{\overline{n}|i}\,\frac{i}{J(k)}(1+i)^{A}(1+i)^{1/k}$$

Figure 7.5 summarised the different deferment, anticipation and pre-payability factors:

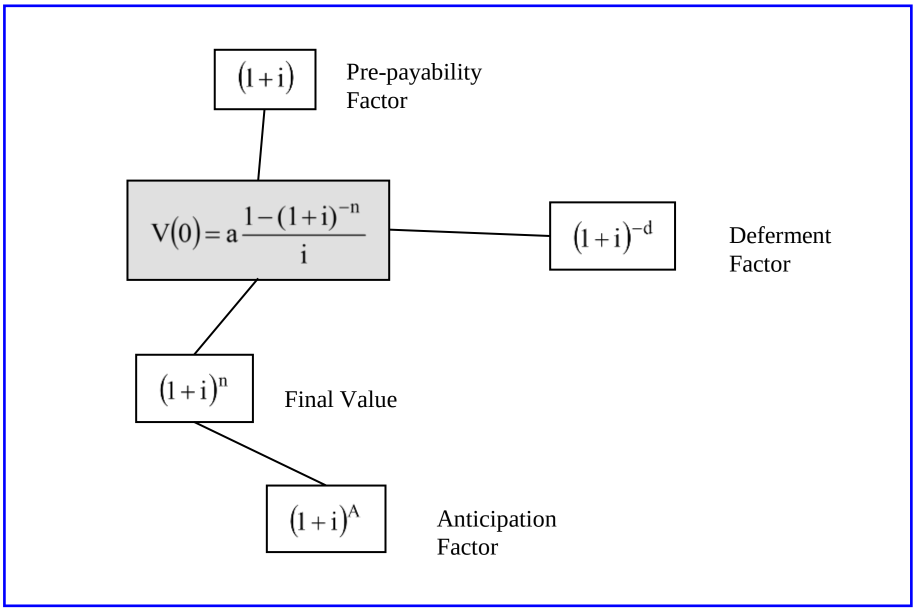


Figure 7.5

7.5. Perpetual fractional income

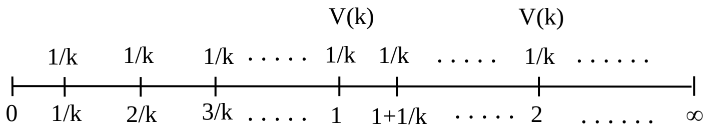


Figure 7.6

7.5.1. Post-payable income. Current value

$$a^{(k)}_{\overline{\infty}|\, i} = a_{\overline{\infty}|\, i} \frac{i}{J(k)} = \frac{1}{i}\frac{i}{J(k)}$$

If the income amounts to “a” monetary units and is divided "k" times,

$$V(0)_{\infty} = a\,k\,\frac{1}{i}\frac{i}{J(k)} = \frac{a}{i_k}$$

7.5.2. Deferred and post-payable income. Current value

$$d/V(0)_{\infty} = a\,k\,\frac{1}{i}\frac{i}{J(k)}(1+i)^{-d}$$

7.5.3. Pre-payable income. Current value

$$\ddot{a}^{(k)}_{\overline{\infty}|\, i} = \ddot{a}_{\overline{\infty}|\, i} \frac{i}{J(k)} = \frac{1}{i}\frac{i}{J(k)}(1+i)^{1/k}$$

If the income amounts to “a” monetary units and is divided "k" times,

$$\ddot{V}(0)_{\infty} = a\,k\,\frac{1}{i}\frac{i}{J(k)}(1+i)^{1/k}$$

7.5.4. Deferred and pre-payable income. Current value

$$d/\ddot{V}(0)_{\infty} = a\,k\,\frac{1}{i}\frac{i}{J(k)}(1+i)^{-d}(1+i)^{1/k}$$

Exercise 7.1. Calculate the current and final value of an income of 120 monthly terms of 500€ if the valuation rate is 12%, assuming income is:

A) Immediate post-payable

B) Immediate pre-payable

C) Deferred 5 years and post-payable.

D) Anticipated 8 years and post-payable.

Solution.

A)

$$V(0) = 12 \times 500\, a_{\overline{10}|0.12} \frac{0.12}{J(12)} = 35{,}727.76\,€$$

$$V(n) = 12 \times 500\, s_{\overline{10}|0.12} \frac{0.12}{J(12)} = 110{,}965.02\,€$$

B)

$$\ddot{V}(0) = 12 \times 500\, a_{\overline{10}|0.12} \frac{0.12}{J(12)} (1+0.12)^{1/12} = 36{,}066.77\,€$$

$$\ddot{V}(n) = \ddot{V}(0)(1+0.12)^{10} = 112{,}017.94\,€$$

C)

$$5/\ddot{V}(0) = 12 \times 500\, a_{\overline{10}|0.12} \frac{0.12}{J(12)} (1+0.12)^{-5} = 20{,}465.26\,€$$

$$V(n) = V(0)(1+0.12)^{15} = 112{,}017.93\,€$$

D)

$$V(0) = 12 \times 500\, a_{\overline{10}|0.12} \frac{0.12}{J(12)} = 35{,}727.76\,€$$

$$8/V(n) = 12 \times 500\, s_{\overline{10}|0.12} \frac{0.12}{J(12)} (1+0.12)^{8} = 274{,}745.25\,€$$

7.6. Fractional income according to the equivalent rate

When considering and resolving the practical exercises with fractional income, it is more operative to work with the equivalent i_k rate that is inferred from the equality:

$$(1+i_k)^k = (1+i) \quad \rightarrow i_k = (1+i)^{1/k} - 1$$

Hence, the current value of the post-payable immediate income is:

$$V(0) = a\, a_{\overline{n \times k}|i_k}$$

For example, the current value of the 3,000 € monthly income that will be paid over a 5-year period at 8%, is given by:

$$V(0) = 3{,}000\, a_{\overline{60}|0.006434} = 148{,}934.71\ €$$

Being

$$i_{12} = (1+0.08)^{1/12} - 1 = 0.06434; \qquad n = 12 \times 5 = 60$$

Exercise 7.2. Perform a valuation at 8% of the current net profit of the following expense and income forecast for the next 10 years.

Charges:

• 42,000 € in one lump payment at the current time.

• 590 € monthly instalments over the next 5 years.

• 900 € monthly over the next 5 years.

Income:

• 10,000 € half-yearly with the first payment starting at the beginning of the third year.

Solution.

$$V(0)_{Ing} = 10{,}000\, a_{\overline{17}|0.03923}(1+0.08)^{1/2}(1+0.08)^{-2} = 109{,}042.53\,€$$

$$V(0)_{Gtos} = 42{,}000 + 590\, a_{\overline{60}|0.006434} + 900\, a_{\overline{60}|0.006434}(1+0.08)^{-5} = 101{,}699.23\,€$$

$$\text{Beneficio neto total} = V(0)_{Ing} - V(0)_{Gtos} = 109{,}042.53 - 101{,}699.23 = 7{,}343.30\,€$$

Exercise 7.3. An urban property provides its owner returns at the beginning of every month of 2,800€. The owner also possesses a country property that provides annual returns of 30,000€ and certain half-yearly operating income during a period of 6 years. Calculate the amount of the latter assuming that the current values of the two properties are equivalent. Valuation rate, 6%.

Solution.

$$V(0)_{F.Urb} = 2{,}800\, a_{\overline{\infty}|0.00486755}(1+0.06)^{1/12} = 578{,}037.98\,€$$

$$V(0)_{F.Rúst} = 30{,}000\, a_{\overline{\infty}|0.06} + X\, a_{\overline{12}|0.029563} = 500{,}000 + 9.980019\,X$$

$$V(0)_{F.Urb} = V(0)_{F.Rúst}$$

$$578{,}037.98 = 500{,}000 + 9.980019\,X \quad \Rightarrow \quad X = 7{,}819.42\,€$$

Exercise 7.4. For the purchase of a machine the supplier offers two forms of equivalent payments:

A) Payment in cash amounting to 33,000 €.

B) Monthly instalments of 1000 € during 3 years and the possibility of buying it at the end of this period, paying at that time a certain amount of money.

If the valuation rate is 8%, calculate the amount to be paid in the third year so that it makes no difference which of the two above-mentioned methods is used to purchase the machine.

Solution.

$$33{,}000 = 1{,}000\, a_{\overline{36}|0.006434} + X(1+0.08)^{-3}$$

$$X = 1{,}205.12\,€$$

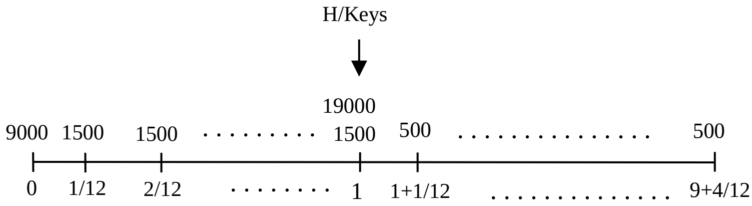
Exercise 7.5. A construction company sells apartments with the following method of payment: on signing the contract: 9000 € followed by 12 monthly instalments of 1,500 € each, paying the first monthly instalment on signing the contract. A year after signing the contract the keys are handed over, and at that moment 19,000 € will be paid, and thereon 100 monthly bills of exchange of 500€ each. If the valuation rate is 8%, calculate:

A) Value of the property on handing over the keys.

B) Amount of the cash payment after signing the contract.

C) The construction company offers the possibility of waiving payment when the keys are handed over on the condition that it is compensated subsequently in the corresponding monthly instalments, being aware at that time that 9% interest will be charged for the entire transaction. In this case, what would be the corresponding monthly instalments?

Solution.



A)

$$V(1) = 9{,}000\,(1+0.08) + 1{,}500\, s_{\overline{12}|0.006434} + 19{,}000 +$$

$$+ 500\, a_{\overline{100}|0.006434} = 84{,}160.73\ €$$

B)

$$V(0) = V(1)\,(1+0.08)^{-1} = 84{,}160.736\,(1+0.08)^{-1} = 77{,}926.65\ €$$

C)

$$77{,}926.65 = 9{,}000 + 1{,}500\, a_{\overline{12}|0.00720733} + x\, a_{\overline{100}|0.00720733}(1+0.09)^{-1}$$

$$x = 793.38\ €$$

Exercise 7.6. A construction company considers building a block of apartments and performs the following analysis for this purpose:

Costs:

- Purchase of 1,200 m^2 of land at 500 € per m^2 for the construction of 25 apartments.

- Monthly payments of 40,000 € during 15 months while the work is carried out and 100,000 € at its conclusion.

An individual purchased an apartment when the building work began and agrees to pay for it in the following manner:

- 12,000 € in cash and 18,000 € when the work is completed.

- In the following 6 years, after handing over the keys the person will have to pay certain monthly instalments, knowing that the sale price in cash is 40% higher than the cost.

If the valuation is performed at 10%, calculate the monthly instalments that the buyer will have to pay.

Solution.

Building cost

$$V(0) = 500 \times 1{,}200 + 40{,}000\, a_{\overline{15}|0.00797414} + 100{,}000\,(1+0.1)^{-15/12} = 1{,}252{,}162.01\ €$$

Price of each apartment,

$$\text{Apart.Cost} = \frac{1{,}52{,}162.01}{25} = 50{,}086.48 \text{ €}$$

The sales price of each apartment will be the result of increasing the cost price by 40%,

$$\text{Sale price of Apart.} = 50{,}086.48 \times 1.4 = 70{,}121.07 \text{ €}$$

The monthly instalments are calculated by,

$$70{,}121.07 = 12{,}000 + 18{,}000(1+0.1)^{-15/12} + X\, a_{\overline{72}|0.00797414}(1+0.1)^{-15/12}$$

$$X = 869.22 \text{ €}$$

Figure 7.7 shows the transformation of the annual post-payable immediate income in a fractional post-payable immediate income:

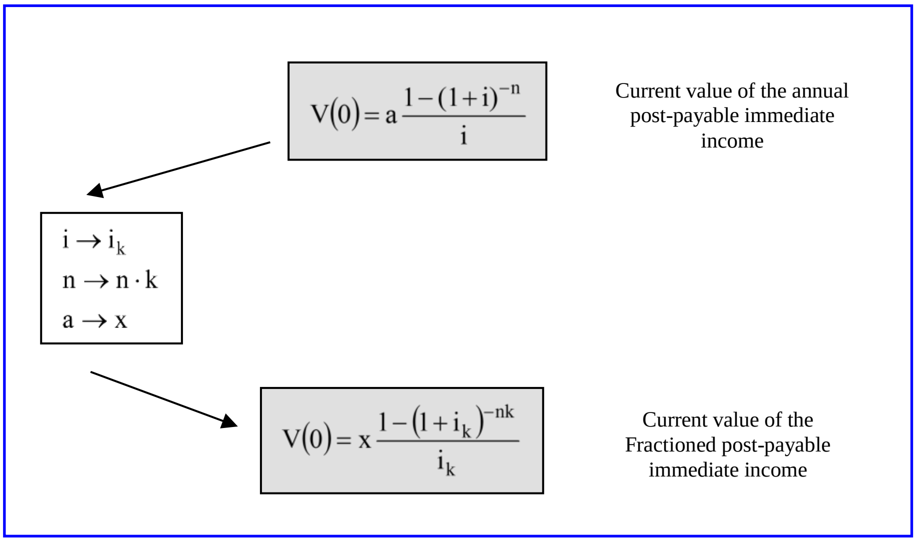


Figure 7.7

7.7. Income with periodicity of more than one year: Immediate, deferred, anticipated and perpetual

Sometimes we find income in which the terms become effective in periods of more than one year, that is, in periods $p > 1$, where "p" they can be two-year, five-ear, six-year periods, etc. Assuming that income is constant, immediate and post-payable, the diagram is as follows:

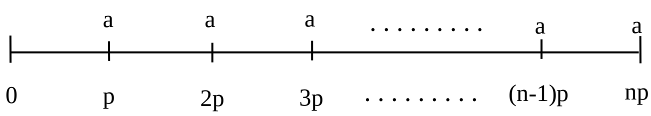


Figure 7.8

7.7.1. Post-payable immediate income

A) Current value:

$$V(0) = a(1+i)^{-p} + a(1+i)^{-2p} + \dots\dots\dots + a(1+i)^{-np} = a\left[\frac{(1+i)^{-p} - (1+i)^{-np}(1+i)^{-p}}{1-(1+i)^{-p}}\right]$$

$$V(0) = a\,\frac{(1+i)^{-p}}{(1+i)^{-p}}\left[\frac{1-(1+i)^{-np}}{(1+i)^{p}-1}\right] = a\left[\frac{1-(1+i)^{-np}}{(1+i)^{p}-1}\right]$$

We divide the numerator and denominator by “i”

$$V(0) = a\left[\frac{\dfrac{1-(1+i)^{-np}}{i}}{\dfrac{(1+i)^{p}-1}{i}}\right] = a\,\frac{a_{\overline{np}|i}}{s_{\overline{p}|i}}$$

B) <u>Final value</u>:

$$V(n) = V(0)(1+i)^{np} = a\,\frac{a_{\overline{np}|i}}{s_{\overline{p}|i}}(1+i)^{np} = a\,\frac{s_{\overline{np}|i}}{s_{\overline{p}|i}}$$

7.7.2. Pre-payable immediate income

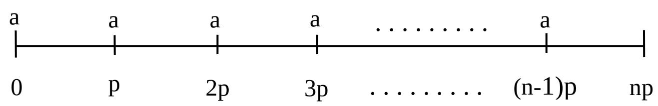


Figure 7.9

A) Current value:

$$\ddot{V}(0) = a + a(1+i)^{-p} + a(1+i)^{-2p} + \dots\dots\dots + a(1+i)^{-(n-1)p} = a\left[\frac{1-(1+i)^{-(n-1)p}(1+i)^{-p}}{1-(1+i)^{-p}}\right] =$$

$$= a\left[\frac{\dfrac{1-(1+i)^{-np}}{i}}{\dfrac{1-(1+i)^{-p}}{i}}\right] = a\left[\frac{a_{\overline{np}|i}}{a_{\overline{p}|i}}\right]$$

B) <u>Final value</u>:

$$\ddot{V}(n) = a(1+i)^{p} + a(1+i)^{2p} + \dots\dots\dots + a(1+i)^{np} = a\left[\frac{(1+i)^{np}(1+i)^{p} - (1+i)^{p}}{(1+i)^{p}-1}\right] =$$

$$a\,\frac{(1+i)^{p}}{(1+i)^{p}}\left[\frac{(1+i)^{np}-1}{1-(1+i)^{-p}}\right] = a\left[\frac{\dfrac{(1+i)^{np}-1}{i}}{\dfrac{1-(1+i)^{-p}}{i}}\right] = a\left[\frac{s_{\overline{np}|i}}{a_{\overline{p}|i}}\right]$$

7.7.3. Deferred income

$$d/V(0) = a\frac{a_{\overline{np}|i}}{s_{\overline{p}|i}}(1+i)^{-d}$$

7.7.4. Anticipated income

$$A/V(n) = a\frac{s_{\overline{np}|i}}{s_{\overline{p}|i}}(1+i)^{A}$$

7.7.5. Perpetuate income

A) <u>Post-payable</u>

$$V(0) = a(1+i)^{-p} + a(1+i)^{-2p} + \dots\dots\dots = a\left[\frac{(1+i)^{-p}}{1-(1+i)^{-p}}\right] =$$

$$= a\frac{(1+i)^{-p}}{(1+i)^{-p}}\left[\frac{1}{(1+i)^{p}-1}\right] = a\left[\frac{\frac{1}{i}}{\frac{(1+i)^{p}-1}{i}}\right] = a\left[\frac{a_{\overline{\infty}|i}}{s_{\overline{p}|i}}\right] = \frac{a}{i}\frac{1}{s_{\overline{p}|i}}$$

B) <u>Pre-payable</u>

$$\ddot{V}(0) = a + a(1+i)^{-p} + a(1+i)^{-2p} + \dots\dots\dots = a\left[\frac{1}{1-(1+i)^{-p}}\right] =$$

$$= a\left[\frac{\frac{1}{i}}{\frac{1-(1+i)^{-p}}{i}}\right] = a\left[\frac{a_{\overline{\infty}|i}}{a_{\overline{p}|i}}\right] = \frac{a}{i}\frac{1}{a_{\overline{p}|i}}$$

7.7.6. Income with periodicity of more than a year according to the equivalent rate

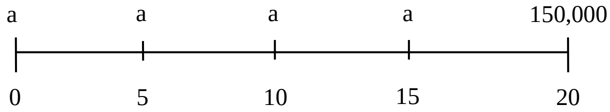
When calculating income with a periodicity of more than a year, we can also use the amount "i^*", equivalent to the annual amount "i" through the expression $(1+i)^p = (1+i^*)$, treating income the same way as in previous chapters. Hence, the calculation of the current value of the post-payable immediate income, for example, would be:

$$V(0) = a(1+i^*)^{-1} + a(1+i^*)^{-2} + \dots\dots\dots + a(1+i^*)^{-n} = a\, a_{\overline{n}|i^*}$$

This reasoning is applicable to all income with a periodicity of more than a year.

Exercise 7.7. A person wishes to create 150.000 € capital in 20 years and to this end, agrees with a financial entity a certain sum every five years, computing interest at 10%. Calculate this amount if the first payment is made immediately and the last one coincides with year 15.

