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On the Entanglement Entropy in Gaussian cMERA

J. J. Fernández-Melgarejo and J. Molina-Vilaplana*

The continuous Multi Scale Entanglement Renormalization Anstaz (cMERA) consists of a variational method which carries out a real space renormalization scheme on the wavefunctionals of quantum field theories. In this work we calculate the entanglement entropy of the half space for a free scalar theory through a Gaussian cMERA circuit. We obtain the correct entropy written in terms of the optimized cMERA variational parameter, the local density of disentanglers. Accordingly, using the entanglement entropy production per unit scale, we study local areas in the bulk of the tensor network in terms of the differential entanglement generated along the cMERA flow. This result spurs us to establish an explicit relation between the cMERA variational parameter and the radial component of a dual AdS geometry through the Ryu-Takayanagi formula. Finally, we argue that the entanglement entropy for the half space can be written as an integral along the renormalization scale whose measure is given by the Fisher information metric of the cMERA circuit. Consequently, a straightforward relation between AdS geometry and the Fisher information metric is also established.

1. Introduction

Entanglement is a key feature to characterize quantum systems. The best known measure of it, entanglement entropy, has been used in a wide range of fields such as condensed matter physics, high energy theory and gravitational physics (see [1] and references therein). Given a system described by a quantum state, for an observer having access only to a subregion A of the total system, all physical predictions are given in terms of the reduced density matrix ρ_A. The entanglement entropy measures the amount of missing information about the total system for

this observer, and is given by the von Neumann entropy of the reduced density matrix ρ_A, e.g.,

$$S_A = -\mathrm{Tr}_A \, \rho_A \log \rho_A. \tag{1}$$

In quantum field theory (QFT), computing S_A has shown to be an extraordinarily difficult task. Noteworthily, in the context of the AdS/CFT[2–4] the entanglement entropy can be computed using one of the central entries in the holographic dictionary, the RyuTakayanagi formula,[5,6]

$$S_A = \frac{\mathrm{Area}(\gamma_A)}{4 \, G_N^{(d+2)}}, \tag{2}$$

which quantifies the entanglement entropy S_A of a region A in a $(d+1)$-QFT admitting a $(d+2)$-gravity dual. Here, γ_A is a codimension-2 static minimal surface in AdS$_{(d+2)}$ anchored to the boundary of the region A.

The holographic formula for the entanglement entropy (2) allows to compute the entanglement entropy in a QFT from the dual bulk geometry. Being the AdS/CFT a duality between theories, it seems reasonable to think that analyzing the entanglement structure of concrete states in a QFT, one would be able to infer the dual bulk geometries related to these states. Interestingly, this strategy has been graciously revealed in terms of tensor networks, concretely in terms of the Multi-scale Entanglement Renormalization Ansatz (MERA).[7] A MERA tensor network,[8] implements a real space renormalization group on the wavefunction of a quantum many body system. A continuous version of MERA (cMERA) has been proposed for free field theories[9,10] and more recently for interacting field theories.[11–14] In [10], to make the connection of cMERA with the AdS/CFT more precise, authors proposed to think in terms of the Fisher information metric defined via quantum distances. Namely, they define the holographic radial