Solution.



$$150{,}000 = a\left[\frac{s_{\overline{20}|0.1}}{a_{\overline{5}|0.1}}\right] \quad \Rightarrow \quad a = 9{,}927.85$$

We can also solve this exercise by calculating the equivalent five-year rate,

$$i^{(5)} = (1+i)^5 - 1 = (1+0.1)^5 - 1 = 0.61051$$

$$150{,}000 = a\ddot{s}_{\overline{4}|0.61051} \quad \Rightarrow \quad a = 9{,}927.85$$

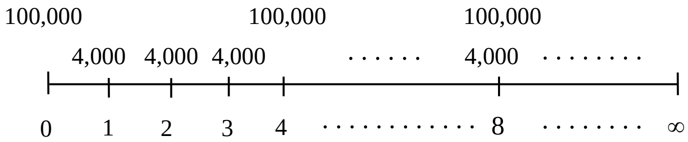
Exercise 7.8. To solve its transport problems, a company can choose between the two following options:

A) Purchase from now on and every four years a vehicle for 100,000€, with maintenance costs of 4,000 € at the end of every year.

B) Acquire every a vehicle every 6 that will cost as of this moment 120.000€ with annual maintenance costs of 4,600 €.

With a valuation rate of 11%, which of the two possibilities is of greater interest for the company, if at the end of the useful life of the vehicles the corresponding indefinite renewals are made?

Solution.



A)

$$\ddot{V}(0) = a\left[\frac{a_{\overline{\infty}|i}}{a_{\overline{p}|i}}\right] + 4{,}000\, a_{\overline{\infty}|0.11} = 100{,}000\,\frac{\frac{1}{0.11}}{a_{\overline{4}|0.11}} + 4{,}000\, a_{\overline{\infty}|0.11} = 329{,}387.59$$

Also, calculating the equivalent four-year rate,

$$i^{(4)} = (1+i)^4 - 1 = (1+0.11)^4 - 1 = 0.51807041$$

$$\ddot{V}(0) = 100{,}000\ \ddot{a}_{\overline{\infty}|0.51807041} + 4{,}000\ a_{\overline{\infty}|0.11} = 329{,}387.59\ €$$

B)

$$\ddot{V}(0) = 120.000\,\frac{\frac{1}{0.11}}{a_{\overline{6}|0.11}} + 4{,}600\, a_{\overline{\infty}|0.11} = 299{,}683.52\ €$$

Also,

$$\ddot{V}(0) = 120{,}000\ a_{\overline{\infty}|0.870414552}\left(1 + 0.870414552\right) + 4{,}600\ a_{\overline{\infty}|0.11} = 299{,}683.52\ €$$

The best option is B).

Exercise 7.9. A railway line has a level crossing that has to be guarded by two guards whose monthly salaries amount to 1,100€.

The possibility of building a bridge at this level crossing is currently being considered, the cost of which would be 200,000€ and which has to be replaced every 20 years. Additionally, its annual maintenance cost amounts to 12,000€ and 20.000€ has to be paid in repairs every two years. Analyse if, a 10% valuation, it would of interest to build the bridge.

Solution.

$$V(0)_{\text{No puente}} = 3 \times 1{,}100\ a_{\overline{\infty}|0.00797414} = 413{,}837.70\ €$$

$$V(0)_{\text{Si puente}} = 200{,}000\frac{a_{\overline{\infty}|0.1}}{a_{\overline{20}|0.1}} + 12{,}000\ a_{\overline{\infty}|0.1} + 20{,}000\frac{a_{\overline{\infty}|0.1}}{s_{\overline{2}|0.1}} = 450{,}157.34\ €$$

$$V(0)_{\text{Si puente}} = 200{,}000\ a_{\overline{\infty}|5.7275}\left(1 + 5.7275\right) + 12{,}000\ a_{\overline{\infty}|0.1} +$$

$$+\ 20{,}000\ a_{\overline{\infty}|0.21} = 450{,}157.34\ €$$

Building the bridge would not be of interest.

7.8. Income broken down into annual blocks

We consider as income by block, income which after being broken down, is not collected in all the periods into which the block was divided (a block being an entire calendar year).

Because of its characteristics, this income cannot be treated as the types of income that we have discussed up to now which is paid in all the fractions of the reference period (if the income is monthly, 12 collections or payments are produced every year). That is, it is fractioned income that is not paid in all periods into which it has been divided. For example, a company that runs a hotel only in the summer months, or the amortization of a loan in which 14 instalments are paid instead of 12 (in this case the two extra instalments r are paid with the collection of the salary bonus), etc.

In all other cases, and provided that income is effective in all the fractions in which we have divided the block, the valuations for fractioned income are applicable. Nevertheless, nothing prevents the valuation from being performed by annual blocks in all cases.

Assuming that income is paid during "m" months every year, being m < 12

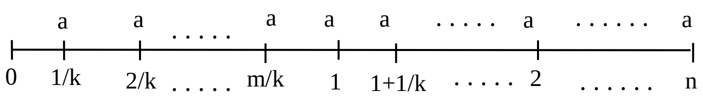


Figure 7.10

Calculate the current value of the first year:

$$V(0)_1 = a\ a_{\overline{m}|i_{12}}$$

If we do the same with all the years in which the income is paid, this will be transformed into a pre-payable annual income (figure 7.11)

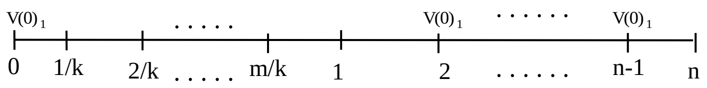


Figure 7.11

Its current value is given by:

$$V(0)_T = V(0)_1\, \ddot{a}_{\overline{n}|i} = V(0)_1\, a_{\overline{n}|i}(1+i)$$

The final value can be calculated through the previously studied associations:

The problem can also be considered by calculating the value acquired in the first year, so that,

$$V(1)_1 = a\, s_{\overline{m}|i_{12}}$$

The current value is given, in this case, by:

$$V(0)_T = V(1)_1\, a_{\overline{n}|i}$$

Exercise 7.10. Calculate the current value of the returns of a nougat confectionary factory over the next 10 years if we know that sales that will be generated only in the months of October, November and December, with the following figures: 150,000 € for October and November and 200,000 € in December. Valuation rate, 5%.

Solution.

$$V(0)_1 = 150{,}000\, a_{\overline{2}|0.00407412}(1+0.05)^{-9/12} + 200{,}000(1+0.05)^{-1} = 477{,}938.89\,€$$

$$V(0)_T = 477{,}938.89\, a_{\overline{10}|0.05}(1+0.05) = 3{,}875{,}043.35\,€$$

Exercise 7.11. Calculate the price a person would have to pay to buy a restaurant considering that it generates profits amounting to €20,000 per month, except in the months of July, August and September when it is closed, in the following cases:

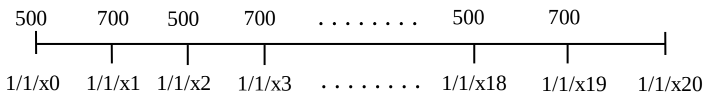
a) The study is conducted for a period of 10 years.

b) Perpetuity income is considered.

Solution: a) €1,424,408.62; b) €3,689,348.66

Exercise 7.12. On 01/01/x0 500€ is deposited in a particular entity; 700 € on 01/01/x1; 2500 € on 01/01/; and so on alternating 500 and 700 €. Calculate the capital that will have been created at the end of 20 years at 4% annually.

Solution.



$$V_1(20) = 500(1+i)^2 + 500(1+i)^4 + ... + 500(1+i)^{20} = 500\left[\frac{(1+i)^{20}(1+i)^2 - (1+i)^2}{(1+i)^2 - 1}\right] =$$

$$= 500(1+i)^2\left[\frac{(1+i)^{20} - 1}{(1+i)^2 - 1}\right] = 500(1+0.04)^2\left[\frac{s_{\overline{20}|i}}{s_{\overline{2}|i}}\right] = 500(1+0.04)^2\left[\frac{s_{\overline{20}|0.04}}{s_{\overline{2}|0.04}}\right] = 7{,}894.1€$$

$$V_2(20) = 700(1+i) + 700(1+i)^3 + ... + 700(1+i)^{19} = 700\left[\frac{(1+i)^{19}(1+i)^2 - (1+i)}{(1+i)^2 - 1}\right] =$$

$$= 700(1+i)\left[\frac{(1+i)^{20} - 1}{(1+i)^2 - 1}\right] = 700(1+0.04)\left[\frac{s_{\overline{20}|i}}{s_{\overline{2}|i}}\right] = 700(1+0.04)\left[\frac{s_{\overline{20}|0.04}}{s_{\overline{2}|0.04}}\right] = 10{,}626.6€$$

$$V(20)_{Total} = V_1(20) + V_2(20) = 18{,}520.79\,€$$

Exercise 7.13. Arrange in descending order the following investments for a period of 7 years and 4 months:

A.- 1,000 € pre-payable monthly instalments at 7.3% EAR.

B.- 6,000 € post-payable half-yearly instalments at 6% convertible nominal.

C.- 750 € post-payable quarterly instalments at 3.2% monthly equivalent.

D.- 1,000 € pre-payable half-yearly instalments at 8% E.A.R.

Solution.

A) $V(7 + 4/12) = 1{,}000\, s_{\overline{88}|0.00588881}(1 + 0.00588881) = 115{,}551.70\ €$

B) $i_2 = \frac{J(2)}{2} = 0.03 \quad \Rightarrow \quad i = (1+0.03)^2 - 1 = 0.0609$

$V(7 + 4/12) = 6{,}000\, s_{\overline{14}|0.03}(1 + 0.0609)^{4/12} = 104{,}558.18\ €$

C) $i_4 = (1+0.032)^3 - 1 = 0.099104768$

$V(7 + 4/12) = 750\, s_{\overline{29}|0.099104768}(1 + 0.032) = 113{,}188.60\ €$

D) $V(7 + 4/12) = 25{,}000\, s_{\overline{3}|0.1664}(1 + 0.1664)(1 + 0.08)^{16/12} = 113{,}957.86\ €$

Arranged in descending order: A > D > C > B.

Exercise 7.14. The following information on the purchase of a particular home appliance appears in the advertisement of a certain financial entity: "From 24.89€/monthly in 48 months. E.A.R.: 9.1%, nominal interest: 8.75%".

Calculate the price in cash of the home appliance and check that the E.A.R. advertised is correct.

Solution.

$i_{12} = \frac{J(12)}{12} = \frac{0.0875}{12} = 0.00729166\ ;\ V(0) = 24.89\, a_{\overline{48}|0.00729166} = 1{,}004.98\ €$

$$E.A.R. = i = \left(1 + \frac{0.0875}{12}\right)^{12} - 1 = 0.09109$$

The E.A.R. is calculated by comparing the amount paid with that already received. In this case, the monthly rate coincides with that of the transaction as there are no charges or other characteristics that alter the initial agreement.

Exercise 7.15. A financial entity offers its clients a 6,000€ loan without interest to be paid in 15 monthly instalments, with an opening commission charge of 2% on the borrowed capital. Calculate the E.A.R. of this financial transaction if:

a) Commission is paid when the loan is granted.

b) Commission is paid with the last monthly loan instalment.

Solution.

a) $6{,}000 = 400\, a_{\overline{15|}i_{12}} + 120 \quad \Rightarrow \quad i_{12} = 0.00254$

E.A.R. = i = $(1 + i_{12})^{12} - 1 = (1 + 0.00254)^{12} - 1 = 0.0309$

b) $6{,}000 = 400\, a_{\overline{15|}i_{12}} + 120(1 + i_{12})^{-15} \quad \Rightarrow \quad i_{12} = 0.0024434$

E.A.R. = i = $(1 + i_{12})^{12} - 1 = (1 + 0.0024434)^{12} - 1 = 0.0297$

8. Security Equities in Geometric and Arithmetic Progression

8.1. Variable annual income in geometric progression

The income analysed up to this moment was constant, but nothing prevents the transactions from being negotiated with variable terms. The most common is variable income in geometric and arithmetic progression. Let us consider the case of a new company. Its earnings performance in the first fiscal years will be worse than in subsequent years when the business will have consolidated. If the company acquires machinery, it is not advisable to pay in full at the beginning. It will be better to make payments as the company starts to generate profits, that is, every higher profit than the previous year.

A home buyer could consider that it would be better to start making small payments that will be increasing every year in a proportion similar to what he believes his salary will be increasing.

These cases lead us to consider payment increments in arithmetic or geometric progression.

Variable income is income the terms of which follow a random or known variation (geometric and arithmetic progression). In this chapter we are going to analyse the valuation of variable income when the terms vary following the pre-established law and, specifically, the terms follow a geometric or arithmetic progression.

8.1.1. Post-payable income

Assuming that the income terms vary in geometric progression of reason "q".

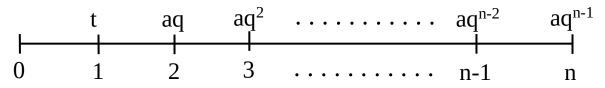


Figure 8.1

A) Current value:

$$V(0) = a(1+i)^{-1} + aq(1+i)^{-2} + \dots\dots\dots + aq^{n-2}(1+i)^{-(n-1)} + aq^{n-1}(1+i)^{-n}$$

$$V(0) = a\left[(1+i)^{-1} + q(1+i)^{-2} + \dots\dots\dots + q^{n-2}(1+i)^{-(n-1)} + q^{n-1}(1+i)^{-n}\right]$$

The expression within the square brackets is a decreasing geometric progression of reason $q(1+i)^{-1}$, whose sum is $S = \frac{x_1 - x_n r}{1-r}$, under the hypothesis that $q < (1+i)$,

$$V(0) = a\left[\frac{(1+i)^{-1} - q^{n-1}(1+i)^{-n} q(1+i)^{-1}}{1-q(1+i)^{-1}}\right] = a\frac{(1+i)^{-1}}{(1+i)^{-1}}\left[\frac{1-q^n(1+i)^{-n}}{(1+i)-q}\right]$$

Therefore,

$$V(0) = A_{(a,q)\overline{n}|i} = a\left[\frac{1-q^n(1+i)^{-n}}{(1+i)-q}\right]$$

B) <u>Final value</u>:

$$V(n) = S_{(a,q)\overline{n}|i} = V(0)(1+i)^n = a\left[\frac{1-q^n(1+i)^{-n}}{(1+i)-q}\right](1+i)^n = a\frac{(1+i)^n-q^n}{(1+i)-q}$$

8.1.2. Pre-payable income

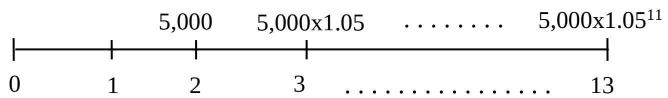
$$\ddot{V}(0) = \ddot{A}_{(a,q)\overline{n}|i} = a\left[\frac{1-q^n(1+i)^{-n}}{(1+i)-q}\right](1+i)$$

8.1.3. Deferred income

$$d/V(0) = d/A_{(a,q)\overline{n}|i} = a\left[\frac{1-q^n(1+i)^{-n}}{(1+i)-q}\right](1+i)^{-d}$$

Exercise 8.1. Calculate the price in cash of the machine if we know that there will be a total of 12 instalment payments, that the first payment amounts to 5,000€, that this payment will be at the beginning of the third year, and that the following payments will be increasing by 5% on a cumulative annual basis. The valuation rate is 8%.

Solution.



$$V(0) = a\left[\frac{1-q^n(1+i)^{-n}}{(1+i)-q}\right] = 5{,}000\left[\frac{1-1.05^{12}(1+0.08)^{-12}}{(1+0.08)-1.05}\right](1+0.08)^{-1} = 44{,}265.54\ €$$

8.1.4. Particular case of indetermination: $q = (1+i)$

A) <u>Post-payable income: Current value</u>

In the case that $q = (1+i)$, on substituting in the formula the current value or final value we find an indetermination that can be solved by calculating its limit,

$$V(0) = \lim_{q\to 1+i} A_{(a,1+i)\overline{n}|i} = \lim_{q\to 1+i} a\left[\frac{1-q^n(1+i)^{-n}}{(1+i)-q}\right] : \frac{0}{0}$$

We apply the L´Hôpital rule,

$$V(0) = a\lim_{q\to 1+i}\frac{-nq^{n-1}(1+i)^{-n}}{-1}$$

We substitute “q”by (1+i),

$$V(0)=A_{(a,1+i)\overline{n|}\,i}=a\times n(1+i)^{n-1}(1+i)^{-n}=a\times n(1+i)^{-1}$$

B) <u>Post-payable income: Final value</u>

$$V(n)=\lim_{q\to 1+i} S_{(a,1+i)\overline{n|}\,i}=\lim_{q\to 1+i} a\left[\frac{(1+i)^{n}-q^{n}}{(1+i)-q}\right]=a\lim_{q\to 1+i}\left[\frac{-nq^{n-1}}{-1}\right]=a\times n(1+i)^{n-1}$$

We can also come to this conclusion from the relation existing between the final and current values,

$$S_{(a,1+i)\overline{n|}\,i}=A_{(a,1+i)\overline{n|}\,i}(1+i)^{n}=a\times n(1+i)^{-1}(1+i)^{n}=a\times n(1+i)^{n-1}$$

C) <u>Pre-payable income: Current value</u>

$$\ddot{V}(0)=\ddot{A}_{(a,1+i)\overline{n|}\,i}=a\times n(1+i)^{-1}(1+i)=a\times n$$

8.1.5. Perpetual income

There could be 3 different cases:

A) $q>(1+i)$

$$V(0)=\lim_{n\to\infty} A_{(a,q)\overline{n|}\,i}=\lim_{n\to\infty} a\left[\frac{1-q^{n}(1+i)^{-n}}{(1+i)-q}\right]=\lim_{n\to\infty} a\left[\frac{1-\dfrac{q^{n}}{(1+i)^{n}}}{(1+i)-q}\right]=\infty$$

The term $\dfrac{q^{n}}{(1+i)^{n}}$ tends towards infinite as $q>(1+i)$, so that the series is divergent.

B) $q=(1+i)$

$$V(0)=\lim_{n\to\infty} A_{(a,q)\overline{\infty|}\,i}=\lim_{n\to\infty} a\left[\frac{1-q^{n}(1+i)^{-n}}{(1+i)-q}\right]=\lim_{n\to\infty} a\times n(1+i)^{-1}=\infty$$

The series is divergent.

C) $q<(1+i)$

$$V(0)=\lim_{n\to\infty} A_{(a,q)\overline{\infty|}\,i}=\lim_{n\to\infty} a\left[\frac{1-q^{n}(1+i)^{-n}}{(1+i)-q}\right]=\lim_{n\to\infty} a\left[\frac{1-\dfrac{q^{n}}{(1+i)^{n}}}{(1+i)-q}\right]=a\frac{1}{(1+i)-q}$$

When o $n\to\infty$ and $q<(1+i)$, the numerator quotient tends towards zero. Later the series is convergent.

Exercise 8.2. Calculate the final and current value of income of 6,000€ during 15 years valued at 8% in the following cases:

A) Income increases a cumulative 7% every year.

B) Income is considered perpetual and generates a cumulative increment of 4%.

C) Income is considered pre-payable and the annual increment is 8%, always higher than the corresponding quantity of the previous year.

Solution.

A)

$$V(0) = a\frac{1-q^n(1+i)^{-n}}{(1+i)-q} = 6{,}000\frac{1-1.07^{15}(1+0.08)^{-15}}{(1+0.08)-1.07} = 78{,}142.91\ €$$

$$V(n) = a\frac{(1+i)^n - q^n}{(1+i)-q} = 6{,}000\frac{(1+0.08)^{15}-1.07^{15}}{(1+0.08)-1.07} = 247{,}882.54\ €$$

Also:

$$V(n) = 78{,}142.91(1+0.08)^{15} = 247{,}882.54\ €$$

B)

$$V(0) = a\frac{1}{(1+i)-q} = 6{,}000\frac{1}{(1+0.08)-1.04} = 150{,}000\ €$$

C)

$$\ddot{V}(0) = a\times n(1+i)^{-1}(1+i) = a\times n = 6{,}000\times 15 = 90{,}000\ €$$

$$\ddot{V}(n) = a\times n(1+i)^n = 90{,}000(1+0.08)^{15} = 285{,}495.22\ €$$

Exercise 8.3. Calculate the final and current value of 10 payments that have to be made for the purchase of a home if it is known that it amounts to 8,000 € annually, with a cumulative annual increment of 8% for the first 5 years and 7% for the remainder. The valuation rate for this transaction is 7%.

Solution. V(0) = 70,677.12€; V(n) = 139,032.59€

8.2. Variable annual income in arithmetic progression

The diagram of this income is as follows:

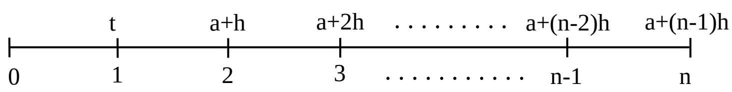
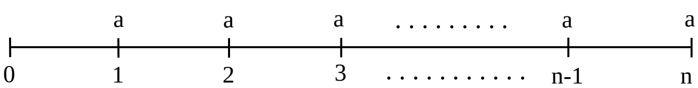


Figure 8.2

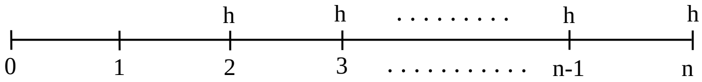
Being "a" the first income term and "h" the reason for the progression, the condition of the last term should be met: $[a+(n-1)h]>0$.

The previous diagram can be broken down into "n" diagrams that represent the "n" deferred and constant income (with the exception of the first) corresponding to the "n" periods in which the income is split.

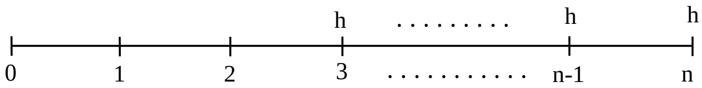
• First income



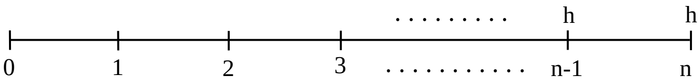
• Second income



• Third income



• Income (n-1)



• Income (n)

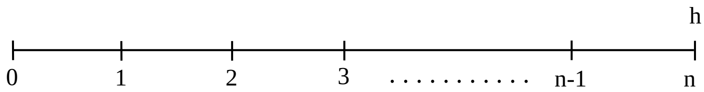


Figure 8.3

The vertical sum of the diagrams of figure 8.3 allows us to obtain the diagram of figure 8.2. Hence, in the first period "a" is generated, in the second period $[a+h]$ is generated, in the third period $[a+h+h=a+2h]$, in the period "n-1" $[a+(n-2)h]$ and in the period "n" $[a+(n-1)h]$ is generated. Therefore, the current value of variable income in arithmetic progression is equal to the sum of the current values of the "n" income that comprise it.

8.2.1. Post-payable income

A) Current value:

$$V(0)=a\,a_{\overline{n|}i}+h\,a_{\overline{n-1|}i}(1+i)^{-1}+h\,a_{\overline{n-2|}i}(1+i)^{-2}+\dots\dots\dots+h\,a_{\overline{1|}i}(1+i)^{-(n-1)}=$$

$$=a\,a_{\overline{n|}i}+h\left[a_{\overline{n-1|}i}(1+i)^{-1}+a_{\overline{n-2|}i}(1+i)^{-2}+\dots\dots\dots+a_{\overline{1|}i}(1+i)^{-(n-1)}\right]=$$

$$=a\,a_{\overline{n|}i}+h\left[(1+i)^{-1}\frac{1-(1+i)^{-(n-1)}}{i}+(1+i)^{-2}\frac{1-(1+i)^{-(n-2)}}{i}+\dots\dots\right.$$

$$\left.\dots\dots\dots\dots+(1+i)^{-(n-1)}\frac{1-(1+i)^{-1}}{i}\right]=$$

$$=a\,a_{\overline{n|}i}+\frac{h}{i}\left[(1+i)^{-1}-(1+i)^{-n}+(1+i)^{-2}-(1+i)^{-n}+\dots\dots\dots+(1+i)^{-(n-1)}-(1+i)^{-n}\right]=$$

$$= a\, a_{\overline{n}|i} + \frac{h}{i}\left[(1+i)^{-1} + (1+i)^{-2} + \dots\dots\dots + (1+i)^{-(n-1)} - (n-1)(1+i)^{-n}\right] =$$

$$= a\, a_{\overline{n}|i} + \frac{h}{i}\left[(1+i)^{-1} + (1+i)^{-2} + \dots\dots\dots + (1+i)^{-(n-1)} - n(1+i)^{-n} + (1+i)^{-n}\right] =$$

$$= a\, a_{\overline{n}|i} + \frac{h}{i}\left[\underbrace{(1+i)^{-1} + (1+i)^{-2} + \dots\dots\dots + (1+i)^{-(n-1)} + (1+i)^{-n}}_{a_{\overline{n}|i}} - n(1+i)^{-n}\right] =$$

$$= a\, a_{\overline{n}|i} + \frac{h}{i}\left[a_{\overline{n}|i} - n(1+i)^{-n}\right] = a\, a_{\overline{n}|i} + \frac{h}{i} a_{\overline{n}|i} - \frac{n\,h(1+i)^{-n}}{i}$$

We add and subtract $\frac{nh}{i}$

$$V(0) = a\, a_{\overline{n}|i} + \frac{h}{i} a_{\overline{n}|i} \underbrace{- \frac{n\,h(1+i)^{-n}}{i} + \frac{n\,h}{i}}_{\text{F.común nh}} - \frac{n\,h}{i}$$

$$V(0) = a\, a_{\overline{n}|i} + \frac{h}{i} a_{\overline{n}|i} + nh\left[\underbrace{\frac{1-(1+i)^{-n}}{i}}_{a_{\overline{n}|i}}\right] - \frac{nh}{i} = a\, a_{\overline{n}|i} + \frac{h}{i} a_{\overline{n}|i} + nh\, a_{\overline{n}|i} - \frac{nh}{i} =$$

$$= \left(a + \frac{h}{i}\right) a_{\overline{n}|i} - \frac{hn}{i}(1+i)^{-n} \qquad (1)$$

We can also grout together the terms as follows.

$$V(0) = A_{(a,h)\overline{n}|i} = a_{\overline{n}|i}\left[a + \frac{h}{i} + nh\right] - \frac{nh}{i}$$

B) <u>Final value</u>:

$$V(n) = V(0)(1+i)^{n} = \left[a_{\overline{n}|i}\left(a + \frac{h}{i} + nh\right) - \frac{nh}{i}\right](1+i)^{n} =$$

$$= a\, S_{\overline{n}|i} + \frac{h}{i} S_{\overline{n}|i} + nh\, S_{\overline{n}|i} - \frac{nh(1+i)^{n}}{i} = S_{\overline{n}|i}\left(a + \frac{h}{i}\right) + nh\left[S_{\overline{n}|i} - \frac{(1+i)^{n}}{i}\right] =$$

$$= S_{\overline{n}|i}\left(a + \frac{h}{i}\right) + nh\left[\frac{(1+i)^{n} - 1}{i} - \frac{(1+i)^{n}}{i}\right] = S_{\overline{n}|i}\left(a + \frac{h}{i}\right) + nh\frac{-1}{i}$$

$$V(n) = S_{(a,h)\overline{n}|i} = S_{\overline{n}|i}\left(a + \frac{h}{i}\right) - \frac{nh}{i}$$

8.2.2. Pre-payable terms

$$\ddot{V}(0) = V(0)(1+i) = \ddot{A}_{(a,h)\overline{n}|i} = \left[\left(a+\frac{h}{i}\right)a_{\overline{n}|i} - \frac{nh}{i}(1+i)^{-n}\right](1+i)$$

8.2.3. Perpetual income

To determine the current value of variable income in arithmetic progression when it has infinite terms, we have to calculate the limit when $n \to \infty$ of $A_{(a,h)\overline{n}|i}$, that is,

$$V(0) = \lim_{n\to\infty}\left[a_{\overline{n}|i}\left(a+\frac{h}{i}+nh\right)-\frac{nh}{i}\right] = \left(a+\frac{h}{i}\right)\lim_{n\to\infty}a_{\overline{n}|i} + \lim_{n\to\infty}\left(a_{\overline{n}|i}nh-\frac{nh}{i}\right)$$

We know that $\lim_{n\to\infty} a_{\overline{n}|i} = \frac{1}{i}$, and that $\lim_{n\to\infty}\left(a_{\overline{n}|i}nh - \frac{nh}{i}\right) = 0$

We substitute,

$$V(0) = \left(a+\frac{h}{i}\right)\frac{1}{i}$$

$$V(0) = A_{(a,h)\overline{\infty}|i} = \left(a+\frac{h}{i}\right)\frac{1}{i} = \frac{a}{i}+\frac{h}{i^2}$$

Exercise 8.4. A company wishes to acquire a machine and its manufacturer offers two equivalent methods of payment:

1ª.- Make a first payment of 12,000 € and another payment of 36,000 € in two years.

2ª.- Pay specific sums of money during 10 years, starting with 6,000 € the first year, increasing this amount by 420 € every year, always on top of the amount of the previous year.

If the machine has a residual value in 12 years of 6,000€ and the market applies a 7% rate for this type of transaction,calculate:

A) The most convenient payment method.

B) Which should be the reason of the progression so that both options are equal?

Solution.

A)

$$V(0)_1 = 12{,}000 + 36{,}000(1+0.07)^{-2} - 6{,}000(1+0.07)^{-12} = 40{,}779.72\,€$$

$$V(0)_2 = a_{\overline{10}|0.07}\left[6{,}000+\frac{420}{0.07}+10\times 420\right] - \frac{10\times 420}{0.07} - 6{,}000(1+0.07)^{-12} = 51{,}117.95$$

B)

$$40{,}779.72 = a_{\overline{10}|0.07}\left[6{,}000+\frac{h}{0.07}+10\times h\right] - \frac{10\times h}{0.07} - 6{,}000(1+0.07)^{-12}$$

$$h = 46.98\,€$$

Exercise 8.5. Calculate the cash value of a property taking into account that a total of 15 payments are to be made. At the beginning, the amount of these payments will be €4,000 and they will increase cumulatively over the first 5 years at a rate of 5%. Over the next 5 years, this cumulative increase will be at a rate of 8% and during the final tranche there will be a linear increase of €200. A transaction interest rate of 8%. Calculate also the value obtained from all these payments. *Solution: V(0) = €48,327.71; V(n) = €153,303.67*

Exercise 8.6. Calculate the value today of the future income of a factory considering that at the beginning they amount to €3,000 and that over the first 6 years they will increase linearly by €500. Over the next 8 years, the increase will be cumulative, at a rate of 3%, and during the remainder of the period there will be a linear increase of €50. Transaction interest rate: 5%. *Solution: €132,336.79*

8.3. Ractioned variable income

When the increase in the income terms is produced every k-th instead of every year, income is valued at the equivalent i_k with duration of $n \times k$ periods.

A) <u>Variable income in geometric progression</u>

$$V(0) = A_{(a,q)\overline{nk}|i_k} = a\,\frac{1-q^{nk}(1+i_k)^{-nk}}{(1+i_k)-q}$$

In the particular case where $q = (1+i_k)$,

$$V(0) = a \cdot n \cdot k(1+i_k)^{-1}$$

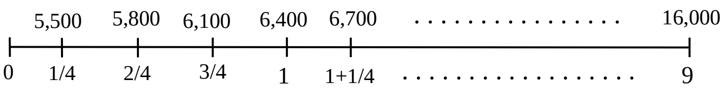
B) <u>Variable income in arithmetic progression</u>

$$V(0) = A_{(a,h)\overline{nk}|i_k} = a_{\overline{nk}|i_k}\left[a + \frac{h}{i_k} + nkh\right] - \frac{nkh}{i_k}$$

If income is pre-payable, it one only needs to be multiplied by $(1+i_k)$

Exercise 8.7. Calculate the current value of the quarterly income of 9 years whose first term amounts to 5,500 € and the reason of quarterly increase is 300€. Annual valuation rate of 8%.

Solution.

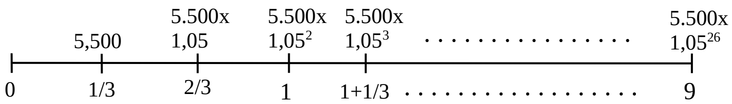


$$i_4 = (1+i)^{1/4} - 1 = (1+0.08)^{1/4} - 1 = 0.01942654$$

$$V(0) = a_{\overline{36}|0.0194265}\left[5{,}500 + \frac{300}{0.0194265} + 36 \times 300\right] - \frac{36 \times 300}{0.0194265} = 260{,}648\ €$$

Exercise 8.8. Repeat the previous exercise in the event that the income increase is 5% cumulative quarterly.

Solution.



$$i_3 = (1+i)^{1/3} - 1 = (1+0.08)^{1/3} - 1 = 0.02598556$$

$$V(0) = 5{,}500 \frac{1 - 1.05^{27}(1+0.02598556)^{-27}}{(1+0.02598556) - 1.05} = 198{,}718.71 \text{ €}$$

8.4. Fractioned variable income by annual blocks

Variable income divided into “blocks” is income the fractioned terms of which increase and these increments are maintained throughout the year, for example, a worker’s salary is constant during the entire year; however, if this salary increases in line with the CPI, the increase will be produced from January to December of every year and, therefore, the increase would have been according to annual blocks. Nevertheless, blocks can also consist of periods of more than one year, such as, for example, the case of a worker whose wages are reviewed every two years; however, a defined percentage is increased every year. In this case the block would be formed by the two-year period.

A) <u>Variable income in geometric progression</u>

• Valuation by annual blocks.

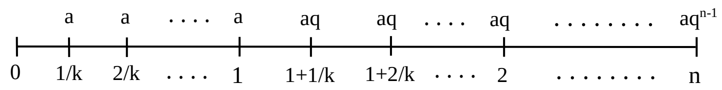


Figure 8.4

The method would be as follows:

a) We value the “k” payments that make up the first year (first block) at its origin.

$$V(0)_1 = a\, a_{\overline{k|}i_k} = a \frac{1-(1+i)^{-1}}{i_k}$$

b) We value the second block at the beginning of the year,

$$V(0)_2 = a\, q\, a_{\overline{k|}i_k} = V(0)_1\, q$$

c) The third block shows the following valuation at the beginning of the third year,

$$V(0)_3 = a\, q^2\, a_{\overline{k|}i_k} = V(0)_1\, q^2$$

If we continue to proceed in this manner, figure 8.4 will be transformed into the diagram in figure 8.5, that is, a pre-payable income of the first term $V(0)_1$, variable in geometric progression of reason "q",

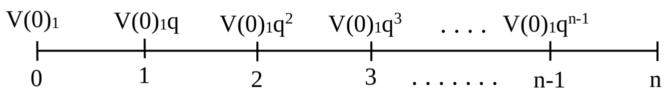


Figure 8.5

Calling $V(0)_T$ the current value of the income in its group,

$$V(0)_T = V(0)_1 \frac{1-q^n(1+i)^{-n}}{(1+i)-q}(1+i) = a\, \mathbf{a}_{\overline{k|}i_k} \frac{1-\left(\frac{q}{1+i}\right)^n}{(1+i)-q}(1+i) \qquad (2)$$

8.4.1. Particular case of indetermination: $\mathbf{q = (1+i)}$

• Valuation by annual blocks.

$$V(0)_1 = a\, \mathbf{a}_{\overline{k|}i_k}$$

$$V(0)_T = \left[V(0)_1 \cdot n(1+i)^{-1}\right](1+i) = V(0)_1 \times n$$

Exercise 8.9. Calculate the current value of the bi-monthly income of 400€ with an increase by annual block of a cumulative 8% rate, if it is going to be earned during 10 years and is valued at a rate of 8%.

Solution.

$$V(0)_1 = 400\, \mathbf{a}_{\overline{6|}0.01290946} = 2{,}295.18\ €$$

$$V(0)_T = \left[2{,}295.18 \times 10(1+0.08)^{-1}\right](1+0.08) = 22{,}951.80\ €$$

Exercise 8.10. Calculate the current value the worker's wage for the next 15 years if we know that the first wage to be earned will amount to 1,200 € each month and the company offers the worker cumulative 7% increments per annual block. Valuation of the transaction at a rate of 5%. *Solution. V(0) = 240,892.05.12 €; V(n) = 500,797.27.59 €*

B) <u>Variable income in arithmetic progression</u>

• Valuation by annual blocks.

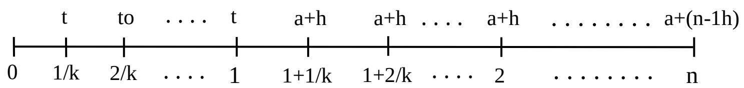


Figure 8.6

The calculation method would be as follows:

a) We value in the "k" payments of the first year (first block) at its origin.

$$V(0)_1 = a\, \mathbf{a}_{\overline{k|}i_k}$$

b) We value the second block at the beginning of the year,

$$V(0)_2 = (a+h)\, \mathbf{a}_{\overline{k|}i_k} = V(0)_1 + h\, \mathbf{a}_{\overline{k|}i_k} = V(0)_1 + V(0)_h$$

c) The third block valued at the origin will be,

$$V(0)_3 = (a+2h)\, \mathbf{a}_{\overline{k|}i_k} = V(0)_1 + 2h\, \mathbf{a}_{\overline{k|}i_k} = V(0)_1 + 2V(0)_h$$

In this way, figure 8.6 is transformed into figure 8.7, that is, a pre-payable income of the first term $V(0)_1$, variable in arithmetic progression of reason $V(0)_h$.

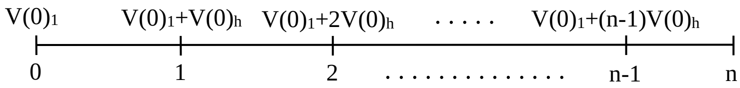


Figure 8.7

Being $V(0)_h = h\,a_{\overline{k}|i_k}$

Referring to the current value of income as a whole as $V(0)_T$,

$$V(0)_T = \left[a_{\overline{n}|i}\left(V(0)_1 + \frac{V(0)_h}{i} + V(0)_h\, n\right) - \frac{V(0)_h\, n}{i}\right](1+i)$$

We can also express this by using the same grouping as in (l), with $a = V(0)_1$ and $h = V(0)_h$.

$$V(0)_T = \left[\left(V(0)_1 + \frac{V(0)_h}{i}\right)a_{\overline{n}|i} - \frac{V(0)_h \times n}{i}(1+i)^{-n}\right](1+i)$$

Exercise 8.11. Calculate the current value of income of 100€ a month that will accrue during 5 years at 6% if there is a 5 € linear increase per annual block every year .

Solution.

$$V(0)_1 = 100\ a_{\overline{12}|0.00486755} = 1{,}162.88\ €$$

$$V(0)_h = 5\ a_{\overline{12}|0.00486755} = 58.14\ €$$

$$V(0)_T = \left[a_{\overline{5}|0.06}\left(1{,}162.88 + \frac{58.14}{0.06} + 58.144 \times 5\right) - \frac{58.14 \times 5}{0.06}\right](1+0.06) = 5{,}681.41\ €$$

Exercise 8.12. Calculate the current value of a variable bi-monthly income in geometric progression with an increase of 4% per annual block. Its first term is 450€, it has duration of 6 years and the valuation rate is 7%.

Solution.

$$V(0)_1 = 450\,a_{\overline{6}|0.01134026} = 2{,}596.00\ €$$

$$V(0)_T = 2{,}596.00\,\frac{1 - 1.04^6(1+0.07)^{-6}}{(1+0.07) - 1.04}(1+0.07) = 14{,}524.18\ €$$

8.5. Variable fractioned perpetual income by annual blocks

A) <u>Variable income in geometric progression</u>

The condition that $q < (1+i)$ has to be met,

- Valuation by annual blocks.

$$V(0)_1 = a\ a_{\overline{k}|i_k}$$

$$V(0)_T = \left[V(0)_1 \frac{1}{(1+i)-q} \right](1+i)$$

B) <u>Variable income in arithmetic progression</u>

• Valuation by annual blocks.

$$V(0)_1 = a\, a_{\overline{k}|i_k} \; ; \; V(0)_h = h\, a_{\overline{1}|i}$$

$$V(0)_T = \left[\frac{V(0)_1}{i} + \frac{V(0)_h}{i^2} \right](1+i)$$

Exercise 8.13. Calculate the current value of a perpetual bi-monthly income of 400€ if a linear increase of 50 € by annual block is applied every year. Valuation at a rate of 8%.

Solution.

$$V(0)_1 = 400\, a_{\overline{6}|0.01290946} = 2{,}295.18\ €$$

$$V(0)_h = 50\, a_{\overline{6}|0.01290946} = 286.89\ €$$

$$V(0)_T = \left[\frac{2{,}295.18}{0.08} + \frac{286.89}{0.08^2} \right](1+0.08) = 79{,}399.15\ €$$

Exercise 8.14. Repeat the previous exercise if the increase by annual block is a cumulative 6%.

Solution.

$$V(0)_1 = 400\, a_{\overline{6}|0.01290946} = 2{,}295.18\ €$$

$$V(0)_T = 2{,}295.18 \frac{1}{(1+0.08)-1.06}(1+0.08) = 123{,}940.14\ €$$

Exercise 8.15. Calculate the updated cost of the salary of a worker for the next 6 years if their first monthly payment amounts to €1,200 and the employer has made a commitment to raise their salary each year cumulatively, by 4%, in the following cases:

a) The valuation is performed based on a 3% rate

b) The valuation is performed based on a 4% rate

c) The contract is permanent and the valuation is performed based on a 5% rate.

Solution: a) €87,121.50; b) €84,589.21; c) €1,472,709.30.

Exercise 8.16. The buyer of a particular tangible fixed asset has agreed with the seller to pay the price in the following way:

- At the end of the following 10 years he will pay increasing amounts of money starting with 1,500€ and increasing every year by 100€.

- From the fifteenth year the instalments will begin to decrease to 400€ every year with respect to the previous year.

At a valuation rate of 10%, calculate:

A) Value in the sixth year of the quantities to be paid to settle the debt.

Nevertheless, after having made the sixth payment, the buyer decides to replace the remaining payments with constant amounts for the same number of payments as those that are still due.

B) What constant amount would have to be paid after the sixth payment in substitution of the variables?

Solution.

	1,500	1,600	. . . . .	2,400	2,400	. . . . .	2,400	2,000	. . . . .	400
0	1	2	. . . . .	10	11	. . . . .	15	16	. . . . .	20

A)

$$\mathrm{V}(6) = s_{\overline{6}|0.1}\left[1{,}500+\frac{100}{0.1}\right]-\frac{6\times 100}{0.1}+a_{\overline{4}|0.1}\left[2{,}100+\frac{100}{0.1}+4\times 100\right]-\frac{4\times 100}{0.1}+$$

$$+2{,}400\,a_{\overline{5}|0.1}(1+0.1)^{-4}+\left[a_{\overline{5}|0.1}\left(2{,}000-\frac{400}{0.1}-5\times 400\right)+\frac{5\times 400}{0.1}\right](1+0.1)^{-9}=$$

$$=13{,}289.025+7{,}094.529+6{,}213.98+2{,}051.297=28{,}648.83\,€$$

B)

$$28{,}648.83 = s_{\overline{6}|0.1}\left[1{,}500+\frac{100}{0.1}\right]-\frac{6\times 100}{0.1}+x\,a_{\overline{14}|0.1}$$

$$x = 2{,}085.03\ €$$

Exercise 8.17. A company is considering the purchase of a hotel that will generate 150,000€ in net revenues at the end of every month during the peak season (June, July, August and September) and 80,000 € in the low season (April, May and October), breaking even for the rest of the year.

It is estimated that these revenues will increase in successive years at a cumulative rate of 7%. Calculate the maximum price that the company would be willing to pay if it hopes to obtain a 10% return on the capital invested.

Solution.

The annual block technique should be used to find the solution to this exercise since constant revenues are not generated during the entire annual block.

$$\mathrm{V}(0)_1 = 150{,}000\,a_{\overline{4}|0{,}00797414}(1+0{,}1)^{-5/12}+$$

$$+80{,},000\left[a_{\overline{2}|0.00797414}(1+0.1)^{-3/12}+(1+0.1)^{-10/12}\right]=793{,}600.04\,€$$

$$\mathrm{V}(0)_{\mathrm{T}} = \left[793{,}600.04\frac{1}{(1+0.1)-1.07}\right](1+0.1)=29{,}098{,}668.37\,€$$

Exercise 8.18. Calculate the current value of the salary that a worker will earn during the next ten years if we know that it amounts to 1,200€ per month during the first year and that it will be increasing at a cumulative rate of 5% per annual block over the coming years. In this operation, a 4% valuation rate is applied for the first five years and 5% for the remaining years

Solution.

$$\mathrm{V}(0)_1 = 1{,}200\,a_{\overline{12}|0.00327374}=14{,}098.20\,€$$

$$\mathrm{V}(0)_{\mathrm{T1}} = 14{,}098.20\frac{1-1.05^5(1+0.04)^{-5}}{1.04-1.05}(1+0.04)=71{,}859.69\,€$$

$$V(0)'_1 = 1{,}00 \times 1.05^5 \, a_{\overline{12}|0.00407412} = 17{,}900.87\,€$$

$$V(0)_{T2} = 17{,}900.87 \times 5(1+0.04)^{-5} = 73{,}566.05\,€$$

$$V(0)_T = 71{,}859.69 + 73{,}566.05 = 145{,}425.74\ €$$

Exercise 8.19. Repeat the previous exercise in the case that the increase per annual block is 120€ on a straight-line basis.

Solution.

$$V(0)_1 = 1{,}200\, a_{\overline{12}|0.00327374} = 14{,}098.20\,€$$

$$V(0)_h = 120\, a_{\overline{12}|0.00327374} = 1{,}409.82\,€$$

$$V(0)_{T1} = \left[a_{\overline{5}|0.04} \left[14{,}098.20 + \frac{1{,}409.82}{0.04} + 5 \times 1{,}409.82 \right] - \frac{1{,}409.82 \times 5}{0.04} \right] (1+0.04) =$$

$$= 77{,}816.18\,€$$

$$V(0)'_1 = 1{,}800\, a_{\overline{12}|0.00407412} = 21{,}038.70\,€$$

$$V(0)'_h = 120\, a_{\overline{12}|0.00407412} = 1{,}402.58\ €$$

$$V(0)_{T2} = \left[a_{\overline{5}|0.05} \left[21{,}038.70 + \frac{1{,}402.58}{0.05} + 5 \times 1{,}402.58 \right] - \frac{1{,}402.58 \times 5}{0.05} \right] \times$$

$$\times (1+0.05)(1+0.04)^{-5} = 88{,}580.31\,€$$

$$V(0)_T = 77{,}816.18 + 88{,}580.31 = 166{,}396.49\ €$$

Exercise 8.20. Calculate what price an investor would have to pay to buy a beach bar considering that its income amounts to €3,000 in July and August, €2,000 in September and October and €1,000 in June. This income is expected to increase each year at a cumulative rate of 3%. Perform the study based on a 4% valuation rate for the following cases:

a) 15-year term.

b) Perpetual income.

Solution: a) €150,317.11; b) €1,114,153.40

Exercise 8.21. Repeat the previous exercise for a scenario whereby there is a linear block annual increase of €500. *Solution: €300,510.28*

Exercise 8.22. Calculate the accumulated value after 10 years of contributions to be made to a pension plan if €100 are to be contributed every two months, with a bimonthly increase of 2%, and the financial institution pays 3% interest. *Solution: a) €12,861.46.*

Exercise 8.23. A construction company sells flats with the payment method being as follows: €9,000 are to be paid upon signing the agreement and a further €20,000 upon completion of construction (scheduled for 15 months from the signing of the agreement). From that moment onwards, 14 instalments are to be made each year, with payments of €750 being made each month, except in the months of June and December when, coinciding with extra salary payments, the instalments to be met will be double this amount. These payments will be made during the 10-year financing period. If the interest rate applied to the financing is 8% annual APR, what would the cash price of the flat? *Solution: €93,321.95*

Exercise 8.24. An insurance company is conducting a study on the expenses to be claimed from another insurance company due to an accident suffered by an insured party resulting in permanent disability. The expenses envisaged are:

- Care workers: €600 monthly with annual CPI increases of 0.5%.
- Adaptation of home: €1,500 every 6 months for 3 years.
- Apparatus for mobility, washing and going to the toilet, etc.: €500 every four months with four-monthly increases of €40.

If the valuation interest rate is 5%, determine the updated value of the above expenses. *Solution: €351,186.65*

Exercise 8.25. Determine the updated salary cost of a worker with a permanent contract taking into consideration that during the first year they earned €1,200 / month and that this salary will be increased based on an annual CPI rate of 2%. The interest rate applied to the valuation shall be 2% for the first 3 years, 3% during the remaining 7 years and 4.5% for the remainder of the period. *Solution: €688,083.06*

9. Amortization: General Case. Loans with Single Repayment

9.1. Amortization. General concepts

A loan agreement is a financial transaction whereby a person called the lender grants capital to another called a borrower, obliging the latter to reimburse this capital in a defined period together with the interest accrued by this capital throughout the term of the agreement.

A the loan agreement may stipulate different terms and conditions for the repayment of capital or for the interest payment method.

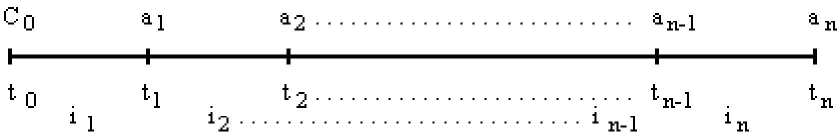


Figure 9.1

The payment is formed by the capital (C_0, t_0). The compensation is formed by $(a_1, t_1), (a_2, t_2), \ldots\ldots\ldots, (a_{n-1}, t_{n-1}), (a_n, t_n)$

We call $a_1, a_2, \ldots\ldots\ldots, a_{n-1}, a_n$ the amortized terms of the periods $(t_0, t_1), (t_1, t_2), \ldots\ldots\ldots, (t_{n-2}, t_{n-1}), (t_{n-1}, t_n)$ and the proceeds $i_1, i_2, \ldots\ldots\ldots, i_{n-1}, i_n$ respectively.

We formulate the equivalence between the payment and compensation at the origin and calculate the reserve by the recurrent, retrospective and prospective methods in an intermediate “s” moment.

Interest could be paid upon maturity or in advance (post-payable interest or pre-payable interest).

9.2. Loan classification

1) According to the method for the payment of interest and the repayment of capital:

A) Loans repaid in a lump-sum payment.

• Loans amortized by means of a single payment of capital and interest: Basic loan transaction.

• Loans repaid in a single reimbursement of capital, paying interest periodically: American System or Amortization Fund.

B) Loans amortized by means of periodical payments of capital and interest, representing income throughout the term of the transaction.

2) According to the amortization system used:

• French system or progressive amortization.
• German system or anticipated interest.
• Instalment system of constant amortization.
• Amortization system with variable annual payments.
• American amortization system (capital reconstruction).
• Loans repaid by means of fractioned income.

3) According to the type or types of interest applied to the loan:

• Loans at reviewable and variable interest.
• Loans at fixed interest.

9.3. General case

9.3.1. Post-payable interest

We can represent the amortization transaction by means of the following diagram:

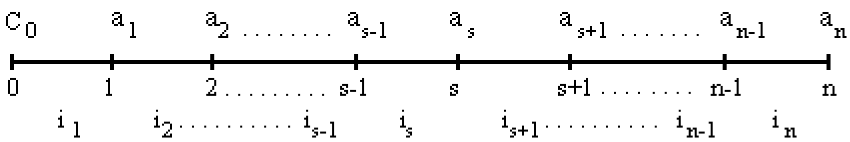


Figure 9.2

A) Equivalence between payment and compensation:

$$C_0 = \sum_{r=1}^{n} a_r \prod_{h=1}^{r} (1+i_h)^{-1}$$

B) Reserve in "S"

• <u>Retrospective Method</u>:

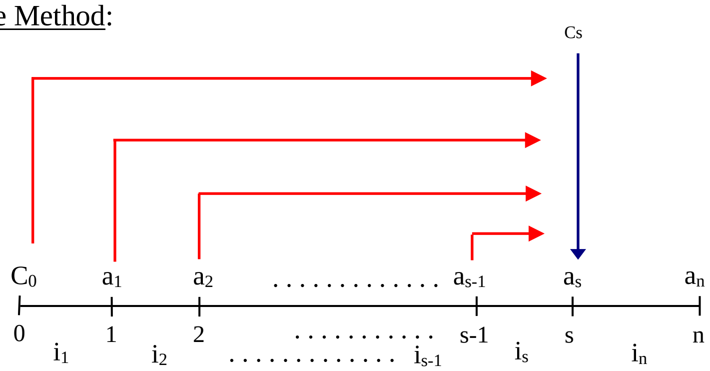


Figure 9.3

$$C_s = C_0 \prod_{h=1}^{s} (1+i_h) - \left[\sum_{r=1}^{s-1} a_r \prod_{h=r+1}^{s} (1+i_h) + a_s \right]$$

- <u>Prospective Method</u>:

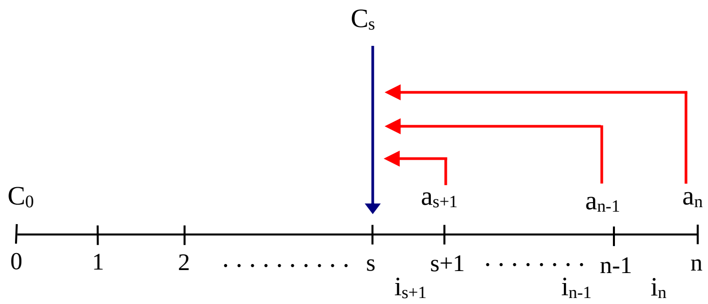


Figure 9.4

$$C_s = \sum_{r=s+1}^{n} a_r \prod_{h=s+1}^{r} (1+i_h)^{-1}$$

- <u>Recurrent Method</u>:

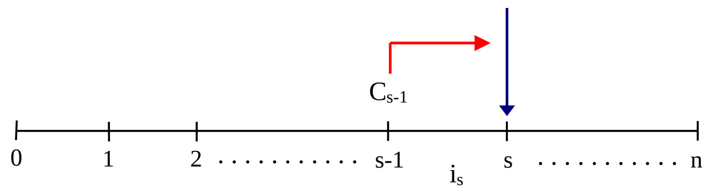


Figure 9.5

$$C_s = C_{s-1}(1+i_s) - a_s$$

$$C_s = C_{s-1} + C_{s-1} i_s - a_s$$

$$a_s = \underbrace{C_{s-1} - C_s}_{A_s} + \underbrace{C_{s-1} i_s}_{I_s}$$

We call:

A_s the instalment of the "s" period (calculated as difference between outstanding capital in two consecutive moments).

I_s the interest payable in period "s".

$$a_s = A_s + I_s$$

On the other hand,

$$A_1 = C_0 - C_1$$
$$A_2 = C_1 - C_2$$
$$\dots\dots\dots\dots\dots\dots$$
$$A_{n-1} = C_{n-2} - C_{n-1}$$
$$A_n = C_{n-1} - C_n$$

We add:

$$\sum_{r=1}^{n} A_r = C_0 - C_1 + C_1 - C_2 + \dots\dots\dots + C_{n-2} - C_{n-1} + C_{n-1} - C_n = C_0$$

Calling M_s the <u>amortized capital</u> until the end of the "s" period,

$$M_s = C_0 - C_s = \sum_{r=1}^{s} A_r$$

C) Theoretical Amortization Table

S	i_s	**Annuity** a_s	**Amortization Quota** A_s	**Quota Interest** I_s	**Amortized Capital** M_s	**Capital Pend. Amortiz.** C_s
0	---	----	----	----	----	C_0
1	i_1	a_1	$A_1 = a_1 - I_1$	$I_1 = C_0 i_1$	$M_1 = A_1$	$C_1 = C_0 - M_1$
2	i_2	a_2	$A_2 = a_2 - I_2$	$I_2 = C_1 i_2$	$M_2 = M_1 + A_2$	$C_2 = C_0 - M_2$
3	i_3	a_3	$A_3 = a_3 - I_3$	$I_3 = C_2 i_3$	$M_3 = M_2 + A_3$	$C_3 = C_0 - M_3$
4	i_4	a_4	$A_4 = a_4 - I_4$	$I_4 = C_3 i_4$	$M_4 = M_3 + A_4$	$C_4 = C_0 - M_4$

Exercise 9.1. A person undertakes to return the following capital units as the repayment of capital and interest over a 4-year period:

1st year: 10,000 € at 6%.

2nd year: 20,000 € at 7%.

3rd year: 5,000€ at 4%.

4th year: 6,000 € at 5%.

The interest is considered as due. Calculate:

1º) Capital loaned.

2º) Amount of the reserve or outstanding capital in the 3rd year by the prospective and retrospective methods.

3º) Create the amortization table.

Solution.

1st)

$$C_0 = \sum_{r=1}^{n} a_r \prod_{h=1}^{r} (1+i_h)^{-1}$$

$$C_0 = 10{,}000\,(1+0.06)^{-1} + 20{,}000\,(1+0.07)^{-1}(1+0.06)^{-1} +$$
$$+ 5{,}000\,(1+0.04)^{-1}(1+0.07)^{-1}(1+0.06)^{-1} +$$

$$+6{,}000\,(1+0.05)^{-1}(1+0.04)^{-1}(1+0.07)^{-1}(1+0.06)^{-1} = 36{,}150.76\text{ €}$$

2nd)

• Retrospective Method:

$$C_3 = 36{,}150.765(1+0.06)(1+0.07)(1+0.04) -$$
$$-[10{,}000(1+0.07)(1+0.04) + 20{,}000(1+0.04)] = 10{,}714.27\text{ €}$$

• Prospective Method:

$$C_3 = 6{,}000\cdot\cdot(1+0.05)^{-1} + 5{,}000 = 10{,}714.27\text{ €}$$

3rd) Amortization table:

S	i_s	a_s	A_s	I_s	M_s	C_s
0		----	----	----	----	36,150.76
1	6%	10,000.00	7,830.95	2,169.04	7,830.95	28,319.80
2	7%	20,000.00	18,017.61	1,982.38	25,848.56	10,302.19
3	4%	5,000.00	4,587.91	412.08	30,436.47	5,714.28
4	5%	6,000.00	5,714.28	285.71	36,150.75	0.00

9.3.2. Pre-payable interest

The diagram of the transaction is as follows:

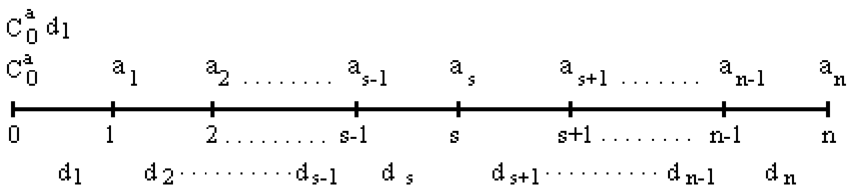


Figure 9.6

Being C_0^a the capital loaned and $d_1, d_2, d_3, \ldots\ldots, d_n$ the anticipated amounts corresponding to periods 1, 2, 3,........,n, respectively.

As shown in the chart, in the "0" period (moment when the loan was granted) the corresponding interests are paid in the first period.

A) Equivalence between payment and compensation:

$$C_0^a(1-d_1) = \sum_{r=1}^{n} a_r \prod_{h=1}^{r}(1-d_h)$$

B) Reserve in "S":

• <u>Retrospective Method</u>:

$$C_s^a = C_s^a d_{s+1} + C_0^a \prod_{h=1}^{s}(1-d_h)^{-1} - \left[C_0^a d_1 \prod_{h=1}^{s}(1-d_h)^{-1} + \sum_{r=1}^{s-1} a_r \prod_{h=r+1}^{s}(1-d_h)^{-1} + a_s\right] =$$

$$=C_s^a d_{s+1}+C_0^a\prod_{h=2}^{s}(1-d_h)^{-1}-\left[\sum_{r=1}^{s-1}a_r\prod_{h=r+1}^{s}(1-d_h)^{-1}+a_s\right]$$

- Prospective Method:

$$C_s^a=C_s^a d_{s+1}+\sum_{r=s+1}^{n}a_r\prod_{h=s+1}^{r}(1-d_h)$$

- Recurrent Method:

$$C_s^a=C_s^a d_{s+1}+C_{s-1}^a-a_s$$

$$C_s^a(1-d_{s+1})=C_{s-1}^a-a_s$$

We calculate the amortized capital by accumulating the amortization instalments,

$$M_s^a=\sum_{r=1}^{s}A_r^a$$

Interest instalments,

$$I_0^a=C_0^a d_1$$

$$I_1^a=C_1^a d_2$$

$$I_n^a=0$$

The annual payments can be calculated by adding the interest and the amortization instalments,

$$a_0=I_0^a$$

$$a_1=A_1^a+I_1^a$$

$$a_2=A_2^a+I_2^a$$

$$a_n=A_n^a$$

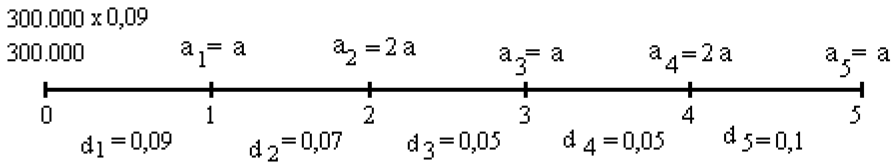
C) Theoretical Amortization Table

S	d_s	**Annuity** a_s	**Amortization Quota** A_s^a	**Interest Quota** I_s^a	**Amortized Capital** M_s^a	**Capital Pend. Amortiz.** C_s^a
0	d_1	$a_0=I_0^a$	----	$I_0^a=C_0^a d_1$	----	C_0^a
1	d_2	$a_1=A_1^a+I_1^a$	$A_1^a=C_0^a-C_1^a$	$I_1^a=C_1^a d_2$	$M_1^a=A_1^a$	$C_1^a=C_0-M_1^a$
2	d_3	$a_2=A_2^a+I_2^a$	$A_2^a=C_1^a-C_2^a$	$I_2^a=C_2^a d_3$	$M_2^a=M_1^a+A_2^a$	$C_2^a=C_0-M_2^a$
3	d_4	$a_3=A_3^a+I_3^a$	$A_3^a=C_2^a-C_3^a$	$I_3^a=C_3^a d_4$	$M_3^a=M_2^a+A_3^a$	$C_3^a=C_0-M_3^a$
4	---	$a_4=A_4^a+I_4^a$	$A_4^a=0$	0	$M_4^a=M_3^a+A_4^a$	$C_5^a=C_0-M_4^a$

Exercise 9.2. A 300,000 € loan is granted to be amortized in five years with anticipated interest payments, the first year at 9%, the second at 7%, the third and fourth at 5% and the fifth at 10%. It is known that the annual payments corresponding to even years will be double the amount corresponding to uneven years. Create the amortization table.

S	d_s	a_s	A_s^a	I_s^a	M_s^a	C_s^a
0	9%	27,000.00	----	27,000.00	----	300,000.00
1	7%	48,563.00	29,637.64	18,925.36	29,637.64	270,362.36
2	5%	97,126.00	88,008.20	9,117.80	117,645.84	182,354.16
3	5%	48,563.00	41,521.36	7,041.64	159,167.20	140,832.69
4	10%	97,126.00	92,269.70	4,856.30	251,436.90	48,563.00
5	---	48,563.00	48,563.00	0.00	300,000.00	0.00

Solution.



$$300{,}000(1-0.09)=a(1-0.09)+2a(1-0.07)(1-0.09)+$$
$$+a(1-0.05)(1-0.07)(1-0.09)+$$
$$+2a(1-0.05)^2(1-0.07)(1-0.09)+a(1-0.1)(1-0.05)^2(1-0.07)(1-0.09)$$

$$a=48{,}563 \text{ € in uneven years: 1, 3, 5}$$

$$2a=97{,}126 \text{ € in even years: 2, 4}$$

Reserves,

- By the prospective method:

$$C_1^a=C_1^a 0.07+97{,}126\ (1-0.07)+48{,}563\ (1-0.05)(1-0.07)+$$
$$+97{,}126\ (1-0.05)^2(1-0.07)+48{,}563\ (1-0.1)(1-0.05)^2(1-0.07)$$
$$C_1^a=270{,}362.36\,€$$

- By the retrospective method:

$$C_1^a=C_1^a d_2+C_0^a-a_1 \quad\Rightarrow\quad C_1^a(1-0.07)=300{,}000-48{,}563$$
$$C_1^a=270{,}362.36\,€$$
$$C_2^a=C_2^a d_3+C_0^a(1-d_2)^{-1}-a_1(1-d_2)^{-1}-a_2$$
$$C_2^a(1-0.05)=300{,}000(1-0.07)^{-1}-48{,}563(1-0.07)^{-1}-97{,}126$$

$$C_2^a = 182,354.07\,€$$

- By the recurrent method:

$$C_1^a(1-0.07) = C_0^a - 48,563 \quad \Rightarrow \quad C_1^a = 270,362.36\,€$$

$$C_2^a = 182,354.07\,€; \quad C_3^a = 140,832.69\,€; \quad C_4^a = 48,563\,€$$

Interest instalments,

$$I_0^a = C_0^a d_1 = 300,000 \times 0.09 = 27,000\,€$$

$$I_1^a = 18,925.36\,€; \quad I_2^a = 9,117.70\,€; \quad I_3^a = 7,041.63\,€; \quad I_4^a = 4,856.3\,€$$

The amortization instalments are calculated by the difference between the reserves,

$$A_1^a = C_0^a - C_1^a = 300,000 - 270,362.36 = 29,637.64\,€$$

$$A_2^a = 88,008.2€; \quad A_3^a = 41,521.36€; \quad A_4^a = 92,269.7€; \quad A_5^a = 48,563€$$

We calculate the amortized capital by accumulating the instalments:

$$M_1^a = A_1^a = 29,637.64\,€$$

$$M_2^a = 117,645.93€; \quad M_3^a = 159,167.3€; \quad M_4^a = 251,437.03€; \quad M_5^a = 300,000\,€$$

9.4. Amortizable loans through single reimbursement.

9.4.1. Loans amortized by means of a single payment including capital and interest: Basic loan transaction.

This loan is characterised because the capital is returned at the end of the life of the loan, together together with the interest accrued in a single payment.

Calling:

a_s : The annuity corresponding to “s” period.

C_0 : The capital loaned.

“i”: The interest rate that we assume as constant

I_s : The interest quota of the period “s”.

A_s : Amortization instalment of the period “s”

C_s : Outstanding capital at the end of period "s".

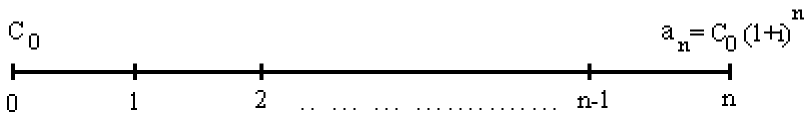


Figure 9.7

The annual payments are all equal to zero, except the last,

$$a_n = C_0(1+i)^n$$

The interest of every period is zero, except for the last period,

$$I_n = a_n - C_0 = C_0(1+i)^n - C_0 = C_0\left[(1+i)^n - 1\right]$$

The amortization instalments:

$$A_1 = A_2 = \dots\dots\dots = A_{n-1} = 0$$

$$A_n = C_0$$

Outstanding capital or pending amortization:

$$C_s = C_0(1+i)^s$$

Exercise 9.3. A loan of 25,000€ is granted that is amortizable through a single reimbursement at 9% in 8 years. Calculate:

A) The amount of the 3rd and 8th annual payments
B) The interest instalment in the 4th and 8th year.

Solution.

A)

$$a_3 = 0$$

$$a_8 = C_0(1+i)^n = 25{,}000(1+0.09)^8 = 49{,}814.06\,€$$

B)

$$I_4 = 0$$

$$I_4 = C_0(1+i)^4 - C_0 = C_0\left[(1+i)^4 - 1\right] = 25{,}000\left[(1+0.09)^4 - 1\right] = 10{,}289.54\,€$$

9.4.2. Amortization through single reimbursement of tcapital, and interest paid periodically: American Loan Operation.

This amortization system is characterised by the periodical payment of interest and amortization at the end of the amortized period.

The annual payment of each period is equal to the interest instalments of the loan for the same period, except for the last one that is equal to the interest instalment plus the amortization quota.

$$a_1 = I_1 = C_0 i_1$$

$$a_2 = I_2 = C_0 i_2$$

$$a_{n-1} = I_{n-1} = C_0 i_{n-1}$$

$$a_n = I_n + C_0 = C_0 i_n + C_0 = C_0(1+i_n)$$

The amortization instalments are all equal except for the last one that is equal to the loaned capital,

$$A_1 = A_2 = \dots\dots\dots = A_{n-1} = 0$$

$$A_n = C_0$$

The interest instalments:

$$I_1 = C_0 i_1$$

$$I_2 = C_0 i_2$$
$$I_n = C_0 i_n$$

Amortized capital:

$$M_1 = 0$$
$$M_2 = 0$$
$$M_n = C_0$$

Capital pending amortization:

$$C_1 = C_2 = \dots\dots\dots = C_{n-1} = 0$$
$$C_n = C_0$$

Theoretical Amortization Table

s	i_s	**Annuity** a_s	**Amortization Quota** A_s	**Interest Quota** I_s	**Amortized Capital** M_s	**Capital Pend. Amortization** C_s
0	--	--	--	--	--	C
1	i_1	$a_1 = I_1$	$A_1 = 0$	$I_1 = C_0 i_1$	$M_1 = 0$	$C_1 = C_0$
2	i_2	$a_2 = I_2$	$A_2 = 0$	$I_2 = C_1 i_2$	$M_2 = 0$	$C_2 = C_0$
3	i_3	$a_3 = I_3$	$A_3 = 0$	$I_3 = C_2 i_3$	$M_3 = 0$	$C_3 = C_0$
4	i_4	$a_4 = C_0 + I_4$	$A_4 = C_0$	$I_4 = C_3 i_4$	$M_4 = C_0$	$C_4 = 0$

The borrower can enter into a transaction to raise capital to ensure that he has the capital on the due of the American loan and to benefit from the interest accrued in the capital procurement transaction since the contributions made to the capital-raising fund could be construed as:

a) A capital-raising transaction, if the capital returns to the person or entity that raises the capital.

b) An amortization transaction, if the purpose of the capital raised is to repay the loan in the event that the loan is according to the American method.

In the event that the borrower decides to enter into a capital-raising transaction with pre-payable constant terms, if we call it i' at the amount of the transaction, the capital-raising term will be determined by,

$$a\,\ddot{S}_{\overline{n}|i'} = C \quad \Rightarrow \quad a = \frac{C}{\ddot{S}_{\overline{n}|i'}}$$

Capital raised up to the moment "s",

$$C^{-}{}_{s} = a\,\ddot{S}_{\overline{s}|i'}$$

Exercise 9.4. Create the amortization table for a 100,000 € loan amortizable in 5 years at 10% annually for the first two years and at 12% annually the remaining three by the American method.

Solution.

S	i_s	a_s	A_s	I_s	M_s	C_s
0		----	----	----	----	100,000.00
1	10%	10,000.00	0.00	10,000.00	0.00	100,000.00
2	10%	10,000.00	0.00	10,000.00	0.00	100,000.00
3	12%	12,000.00	0.00	12,000.00	0.00	100,000.00
4	12%	12,000.00	0.00	12,000.00	0.00	100,000.00
5	12%	112,000.00	100,000.00	12,000.00	100,000.00	0.00

Exercise 9.5. A person requests a 50,000 € loan amortizable in 4 years at 9% by the American method. At the same time, the person goes to another financial entity to raise capital for he same amount and term but at 5%, through constant pre-payable deposits. Create the capital-raising and amortization tables.

Solution.

- **American loan**:

S	a_s	A_s	I_s	M_s	C_s
0	----	----	----	----	50,000.00
1	4,500.00	0.00	4,500.00	0.00	50,000.00
2	4,500.00	0.00	4,500.00	0.00	50,000.00
3	4,500.00	0.00	4,500.00	0.00	50,000.00
4	54,500.00	50,000.00	4,500.00	50,000.00	0.00

- **Capital-raising transaction**:

S	a_s	$\bar{A}_s$	$\bar{I}_s$	$\bar{C}_s$	$\bar{M}_s$
0	0.00	0.00	0.00	0.00	50,000.00
1	11,048.18	11,600.58	552.41	11,600.58	38,399.41
2	11,048.18	12,180.61	1,132.43	23,781.19	26,218.80
3	11,048.18	12,789.64	1,741.46	36,570.83	13,429.16
4	11,048.18	13,429.15	2,380.95	50,000.00	0.00

$$50{,}000 = a\ddot{S}_{\overline{4|}0.05} \quad \Rightarrow \quad a = 11{,}048.18\,€$$

Column $\bar{M}_s$ shows the <u>combined balance</u> of the borrower's two transactions.

10. Amortization of Constant Revenue Loans

10.1. French or progressive amortization system

This amortization method is characterised by annuities and post-payable revenue. Likewise, the interest instalments decrease during the amortized period, as these are calculated for every period according to the outstanding balance at the beginning of the period immediately before. For this same reason, and due to the fact that the annual payments are constant, the amortization instalments increase throughout the entire amortization period. Accordingly, this amortization system is also known as the "progressive amortization system".

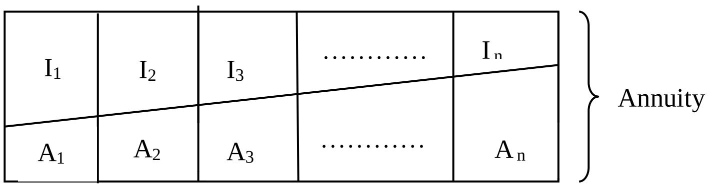


Figure 10.1

We call:

C_0 : Loaned capital.

a_s : Constant annual repayment.

A_s : Amortization instalment of the "s" period

I_s : Interest instalment of the "s" period

M_s : Amortized capital at the end of "s" period

C_s : Outstanding capital at the end of the "s" period (start of the "s+1" period).

$$a_1 = a_2 = a_3 = \dots\dots\dots = a_{n-1} = a_n = a$$
$$i_1 = i_2 = i_3 = \dots\dots\dots = i_{n-1} = i_n = i$$

Graphically,

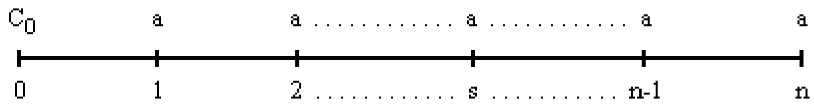


Figure 10.2

A) Equivalence between payment and compensation

$$C_0 = a\,a_{\overline{n|}i} \quad \Rightarrow \quad a = \frac{C_0}{a_{\overline{n|}i}}$$

B) Reserve in “S”

● <u>Retrospective Method:</u>

$$C_s = C_0(1+i)^s - a\,s_{\overline{s|}i}$$

● <u>Prospective Method:</u>

$$C_s = a\,a_{\overline{n-s|}i}$$

● <u>Recurrent Method:</u>

Outstanding capital in “s”: $C_s = C_{s-1}(1+i) - a$

Outstanding capital in “s+1”: $C_{s+1} = C_s(1+i) - a$

We operate and reflect "a" in the equation corresponding to the outstanding capital in "s",

$$a = \underbrace{C_{s-1} - C_s}_{A_s} + \underbrace{C_{s-1}i}_{I_s}$$

$$a = A_s + I_s$$

The annual payment for the "s" period is divided into the amortization instalment and the interest instalment

We subtract C_s and C_{s+1}:

$$\underbrace{C_s - C_{s+1}}_{A_{s+1}} = \left(\underbrace{C_{s-1} - C_s}_{A_s}\right)(1+i)$$

$$A_{s+1} = A_s(1+i)$$

The amortization instalments follow a geometric progression of reason (1+i), and since the sum of the amortization instalments is equal to the loaned capital, $\sum_{r=1}^{n} A_r = C_0$ we will have to,

$$A_1 + A_1(1+i) + A_1(1+i)^2 + ... + A_1(1+i)^{n-1} = C_0$$

$$A_1\left[\underbrace{1+(1+i)+(1+i)^2+..........+(1+i)^{n-1}}_{\text{Increasing Geometric Progression}}\right]=C_0$$

$$A_1 s_{\overline{n}|i}=C_0 \quad \Rightarrow \quad A_1=\frac{C_0}{s_{\overline{n}|i}}$$

The <u>amortized capital</u> at the end of the "s" period (in the "s" first years) (M_s)is calculated by the difference between the loaned capital and the outstanding capital, that is,

$$M_s=C_0-C_s=C_0-\sum_{r=s+1}^{n}A_r=C_0-\left[A_{s+1}+A_{s+2}+..........+A_n\right]$$

Applying the instalment recurrence law by which $A_{s+1}=A_1(1+i)^s$, we express all the instalments according to A_1:

$$M_s=C_0-A_1(1+i)^s\left[1+(1+i)+(1+i)^2+..........+(1+i)^{n-1-s}\right]=$$

$$=C_0-\frac{C_0}{s_{\overline{n}|i}}(1+i)^s s_{\overline{n-s}|i}=C_0\left[1-\frac{s_{\overline{n-s}|i}}{s_{\overline{n}|i}}(1+i)^s\right]$$

S is expressed according to the current value of revenue,

$$M_s=C_0\left[1-\frac{a_{\overline{n-s}|i}(1+i)^{n-s}(1+i)^s}{a_{\overline{n}|i}(1+i)^n}\right]=C_0\left[1-\frac{a_{\overline{n-s}|i}(1+i)^n}{a_{\overline{n}|i}(1+i)^n}\right]$$

$$M_s=C_0\left[1-\frac{a_{\overline{n-s}|i}}{a_{\overline{n}|i}}\right]$$

We can also calculate M_s based on the recurrence law followed by the amortization instalments, hence

$M_1=A_1$

$M_2=M_1+A_2=A_1+A_2=A_1+A_1(1+i)=A_1[1+(1+i)]=A_1 s_{\overline{2}|i}$

$M_3=M_2+A_3=A_1[1+(1+i)]+A_1(1+i)^2=A_1\left[1+(1+i)+(1+i)^2\right]=A_1 s_{\overline{3}|i}$

$M_s=M_{s-1}+A_s=A_1\left[1+(1+i)+.....+(1+i)^{s-2}\right]+A_1(1+i)^{s-1}=$

$=A_1\left[1+(1+i)+(1+i)^2+.....+(1+i)^{s-1}\right]=A_1 s_{\overline{s}|i}$

Therefore,

$$M_s=A_1 s_{\overline{s}|i}$$

C) Theoretical Amortization Table

S	a_s	A_s	I_s	M_s	C_s
0	----	----	----	----	C_0
1	$a = \dfrac{C_0}{a_{\overline{n}\rceil i}}$	$A_1 = a - I_1$	$I_1 = C_0 i$	$M_1 = A_1$	$C_1 = C_0 - M_1$
2	to	$A_2 = a - I_2$	$I_2 = C_1 i$	$M_2 = M_1 + A_2$	$C_2 = C_0 - M_2$
3	to	$A_3 = a - I_3$	$I_3 = C_2 i$	$M_3 = M_2 + A_3$	$C_3 = C_0 - M_3$
4	to	$A_4 = a - I_4$	$I_4 = C_3 i$	$M_4 = M_3 + A_4$	$C_4 = C_0 - M_4$

We can also calculate the column corresponding to the outstanding capital or capital pending amortization by the difference between the outstanding capital in the previous period and the amortization instalment of this period,

$$C_1 = C_0 - A_1;\; C_2 = C_1 - A_2;\; C_n = C_{n-1} - A_n$$

Exercise 10.1. Create the amortization schedule according to the French method of a 28.000€ loan amortizable in 5 years at 8%.

Solution

S	a_s	A_s	I_s	M_s	C_s
0	----	----	----	----	28,000.00
1	7,012.78	4,772.78	2,240.00	4,772.78	23,227.21
2	7,012.78	5,154.60	1,858.17	9,927.38	18,072.61
3	7,012.78	5,566.97	1,445.80	15,494.35	12,505.64
4	7,012.78	6,012.32	1,000.45	21,506.67	6,493.32
5	7,012.78	6,493.32	519.46	28,000.00	0.00

$$C_0 = a\, a_{\overline{n}\rceil i} \quad \Rightarrow \quad 28{,}000 = a\, a_{\overline{5}\rceil 0.08}$$

$$a = 7{,}012.78 \;\text{€}$$

Exercise 10.2. A person requests a 32.500€ loan amortizable in 12 years at 9% by the French method. Calculate:

Constant annual loan repayments.

B) Interest instalment of the 8th year.

C) Amortization instalment of the 9th year.

D) Amortized capital after payment of the 6th annuity.

E) Capital pending amortization at the beginning of the 4th year (end of the 3rd year).

Solution:

A)

$$C_0 = a\,a_{\overline{n}|i} \quad \Rightarrow \quad 32{,}500 = a\,a_{\overline{12}|0.09}$$

$$a = \frac{32{,}500}{7.160725277} = 4{,}538.64€$$

B)

$$C_s = a\,a_{\overline{n\text{-}s}|i} \quad \Rightarrow \quad C_7 = 4{,}538.64\,a_{\overline{5}|0.09} = 17{,}653.72€$$

$$I_8 = C_7 i = 17{,}653.72 \times 0.09 = 1{,}588.83€$$

C)

$$A_s = A_{s-1}(1+i) = A_1(1+i)^{s-1}$$

$$A_1 = a - I_1 = 4{,}538.64 - 32{,}500 \times 0.09 = 1{,}613.64€$$

$$A_9 = A_1(1+i)^8 = 1{,}613.64(1+0.09)^8 = 3{,}215.27€$$

D)

$$M_s = C_0\left[1 - \frac{a_{\overline{n-s}|i}}{a_{\overline{n}|i}}\right] \quad \Rightarrow \quad M_6 = C_0\left[1 - \frac{a_{\overline{6}|0.09}}{a_{\overline{12}|0.09}}\right] = 12{,}140€$$

Also,

$$M_6 = A_1 s_{\overline{6}|i} = 1{,}613.64\,s_{\overline{6}|0.09} = 12{,}140€$$

E)

$$C_3 = C_0 - M_3 = 32{,}500 - 32{,}500\left[1 - \frac{a_{\overline{9}|0.09}}{a_{\overline{12}|0.09}}\right] = 27{,}210.30€$$

Also,

$$C_3 = a\,a_{\overline{9}|0{,}09} = 4{,}538.64\,a_{\overline{9}|0.09} = 27{,}210.3€$$

Also,

$$C_3 = C_0(1+0.09)^3 - a\,s_{\overline{3}|0.09} = 32{,}500(1+0.09)^3 - 4{,}538.64\,s_{\overline{3}|0.09} = 27{,}210.32€$$

10.2. German amortization system or anticipated interests

This amortization system is characterised by the anticipated payment of interest, amortizing at the end of each period the corresponding instalment for this period plus de interest of the following period.

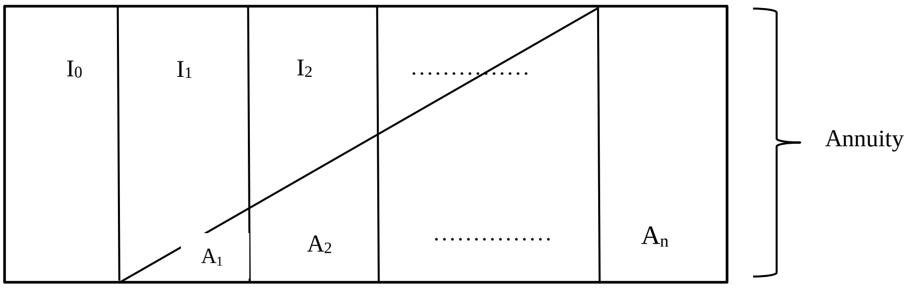


Figure 10.3

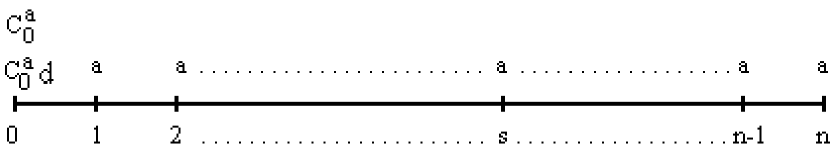


Figure 10.4

The amortization terms (annuities) are constant, as well as the anticipated returns,

$$a_1 = a_2 = a_3 = \ldots\ldots\ldots. = a_n = a$$

$$d_1 = d_2 = d_3 = \ldots\ldots\ldots = d_n = d$$

With "d" referring to the anticipated interest, $C_0^a d$ the anticipated interest of the "0" period.

We call the loaned nominal capital C_0^a.

We formulate the equivalence at the origin to determine the value of the constant annual payment to amortize the loan:

$$C_0^a = C_0^a d + a(1-d) + a(1-d)^2 + \ldots\ldots\ldots + a(1-d)^n =$$

$$= C_0^a d + a(1-d)\left[\underbrace{1 + (1-d) + (1-d)^2 \ldots\ldots\ldots + (1-d)^{n-1}}_{\text{Decreasing Geometric Progression}}\right] =$$

$$= C_0^a d + a(1-d)\left[\frac{1-(1-d)^{n-1}(1-d)}{1-(1-d)}\right] = C_0^a d + a(1-d)\left[\frac{1-(1-d)^n}{d}\right]$$

$$C_0^a(1-d) = a(1-d)\left[\frac{1-(1-d)^n}{d}\right] \quad \Rightarrow \quad a = \frac{C_0^a(1-d)}{\frac{1-(1-d)^n}{d}(1-d)}$$

If we call $a^a_{\overline{n|}d} = \frac{1-(1-d)^n}{d}$,

$$C_0^a = a\, a^a_{\overline{n|}d} \qquad a = C_0^a \frac{d}{1-(1-d)^n}$$

The interest is calculated:

$$I_n^a = 0$$
$$I_{n-1}^a = C_{n-1}^a d = a - A_{n-1}^a$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$I_1^a = C_1^a d = a - A_1^a$$
$$I_0^a = C_0^a d$$

Since the interest of period "n" is the same as zero, the amortization instalment and the annual paymentwill coincide:

$$a = A_n^a$$

We break down the annual payment by years:

1st year: $a = A_1^a + I_1^a = A_1^a + C_1^a d$

2nd year: $a = A_2^a + I_2^a = A_2^a + C_2^a d$

Year "n-2": $a = A_{n-2}^a + I_{n-2}^a = A_{n-2}^a + C_{n-2}^a d$

Year "n-1": $a = A_{n-1}^a + I_{n-1}^a = A_{n-1}^a + C_{n-1}^a d$

Year "n-2": $a = A_n^a$

We look for an expression that interrelates the amortization instalments and makes it possible for us to arrive at the recurrence law for its calculation. . Since the annuities are constant, we equalise those corresponding to the $(n-1)$ and $(n-2)$:

$$A_{n-1}^a + C_{n-1}^a d = A_{n-2}^a + C_{n-2}^a d$$

$$A_{n-1}^a - A_{n-2}^a = C_{n-2}^a d - C_{n-1}^a d$$

$$A_{n-1}^a - A_{n-2}^a = d\left(\underbrace{C_{n-2}^a - C_{n-1}^a}_{A_{n-1}^a}\right)$$

$$A_{n-2}^a = A_{n-1}^a - dA_{n-1}^a$$

$$A_{n-2}^a = A_{n-1}^a(1-d)$$

If we equalise the annuities of the "n-1" and "n" periods.

$$A_n^a = A_{n-1}^a + C_{n-1}^a d$$

$C_{n-1}^a = a$, therefore,

$$A_n^a = A_{n-1}^a + ad$$

But $a = A_n^a$ (as $I_n = 0$) therefore,

$$A_n^a = A_{n-1}^a + A_n^a d$$

$$A^a_{n-1} = A^a_n - A^a_n d = A^a_n (1-d)$$

$$A^a_{n-1} = A^a_n (1-d)$$

Expressing all of them according to the last one:

$$\begin{aligned}
&A^a_{n-1} = A^a_n (1-d) \\
&A^a_{n-2} = A^a_{n-1}(1-d) = A^a_n (1-d)(1-d) = A^a_n (1-d)^2 \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&A^a_1 = A^a_2 (1-d) = A^a_n (1-d)^{n-1}
\end{aligned}$$

Based on this relation and knowing that $a = A^a_n$ we can calculate the interest instalment for the “s” periodaccording to the annual payment,

$$I^a_s = a - A^a_s = a - a(1-d)^{n-s} = a\left[1-(1-d)^{n-s}\right]$$

We calculate the outstanding capital as follows:

$$C^a_s = \sum_{r=s+1}^{n} A^a_r = A^a_{s+1} + A^a_{s+2} + \dots\dots\dots + A^a_{n-1} + A^a_n$$

We express all the instalments according to the instalment corresponding to year "n":

$$C^a_s = A^a_n + A^a_n(1-d) + A^a_n(1-d)^2 + \dots\dots\dots + A^a_n(1-d)^{n-s-1} =$$

$$= A^a_n \left[\underbrace{1 + (1-d) + (1-d)^2 + \dots\dots + (1-d)^{n-s-1}}_{\text{Progresión Geométrica Decreciente}} \right] =$$

$$= A^a_n \left[\frac{1-(1-d)^{n-s-1}(1-d)}{1-(1-d)} \right] = A^a_n \left[\frac{1-(1-d)^{n-s}}{d} \right] = a \left[\frac{1-(1-d)^{n-s}}{d} \right] \quad C^a_0 = a\, a^a_{\overline{n-s}|d}$$

If we want to express C^a_s according to C^a_0, we sustitute “a” by the value arrived at by comparing the payment and the compensation at the origin:

$$C^a_s = C^a_0 \frac{1}{a^a_{\overline{n}|d}} a^a_{\overline{n-s}|d} = C^a_0 \frac{d}{1-(1-d)^n} \left[\frac{1-(1-d)^{n-s}}{d} \right] = C^a_0 \left[\frac{1-(1-d)^{n-s}}{1-(1-d)^n} \right]$$

We calculate the <u>amortized capital</u> :

$$M^a_s = C^a_0 - C^a_s = C^a_0 - C^a_0 \left[\frac{1-(1-d)^{n-s}}{1-(1-d)^n} \right] = C^a_0 \left[1 - \frac{1-(1-d)^{n-s}}{1-(1-d)^n} \right] =$$

$$= C^a_0 \left[\frac{1-(1-d)^n - 1 + (1-d)^{n-s}}{1-(1-d)^n} \right] = C^a_0 \left[\frac{(1-d)^{n-s} - (1-d)^n}{1-(1-d)^n} \right]$$

Theoretical Amortization Table

S	a_s	A_s^a	I_s^a	M_s^a	C_s^a
0			$I_0^a = C_0^a d$		C_0^a
1	a	$A_1^a = A_2^a(1-d)$	$I_1^a = a - A_1^a = C_1^a d$	$M_1^a = A_1^a$	$C_1^a = C_0 - M_1^a$
2	a	$A_2^a = A_3^a(1-d)$	$I_2^a = a - A_2^a = C_2^a d$	$M_2^a = M_1^a + A_2^a$	$C_2^a = C_0 - M_2^a$
3	a	$A_3^a = A_4^a(1-d)$	$I_3^a = a - A_3^a = C_3^a d$	$M_3^a = M_2^a + A_3^a$	$C_3^a = C_0 - M_3^a$
4	a	$A_5^a = a$	0	$M_5^a = M_4^a + A_5^a$	$C_5^a = C_0 - M_5^a$

The value of "a" is obtained from the equivalence between payment and compensation.

Exercise 10.3. Present the amortization table for a 6,450€ loan amortizable in 4 years at 8% by the German method.

Solution:

S	a_s	A_s^a	I_s^a	M_s^a	C_s^a
0	516.00	0.00	516.00		6,450.00
1	1,819.42	1,416.76	402.65	1,416.76	5,033.24
2	1,819.42	1,539.95	279.46	2,956.71	3,493.29
3	1,819.42	1,673.86	145.55	4,630.57	1,819.43
4	1,819.42	1,819.42	0.00	6,450.00	0.00

Calculation of the annuity,

$$a = C_0^a \frac{d}{1-(1-d)^n} = 6{,}450 \frac{0.08}{1-(1-0.08)^4} = 1{,}819.42\,€$$

$$a = A_4^a = 1{,}1819.42\,€$$

$$A_3^a = A_4^a(1-d) = 1{,}1819.42(1-0.08) = 1{,}673.86\,€$$

$$A_2^a = A_3^a(1-d) = A_4^a(1-d)^2 = 1{,}673.86(1-0.08) = 1{,}539.95$$

$$A_1^a = A_2^a(1-d) = 1{,}539{,}95(1-0.08)1{,}416.76\,€$$

The interest is calculated by the difference between the annuities and the amortization instalments, except for the instalment corresponding to the initial moment.

$$I_0^a = C_0^a d = 6{,}450 \times 0.08 = 516\,€$$

$$I_1^a = a - A_1^a = 1{,}819.42 - 1{,}416.76 = 402.66\,€$$

Adter calculating the column corresponding to A_s^a, we can also calculate the amortized capital (M_s^a), then the outstanding capital (C_s^a) and, on the basis of the latter, we can calculate the interest.

We can also calculate the annual payment considering that,

$$i = \frac{d}{1-d} \quad \Rightarrow \quad d = \frac{i}{1+i}$$

We subtract the two members of the second equivalence from 1

$$(1-d) = 1 - \frac{i}{1+i} = (1+i)^{-1}$$

We consider the equation of the equivalence between payment and compensation at the origin by substituting (l - d) with ($(1+i)^{-1}$,

$$C_0^a(1-d) = a\left[(1+i)^{-1} + (1+i)^{-2} + ... + (1+i)^{-n}\right]$$

$$C_0(1+i)^{-1} = a\, a_{\overline{n}|i}$$

Or,

$$C_0(1-d) = a\, a_{\overline{n}|i} \quad \Rightarrow \quad a = \frac{C_0(1-d)}{a_{\overline{n}|i}}$$

Exercise 10.4. Calculate the annual payment to repay a 200,000 € loan by the German method in 8 years at the anticipated rate of 10%.

Solution:

$$a = C_0^a \frac{d}{1-(1-d)^n} = 200{,}000 \frac{0.1}{1-(1-0.1)^8} = 35{,}116.50€$$

Or, knowing that $i = \frac{0,1}{1-0.1} = 0.111111$

$$a = \frac{C_0(1-d)}{a_{\overline{n}|i}} = \frac{200{,}000(1-0.1)}{a_{\overline{8}|0.1111111}} = 35{,}116.50€$$

10.3 Particular case: Deferment (grace period)

In certain types of transactions, the repayment of the loan often does not start until after a certain period. This would be a typical case of a loan granted to a construction company that will not begin to repaythe loan until the project has been build completely. It would also be the case of the sales promotion of vehicles such as "buy a car now and start to pay for it in three months".

The period between the time that the loan is granted until the loan starts to be repaid is known as "deferment".

During the deferment period, two things can occur:

1st) That the interest is paid periodically. This would be the most frequent case.

2nd) That no interest is paid during the deferment period.

1st Case:

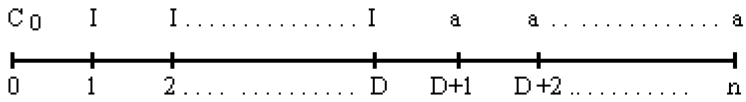


Figure 10.5

In this case, if the loanagreemnt contemplates a term of "n" periods, we have to keep in mind that amortization will only take place in the "n-D" periods, therefore the approach taken in the French and German method remains unchanged. We will only treat the deferment for the French amortization method.

10.3.1. French Method

To calculate the value of the annual loan repayment, we consider the equivalence between payment and the compensation at the origin,

$$C_0 = C_0\, i\, a_{\overline{D}|i} + a\, a_{\overline{n-D}|i}(1+i)^{-D}$$

$$C_0 = C_0\, i\,\frac{1-(1+i)^{-D}}{i} + a\, a_{\overline{n-D}|i}(1+i)^{-D}$$

$$C_0(1+i)^{-D} = a\, a_{\overline{n-D}|i}(1+i)^{-D}$$

$$C_0 = a\, a_{\overline{n-D}|i} \quad \Rightarrow \quad a = \frac{C_0}{a_{\overline{n-D}|i}}$$

Theoretical Amortization Table

We consider the amortization schedule according to the the French system with a duration of five periods and two deferment periods

Calling "D" the deferment periods,

S	a_s	A_s	I_s	M_s	C_s
0	----	----	----	----	C_0
D	$a_D = C_0 i$	----	$I_D = C_0 i$	----	C_0
D	$a_D = C_0 i$	----	$I_D = C_0 i$	----	C_0
1	$a = \dfrac{C_0}{a_{\overline{3}|i}}$	$A_1 = a - I_1$	$I_1 = C_0 i$	$M_1 = A_1$	$C_1 = C_0 - M_1$
2	to	$A_2 = a - I_2$	$I_2 = C_1 i$	$M_2 = M_1 + A_2$	$C_2 = C_0 - M_2$
3	to	$A_3 = a - I_3$	$I_3 = C_2 i$	$M_3 = M_2 + A_3$	$C_3 = C_0 - M_3$

Exercise 10.5. Create the table according to the French method for a 50,000€ loan amortizable in five years at 12% annually with the first two years as a grace period.

Solution:

S	a_s	A_s	I_s	M_s	C_s
0	----	----	----	----	50,000.00
1	6,000.00	0.00	6,000.00	0.00	50,000.00
2	6,000.00	0.00	6,000.00	0.00	50,000.00
3	20,817.45	14,817.45	6,000.00	14,817.45	35,182.55
4	20,817.45	16,595.54	4,221.90	31,413.00	18,587.00
5	20,817.45	18,587.00	2,230.44	50,000.00	0.00

2nd Case:

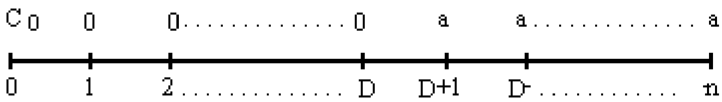


Figure 10.6

The equivalence between payment and compensation will be defined by:

$$C_0 = a\,a_{\overline{n-D}|i}(1+i)^{-D}$$

$$a = \frac{C_0(1+i)^D}{a_{\overline{n-D}|i}}$$

The remainder the loan is the same as the one studied in the French method.

Exercise 10.6. Create the amortization schedule by the French method for a 100,000€ loan amortizable in five years at 10% annually with two grace periods.

Solution:

$$100.000(1+0,10)^2 = a\,a_{\overline{3}|0,10}$$

S	a_s	A_s	I_s	M_s	C_s
0	----	----	----	----	100,000.00
1	0.00	0.00	0.00	0.00	110,000.00
2	0.00	0.00	0.00	0.00	121,000.00
3	48,655.89	36,555.89	12,100.00	36,555.89	84,444.10
4	48,655.89	40,211.48	8,444.41	76,767.37	44,232.62
5	48,655.89	44,232.62	4,423.26	121,000.00	0.00

10.4. Loans with accrued interest and annual amortization: French method

In this case the loan is amortized once a year with periodical half-yearly, monthly, etc., interest payments.

10.4.1. According to the Fractioned Amount

If we assume that the payment of interest is bi-monthly, the diagram of the 3-year transaction is as follows,

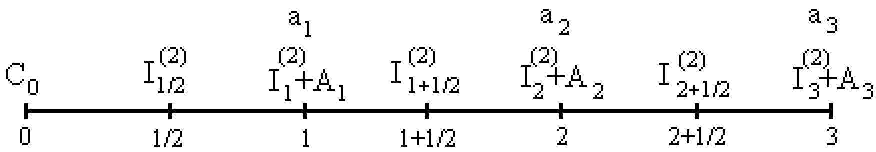


Figure 10.7

Calculate the components of the amortization table:

We calculate the annual instalments by adding up the interest and the amortization instalments:

$$a_{1/k} = I_{1/k}^{(k)}$$

$$a_{2/k} = I_{2/k}^{(k)}$$

$$a_1 = I_1^{(k)} + A_1$$

If we capitalise the interest of every fraction until the instalment is paid, we verify that making the payments for each of the fractions in which the year was divided is the same as making a single lump payment at the end of this period.

Assuming there is only one annual period divided into "k" fractions with its corresponding interest:

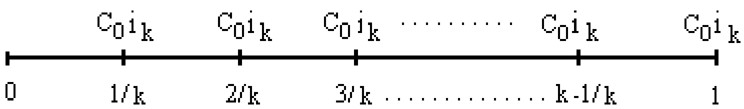


Figure 10.8

Calculate the acquired value in the moment "1"

$$V(1) = C_0 i_k + C_0 i_k (1+i)^{1/k} + C_0 i_k (1+i)^{2/k} + \dots\dots + C_0 i_k (1+i)^{k-1/k} =$$

$$= C_0 i_k \left[1 + (1+i)^{1/k} + (1+i)^{2/k} + \dots\dots + (1+i)^{k-1/k}\right] =$$

$$= C_0 i_k \left[\frac{(1+i)^{k-1/k}(1+i)^{1/k} - 1}{(1+i)^{1/k} - 1}\right] = C_0 i_k \left[\frac{i}{(1+i)^{1/k} - 1}\right] =$$

$$= C_0 i_k \frac{i}{i_k} = C_0 i$$

Therefore,

$$C_0 i_k s_{\overline{k}|i_k} + A_1 = a_1$$

$$C_0 i_k \frac{(1+i_k)^k - 1}{i_k} + A_1 = a_1$$

For the second annuity,

$$C_1 i_k s_{\overline{k}|i_k} + A_2 = a_2$$

The amortization instalments

$$A_1 = \frac{C_0}{s_{\overline{n}|i}}$$

$$A_2 = A_1(1+i)$$

$$A_n = A_{n-1}(1+i)$$

To find the interest instalments, firstly we will have to calculate the actual equivalent amount through the expression,

$$(1+i) = (1+i_k)^k \quad \Rightarrow \quad i_k = (1+i)^{1/k} - 1$$

$$I^{(k)}_{1/k} = C_0 i_k = I^{(k)}_{2/k} = ... = I^{(k)}_1$$

$$I^{(k)}_{1+1/k} = C_1 i_k$$

$$I^{(k)}_n = C_{n-1} i_k$$

No difficulties will be encountered when calculating amortized and outstanding capital.

• **Theoretical Amortization schedule for a three year loan with half-yearly interest**

S	a_s	A_s	I_s	M_s	C_s
0	----	----	----	----	C_0
1/2	$a_{1/2} = I^{(2)}_{1/2}$	---	$I^{(2)}_{1/2} = C_0 i_k$	$M_{1/2} = 0$	$C_{1/2} = C_0$
1	$a_1 = I^{(2)}_1 + A_1$	$A_1 = \frac{C_0}{s_{\overline{n}\|i}}$	$I^{(2)}_1 = C_0 i_k$	$M_1 = A_1$	$C_1 = C_0 - M_1$
1+1/2	$a_{1+1/2} = I^{(2)}_{1+1/2}$	---	$I^{(2)}_{1+1/2} = C_1 i_k$	$M_{1+1/2} = A_1$	$C_{1+1/2} = C_1$
2	$a_2 = I^{(2)}_2 + A_2$	$A_2 = A_1(1+i)$	$I^{(2)}_2 = C_1 i_k$	$M_2 = M_1 + A_2$	$C_2 = C_0 - M_2$
2+1/2	$a_{2+1/2} = I^{(2)}_{2+1/2}$	---	$I^{(2)}_{2+1/2} = C_2 i_k$	$M_{2+1/2} = M_2$	$C_{2+1/2} = C_2$
3	$a_3 = I^{(2)}_3 + A_3$	$A_3 = A_2(1+i)$	$I^{(2)}_3 = C_2 i_k$	$M_3 = M_2 + A_3$	$C_3 = C_0 - M_3$

Exercise 10.7. Create the amortization table by the progressive system for a 26,400€ capital loan amortizable in 3 years at 6% annually with quarterly interest payments.

Solution:

S	a_s	A_s	I_s	M_s	C_s
0	----	----	----	----	26,400.00
1/3	517.77	---	517.77	0.00	26,400.00
2/3	517.77	---	517.77	0.00	26,400.00
1	8,810.27	8,292.50	517.77	8,292.50	18,107.50
1+1/3	355.14	---	355.14	8,292.50	18,107.50
1+2/3	355.14	---	355.14	8,292.50	18,107.50
2	9,145.19	8,790.05	355.14	17,082.55	9,317.45
2+1/3	182.74	---	182.74	17,082.55	9,317.45
2+2/3	182.74	---	182.74	17,082.55	9,317.45
3	9,500.19	9,317.45	182.74	26,400.00	0.00

$$A_1 = \frac{C_0}{s_{\overline{n|}i}} = \frac{26{,}400}{s_{\overline{3|}0.06}} = 8{,}292.50€$$

$$A_2 = A_1(1+i) = 8{,}292.50(1+0.06) = 8{,}790.05\,€$$

$$(1+i) = (1+i_k)^k \quad \Rightarrow \quad i_k = (1+i)^{1/k} - 1; \qquad i_3 = 0.019612822$$

$$I^{(3)}_{1/3} = C_0 i_3 = 26{,}400 \times 0.019612822 = I^{(3)}_{2/3} = I^{(3)}_1 = 517.77\,€$$

$$I^{(3)}_{1+1/3} = C_1 i_3 = 18{,}107.50 \times 0.019612822 = I^{(3)}_{1+2/3} = I^{(3)}_2 = 355.14\,€$$

$$a_{1/3} = I^{(3)}_{1/3} = 517.77 = a_{2/3}$$

$$a_1 = A_1 + I^{(3)}_1 = 8{,}292.50 + 517.77 = 8{,}810.27€$$

$$M_1 = A_1 = 8{,}292.50 = M_{1+1/3} = M_{1+2/3}$$

We verify that the quarterly interest payments plus the corresponding amortization payments are equivalent to the annuities according to the French system at the end of every year.

- By the French method:

$$C_0 = a\,a_{\overline{n|}i} \quad \Rightarrow \quad 26{,}400 = a\,a_{\overline{3|}0.06}; \qquad a = 9{,}876.50\,€$$

If we capitalise the interest instalments and add them to the amortization instalment,

$$a_1 = C_0 i_k s_{\overline{k|}i_k} + A_1 = 26{,}400 \times 0.01961282\; s_{\overline{3|}0.01961282} + 8{,}292.50 = 9{,}876.50\,€$$

$$a_2 = C_1 i_k s_{\overline{k|}i_k} + A_2 = 18{,}107.50 \times 0.01961282\; s_{\overline{3|}0.01961282} + 8{,}790.05 = 9{,}876.50\,€$$

$$a_3 = C_2 i_k s_{\overline{k|}i_k} + A_3 = 9{,}317.45 \cdot 0.01961282\; s_{\overline{3|}0.01961282} + 9{,}317.45 = 9{,}876.50\,€$$

The remaining amountscan be deduced following the theoretical table.

10.4.2. According the Nominal Rate

It is customary for loan offers to be based on the nominal rate that can be capitalised "k" times each year. The analysis performed based on the effective “i” rate would also apply in the event that we use the nominal rate. It would only be necessary to indicate, with respect to the progressive system, the need to calculate "i" according to J(k) when trying to calculate the amortization instalment for the first year.

$$A_1 = \frac{C_0}{S_{\overline{n}|i}}$$

Being,

$$i = \left(1 + \frac{J(k)}{k}\right)^k - 1$$

The other exception that affects the two systems being analysed is that it would be necessary to use the $i_k = \frac{J(k)}{k}$ rate. Otherwise, the theoretical tables for both systems are identical to those studied in the previous section.

Exercise 10.8. Create the amortization schedule according to the progressive system for a 26,400€ capital loan amortizable in 3 years at the nominal rate of 6% with quarterly interest payments.

Solution:

S	a_s	A_s	I_s	M_s	C_s
0	----	----	----	----	26,400.00
1/3	528.00	---	528.00	0.00	26,400.00
2/3	528.00	---	528.00	0.00	26,400.00
1	8,810.68	8,282.68	528.00	8,282.68	18,117.32
1+1/3	362.34	---	362.34	8,282.68	18,117.32
1+2/3	362.34	---	362.34	8,282.68	18,117.32
2	9,151.99	8,789.65	362.34	17,072.33	9,327.65
2+1/3	186.55	---	186.55	17,072.33	9,327.65
2+2/3	186.55	---	186.55	17,072.33	9,327.65
3	9,514.19	9,327.64	186.55	26,400.00	0.00

$$i = \left(1 + \frac{J(k)}{k}\right)^k - 1 = \left(1 + \frac{0.06}{3}\right)^3 - 1 = 0.061208;\; i_3 = \frac{J(3)}{3} = \frac{0.06}{3} = 0.02$$

$$A_1 = \frac{26{,}400}{S_{\overline{3}|0.061208}} = 8{,}282.68\,€$$

$$A_2 = A_1(1+i) = 8{,}282.68(1+0.061208) = 8{,}789.65€$$

$$I^{(3)}_{1/3} = 26{,}400 \times 0.02 = I^{(3)}_{2/3} = I^{(3)}_{1} = 528\,€$$

$$I^{(3)}_{1+1/3} = 18{,}117.32 \times 0.02 = I^{(3)}_{1+2/3} = I^{(3)}_{2} = 362.34\,€$$

The remainder can be solved in the same way as analysed in the case of the effective rate.

We verify that the quarterly interest payments plus the instalment are equivalent to the annuities according to the French system if the payments are made at the end of the year.

$$26{,}400 = a\, a_{\overline{3}|0.061208} \quad \Rightarrow \quad a = 9{,}898.58€$$

$$a_1 = 528\left[(1+0.061208)^{2/3} + (1+0.061208)^{1/3} + 1\right] + 8{,}282.68 =$$
$$= 528\, s_{\overline{3}|0.02} + 8{,}282.68 = 9{,}898.57\,€$$

$$a_2 = 362.34\left[(1+0.061208)^{2/3} + (1+0.061208)^{1/3} + 1\right] + 8{,}789.65 =$$
$$= 362.34\, s_{\overline{3}|0.02} + 8{,}789.65 = 9{,}.898.55€$$

10.5. Loan with fractioned amortization and interest: French method

In this case, besides paying interest in each period, capital is also amortized. In practices, this is the most frequently used system. The financial mathematic treatment is the one used for fractioned income.

The diagram of the transaction in the case of half-yearly payments and a 3-year term is as follows:

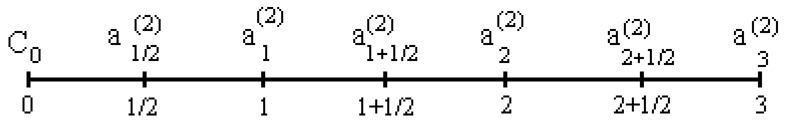


Figura 10.9

10.5.1. According to the Nominal Rate

We calculate the annual constant loan repayment instalment based on the financial equivalence between payment and compensation at the origin, with $i = \left(1 + \frac{J(k)}{k}\right)^k - 1$

$$C_0 = a \times k\, a_{\overline{n}|i}\, \frac{i}{J(k)} = a\, \frac{1-(1+i)^{-n}}{i_k}$$

We will bear in mind that $i_k = \frac{J(k)}{k}$. The rest is identical to the pure French system.

The most practical wayof resolving this type of transaction is to calculate the rate that is equivalent to the nominal rate and to calculate the transaction with a term of "$n \cdot k$" periods by applying this rate. Therefore, knowing that $i_k = \frac{J(k)}{k}$,

$$C_0 = a\, a_{\overline{n\times k}|i_k}$$

Exercise 10.9. Create the amortization schedule of a 500,000€ loan amortizable half-yearly in 4 years at the nominal rate of 10%.

Solution:

S	a_s	A_s	I_s	M_s	C_s
0	----	----	----	----	500,000.00
1/2	77,360.90	52,360.90	25,000.00	52,360.90	447,639.10
1	77,360.90	54,978.94	22,381.95	107,339.84	392,660.15
1+1/2	77,360.90	57,727.89	19,633.00	165,067.73	334,932.26
2	77,360.90	60,614.28	16,746.61	225,682.01	274,317.98
2+1/2	77,360.90	63,645.00	13,715.90	289,327.01	210,672.99
3	77,360.90	66,827.25	10,533.65	356,154.26	143,845.74
3+1/2	77,360.90	70,168.61	7,192.28	426,322.87	73,677.12
4	77,360.90	73,677.04	3,683.85	500,000.00	0.00

$$i_2 = \frac{J(2)}{2} = \frac{0.1}{2} = 0.05$$

$$500{,}000 = a\, a_{\overline{8}|0.05} \quad \Rightarrow \quad a = 77{,}360.90\,€$$

$$I_{1/2} = C_0 i_2 = 500{,}000 \times 0.05 = 25{,}000\,€$$

$$I_1 = C_{1/2} i_2 = 447{,}639.10 \times 0.05 = 22{,}381.95\,€$$

$$A_{1/2} = a - I_{1/2} = 77{,}360.90 - 25{,}000 = 52{,}360.90\,€$$

$$A_1 = A_{1/2}(1 + i_2) = 52{,}360.90(1 + 0.05) = 54{,}978.94\,€$$

10.6. Financial lease: leasing

Leasing is a commercial transaction of a financial nature that is arranged by specialised entities and is based on a lease agreement with a buy option. The leasing company (the lessor) acquires an asset, generally a tangible fixed asset (chosen by the client) and assings its use to a company or to an individual t (lessee) during a period of time. The company or individual undertakes to return this amount plus interest accrued (lease instalments, generally on a monthly basis) and the related transaction expenses in a previously agreed period.

Once the term of the agreement expires, the lessee will be able to either acquire the asset for the fixed residual value or return it.

The lessor should be a company, entrepreneur or professional and the asset should refer to a company or commercial activity.

The leasing contract offers several advantages, including: * It allows financing a percentage, which can reach 100% of the acquisition value, greater than the percentage of other financial formulas.

*It makes it possible to reduce the payment of corporate or personal income tax whether it is an individual entrepreneur or a company.

* In the same way, leasing is a tax deductible expenditure.

In the mathematical-financial aspect, leasing is characterised by annual pre-payable instalments, that is, the first annuity is paid (without interest) when the transaction is executed and the last payment would correspond to its residual value.

The equivalence between payment and compensation is defined by the following equation in which the term $\left[a\,(1+i)^{-n}\right]$ represents the residual value:

$$C_0 = a\,\ddot{a}_{\overline{n}|\,i} + a\,(1+i)^{-n} \Rightarrow a = \frac{C_0}{\ddot{a}_{\overline{n}|\,i} + (1+i)^{-n}}$$

Exercise 10.10. Create the amortization schedule for a leasing transaction characterised by:

* Capital to be financed: 100,000€

* Amortization term: 5 years

* Interest: 15%

* Residual value: 1 instalment.

$$100.000 = a\,\ddot{a}_{\overline{5}|0.15} + a\,(1+0.15)^{-5} \quad \Rightarrow \quad a = 22{,}977.12\ €$$

S	a_s	A_s	I_s	M_s	C_s
0	22,977.12	22,977.12	0.00	22,977.12	77,022.90
1	22,977.12	11,423.68	11,553.43	34,400.81	65,599.19
2	22,977.12	13,137.24	9,839.88	47,538.05	52,461.94
3	22,977.12	15,107.83	7,862.93	62,645.88	37,354.11
4	22,977.12	17,374.01	5,603.11	80,019.88	19,980.11
5 **(RV)**	22,977.12	19,980.10	2,997.01	100,000.00	0.00

It can be established that the residual value is a fixed quantity regardless of the value of the annual instalment.

Exercise 10.11. Create the amortization schedule for a leasing transaction characterised by:

* Capital to be financed: 100,000€

* Amortization term: 5 years

* Interest: 15%

* Residual value: 10,000€

* Value Added Tax: 21%.

$$100{,}000 = a\,\ddot{a}_{\overline{5}|0.15} + 10{,}000(1+0.15)^{-5}$$

$$a = 24{,}650.78 + \text{VAT}(21\%) = 29{,}827.45€$$

S	a_s	VAT	Total	A_s	I_s	M_s	C_s
0	24,650.78	5,176.66	29,827,45	24,650.78	0.00	24,650.78	75,349.22
1	24,650.78	5,176.66	29,827,45	13.348.39	11,302.38	37,999.17	62,000.82
2	24,650.78	5,176.66	29,827,45	15,350.65	9.300.12	53.349.82	46,650.17
3	24,650.78	5,176.66	29,827,45	17,653.25	6,997.52	71,003.07	28,996.92
4	24,650.78	5,176.66	29,827,45	20,301.24	4,349.53	91,304.31	8.695.69
6 (RV)	10,000.00	2,100.00	12,100.00	8.695.65	1.304.35	100,000.00	0.00

Exercise 10.12. Create the amortization schedule for a leasing transaction characterised by:

* Capital to be financed: 50,000€
* Amortization term: 3 years
* Quarterly payments.
* Nominal amount: 8%
* Residual value: 1 instalment.

$$i_4 = \frac{J(4)}{4} = \frac{0.08}{4} = 0.02$$

$$50{,}000 = a\,\ddot{a}_{\overline{12}|0.02} + a(1+0.02)^{-12} \quad \Rightarrow \quad a = 4{,}319.52€$$

S	a_s	A_s	I_s	M_s	C_s
0	4,319.52	4,319.52	0.00	4,319.52	45,680.48
1/4	4,319.52	3,405.91	913.61	7,725.43	42,274.57
2/4	4,319.52	3,474.03	845.49	11,199.46	38,800.54
3/4	4,319.52	3,543.51	776.01	14,742.97	35,257.03
1	4,319.52	3,614.38	705.14	18,357.35	31,642.65
1+1/4	4,319.52	3,686.67	632.85	22,044.01	27,955.99
1+2/4	4,319.52	3,760.40	559.12	25,804.41	24,195.59
1+3/4	4,319.52	3,835.61	483.91	29,640.02	20,359.98
2	4,319.52	3,912.32	407.20	33,552.34	16,447.66
2+1/4	4,319.52	3,990.57	328.95	37,542.91	12,457.09
2+2/4	4,319.52	4,070.38	249.14	41,613.29	8,386.71
2+3/4	4,319.52	4,151.79	167.73	45,765.07	4,234.93
3 (RV)	4,319.52	4,234.82	84.70	50,000.00	0.00

Exercise 10.13. Create the amortization schedule for a leasing transaction characterised by:

Capital to be financed: 50,000 €; Amortization period: 1.5 years; Quarterly payments. Nominal rate: 8%; Residual value: 1 instalment; Value Added Tax: 21%.

$$i_4 = \frac{J(4)}{4} = \frac{0.08}{4} = 0.02$$

$$50{,}000 = a\,\ddot{a}_{\overline{6}|0.02} + a(1+0.02)^{-6}$$

a = 7.574,11 € + VAT (21%) = 9.164,68 €

S	a_s	VAT	**Total**	A_s	I_s	M_s	C_s
0	7,574.11	1,590.56	9,164.68	7,574.11	0.00	7,574.11	42,425.89
1	7,574.11	1,590.56	9,164.68	6,725.59	848.52	14,299.70	35,700.29
2	7,574.11	1,590.56	9,164.68	6,860.10	714.00	21,159.80	28,840.19
3	7,574.11	1,590.56	9,164.68	6,997.30	576.80	28,157.10	21,842.89
4	7,574.11	1,590.56	9,164.68	7,137.25	436.85	35,294.35	14,705.64
5	7,574.11	1,590.56	9,164.68	7,279.99	294.11	42,574.34	7,425.65
6 (RV)	7,574.11	1,590.56	9,164.68	7,425.59	148.51	50,000.00	0.00

$I_1 = 42{,}425.89 \times 0.02 = 848.52$

$I_2 = 35{,}700.29 \times 0.02 = 714.00$

....

$I_{RV} = 7{,}425.65 \times 0.02 = 148.51$

Some financial entities calculate the interest on the initial instalment so that the instalment corresponding to the residual value is not included. Applying the previous amortization schedule we will arrive at:

S	a_s	VAT	**Total**	A_s	I_s	M_s	C_s
0	7,574.11	1,590.56	9,164.68	7,574.11	848.52	6.725.59	43.274.40
1	7,574.11	1,590.56	9,164.68	6,725.59	714.00	13.585.69	36.414.30
2	7,574.11	1,590.56	9,164.68	6,860.10	576.80	20.582.99	29.417.00
3	7,574.11	1,590.56	9,164.68	6,997.30	436.85	27.720.24	22.279.75
4	7,574.11	1,590.56	9,164.68	7,137.25	294.11	35.000.23	14.999.76
5	7,574.11	1,590.56	9,164.68	7,279.99	148.51	42.425.82	7.574.11
6 (RV)	7,574.11	1,590.56	9,164.68	7,425.59	0.00	50,000.00	0.00

$$I_1 = (50{,}000.00 - 7{,}574.11) \times 0.02 = 848.52$$
$$I_2 = (43{,}274.40 - 7{,}574.11) \times 0.02 = 714.00$$
$$I_3 = (36{,}414.30 - 7{,}574.11) \times 0.02 = 576.80$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$I_{RV} = (7{,}574.11 - 7{,}574.11) \times 0.02 = 0.00$$

Appendix

A-1. Logarithms

We define the logarithm of a number “n” in a “b” base as the exponent “x” where the base is elevated to obtain that number, that is,

$$Log_b N = x, \qquad si \ b^x = N$$

The most usual bases are: "10" in Decimal Logarithms (Log) and “e” in the natural or Neperian Logarithms (Ln). e = 2,718281828.

The fundamental rules for calculating logarithms are:

1[a]) The base logarithm is equal to 1 and the Logarithm of 1 is equal to 0:

$$Log\,10 = 1 \qquad Log\ 1 = 0$$

2[nd]) The logarithm of a product is equal to the sum of the logarithms,

$$Log(a \times b) = Log\,a + Log\,b$$

3[rd]) The logarithm of a quotient is equal to the numerator logarithm less the divisor logarithm,

$$Log\left(\frac{a}{b}\right) = Log\ a - Log\ b$$

4[th)] The logarithm of a power is equal to the exponent multiplied by the logarithm of the base,

$$Log\ a^b = b\ Log\ a$$

5[th]) The logarithm of a root is equal to the logarithm of root divided by the root index,

$$Log\ \sqrt[b]{a} = \frac{Log\ a}{b}$$

<u>Antilogarithm:</u>

When we want to know the number “N” corresponding to logarithm “x” in a “b” base, we will have to calculate its antilogarithm.

In the case of decimal logarithms: $N = 10^x$

In the case of decimal logarithms: $N = e^x$

A-2. Second degree equations

A second degree equation with an incognito (x) presents the following form:

$$ax^2 + bx + c = 0$$

If the coefficient "b" or "c" is equal to zero, the second degree equation is incomplete.

If the coefficient $b = 0$, the equation is: $ax^2 + c = 0$, with the following solutions:

$$x^2 = -\frac{c}{a} \qquad \Rightarrow \qquad x = \pm\sqrt{-\frac{c}{a}}$$

If the coefficient $c = 0$, the equation is:

$$ax^2 + bx = 0 \Rightarrow x(ax+b) = 0 \begin{cases} x = 0 \\ ax + b = 0 \Rightarrow x = -\dfrac{b}{a} \end{cases}$$

If $b = 0$ and $c = 0$, the only solution is zero.

The complete second degree equation ($ax^2 + bx + c = 0$) is resolved in the following manner:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \begin{cases} x_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a} \\ x_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a} \end{cases}$$

A-3. Geometric progressions

A geometric progression is a succession of terms in which each one is equal to the previous one multiplied by a fixed amount called reason of the progression. For example: 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024.

Calling the reason of the progression “r”: $\frac{x_2}{x_1} = \frac{x_3}{x_2} = \frac{x_4}{x_3} = ... = \frac{x_n}{x_{n-1}} = r$

The different terms of the progression are:

$$\begin{array}{lllll} x_1 & x_2 & x_3 \ldots\ldots\ldots\ldots\ldots x_{n-1} & & x_n \\ x_1 & x_1 r & x_1 r^2 \ldots\ldots\ldots\ldots x_1 r^{n-2} & & x_1 r^{n-1} \end{array}$$

The last term according to the first is: $x_n = x_1 r^{n-1}$

The first according to the last: $x_1 = \frac{x_n}{r^{n-1}}$

The progression according to the reason:

- If $r > 1$: Growing Progression.
- If $0 < r < 1$: Decreasing progression.
- If $r < 0$: Alternate positive and negative terms.

The progression according to the number of terms:

- If n is finite: Finite geometric progression.
- If n is infinite: Unlimited geometric progression.

Sum of the geometric progression:

A) Growing and finite progression:

The sum of the progression is: $S = \sum_{r=1}^{n} x_r$

$$S = x_1 + x_2 + x_3 + \dots\dots + x_n = x_1 + x_1 r + x_2 r + \dots\dots + x_{n-1} r =$$
$$= x_1 + (x_1 + x_2 + x_3 + \dots\dots + x_{n-1}) r$$
$$S = x_1 + (S - x_n) r$$

The progression is increasing due to $r > 1$,

$$S = x_1 + Sr - x_n r$$
$$Sr - S = x_n r - x_1$$
$$S(r-1) = x_n r - x_1$$
$$S = \frac{x_n r - x_1}{r - 1}$$

B) Decreasing and finite progression:

Applying the same reasoning followed in point A) until reaching:

$$S = x_1 + Sr - x_n r$$

On decreasing the progression, $r < 1$

$$S - Sr = x_1 - x_n r$$
$$S(1 - r) = x_1 - x_n r$$
$$S = \frac{x_1 - x_n r}{1 - r}$$

Expression that enables us to work with positive numbers.

C) Unlimited progression:

We calculate the limit of the sum when $n \to \infty$

$$\text{Si } r > 1; \ \lim_{n\to\infty} S = \lim_{n\to\infty} \frac{x_n r - x_1}{r - 1} = \infty$$
$$\text{Si } r < 1; \ \lim_{n\to\infty} S = \lim_{n\to\infty} \frac{x_1 - x_n r}{1 - r} = \frac{x_1}{1 - r}$$

Example. Calculate term number 15 of the following progressions:

a) 2; 2.4; 2.88; 3.456; ……..

b) 15; 13.50; 12.15; …………

In both cases we are dealing with geometric progressions because each term is formed by multiplying the previous one by a certain amount ("r" progression rate) and it is necessary to determine this rate of progression.

In case a) $r = \frac{2{,}4}{2} = \frac{2{,}88}{2{,}4} = \dots = 1{,}2$ (increasing geometric progression rate of 1.2)

In case b) $r = \frac{13,50}{15} = \frac{12,15}{13,50} = ...= 0,9$ (decreasing geometric progression rate of 0.9).

Term 15 will be determined by: $x_n = x_1 r^{n-1} \quad \rightarrow \quad x_{15} = x_1 r^{14}$

In case a) $x_{15} = x_1 r^{14} = 2 \times 1,2^{14} = 25,67837$

In case b) $x_{15} = x_1 r^{14} = 15 \times 0,9^{14} = 3,4315188$

Ejemplo. Calculate the sum of the first 30 terms of the following progressions:

a) 4; 4,4; 4,84; ……..

b) 8; 7,6; 7,22; ..……

c) Repeat case b) based on the assumption that the number of terms is infinite.

The reason for the progressions will be as follows:

In case a) $r = \frac{4,4}{4} = \frac{4,84}{4,4} = ...= 1,1$ (increasing geometric progression rate of 1.1)

In case b) $r = \frac{7,6}{8} = \frac{7,22}{7,6} = ...= 0,95$ (decreasing geometric progression rate of 0.95).

It is necessary to firstly calculate the amount corresponding to the last term (number 30):

a) $x_{30} = x_1 r^{29} = 4 \times 1,1^{29} = 63,452372$

b) $x_{30} = x_1 r^{29} = 8 \times 0,95^{29} = 1,807484$

The sum is determined by the following mathematical expression:

In case a) $S = \frac{x_n r - x_1}{r - 1} = \frac{63,452372 \times 1,1 - 4}{1,1 - 1} = 657,976$

In case b) $S = \frac{x_1 - x_n r}{1 - r} = \frac{8 - 1,807484 \times 0,95}{1 - 0,95} = 125,6578$

In case c) $S = \frac{x_1}{1 - r} = \frac{8}{1 - 0,95} = 160$

A-4. Newton's binomial

The Newton Binomial expresses the method for calculating the value of the “m” power

The most important characteristics of the Binomial coefficient $\binom{n}{r}$:

1) $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)......(n-r+1)}{1.2.3......r}$

2) $\binom{n}{0} = 1$, as $0! = 1$.

3) $\binom{0}{0}=1$

4) $\binom{n}{n}=1$

5) $\binom{n}{r}=0, \text{si}\, r>n, \text{siendo}\, n>0$.

Newton's Binomial

$$(a+b)^m=\binom{m}{0}a^m b^0+\binom{m}{1}a^{m-1}b^1+\binom{m}{2}a^{m-2}b^2+..........+\binom{m}{m-1}ab^{m-1}+\binom{m}{m}a^0b^m=$$

$$=\sum_{k=0}^{m}\binom{m}{k}a^{m-k}b^k$$

We can also express $(a+b)^m$ in the following way:

$$(a+b)^m=a^m+ma^{m-1}b+\frac{m(m-1)}{2!}a^{m-2}b^2+..........+b^m$$

$$(a-b)^m=\binom{m}{0}a^m b^0-\binom{m}{1}a^{m-1}b^1+\binom{m}{2}a^{m-2}b^2-.......+(-1)^{m-1}\binom{m}{m-1}ab^{m-1}+(-1)^{m-1}\binom{m}{m}a^0b^m=$$

$$=\sum_{k=0}^{m}(-1)^k\binom{m}{k}a^{m-k}b^k$$

The term $(-1)^k$ shows that if "k" is an even number, the corresponding term is positive and if it is an uneven number, the corresponding term is negative, that is, positive for even places and negative for the uneven places.

Example. Calculated $(a+b)^3$ based on Newton's binomial theorem

$$(a+b)^3=\binom{3}{0}a^3b^0+\binom{3}{1}a^2b+\binom{3}{2}ab^2+\binom{3}{3}a^0b^3=$$

$$a^3+\frac{3!}{1!(3-1)!}a^2b+\frac{3!}{2!(3-2)!}ab^2+b^3=a^3+3a^2b+3ab^2+b^3$$

Example. Calculated $(b-2)^5$ based on Newton's binomial theorem

$$(b-2)^5=\binom{5}{0}b^5 2^0-\binom{5}{1}b^4 2+\binom{5}{2}b^3 2^2-\binom{5}{3}b^2 2^3+\binom{5}{4}b2^4-\binom{5}{5}b^0 2^5=$$

$$=b^5-\frac{5!}{1!(5-1)!}b^4 2+\frac{5!}{2!(5-2)!}b^3 2^2-\frac{5!}{3!(5-3)!}b^2 2^3+\frac{5!}{4!(5-4)!}b2^4-2^5=$$

$$=b^5-10b^4+40b^3-80b^2+80b-2^5$$

Example. Calculated $(a+2b)^4$ based on Newton's binomial theorem

$$(a+2b)^4=\binom{4}{0}a^4 2b^0+\binom{4}{1}a^3 2b+\binom{4}{2}a^2 2^2b^2+\binom{4}{3}a^3 2^3b^3+\binom{4}{4}a^0 2^4b^4=$$

$$a^4+\frac{4!}{1!(4-1)!}a^3 2b+\frac{4!}{2!(4-2)!}a^2 2^2 b^2+\frac{4!}{3!(4-3)!}a 2^3 b^3+2^4 b^4=$$
$$=a^4+8a^3b+24a^2b^2+32ab^3+16b^4$$

A-5. The operative sum

Assuming we want to perform the following sum: $x_1+x_2+x_3+x_4+x_5$.

We can represent it using the operator sum: $\sum_{r=1}^{5} x_r$ and we can read them by saying: Sum (or summation) of all the values of x, from "r" is worth 1 until it is worth 5.

The values that appear above and below the operator $\sum$ are called limits of the sum. The sub index "r" only takes whole values.

Properties of the operator:

1ª) $\sum_{r=1}^{n}(x_r+y_r)=\sum_{r=1}^{n}x_r+\sum_{r=1}^{n}y_r$ 2ª) $\sum_{r=1}^{n}a=n\times a$ being "a" constant

3ª) $\sum_{r=1}^{n}(x_r+a)=\sum_{r=1}^{n}x_r+n\times a$ 4ª) $\sum_{r=1}^{n}x_r a=a\sum_{r=1}^{n}x_r$

5ª) $\sum_{r=s}^{n}x_r=\sum_{r=0}^{n-s}x_{r+s}$ 6ª) $\sum_{r=1}^{n}(ax_r+b)=a\sum_{r=1}^{n}x_r-n\times b$, (a and b constants)

When the sum covers all the possible terms, the indication of the limits is usually suppressed representing only $\sum$

Example. $x_4+x_5+x_6+x_7=\sum_{i=4}^{7}x_i$

Example. $x_2^2+x_3^2+x_4^2+x_5^2+x_6^2=\sum_{i=2}^{6}x_i^2$

Example. $\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}+\frac{1}{x_5}+\dots\dots\dots+\frac{1}{x_{20}}=\sum_{i=2}^{20}\frac{1}{x_i}$

Example. $(1+x_1)^2+(1+x_2)^2+(1+x_3)^2+(1+x_4)^2+(1+x_5)^2+(1+x_6)^2=\sum_{i=1}^{6}(1+x_i)^2$

Example. $(1+x)+(1+x)^2+(1+x)^3+(1+x)^4+(1+x)^5=\sum_{i=1}^{5}(1+x)^i$

Example. $5(1+x_1)^8+5(1+x_2)^8+5(1+x_3)^8+5(1+x_4)^8+5(1+x_5)^8+5(1+x_6)^8=5\sum_{i=1}^{6}(1+x_i)^8$

A-6. Elimination of some of the indetermination in the limits of functions.

When the limits of the form $\frac{f(x)}{g(x)}$ tend towards zero or towards an infinite numerator and denominator, that is, they take on the form $\frac{0}{0}$ o $\frac{\infty}{\infty}$, it fulfils,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \qquad (1)$$

- The limits of the form $0.\infty$ can be reduced to the following form:

$$\lim_{x \to \infty} \frac{1}{f(x)} g(x) = \lim_{x \to \infty} \frac{g(x)}{\frac{1}{f(x)}} \qquad \text{is applied (1)}$$

- The limits of the form 1^{∞} can be resolved in the following manner:

$$\lim_{x \to \infty} f(x)^{g(x)} = e^{\lim\limits_{x \to \infty} g(x) Ln f(x)}$$

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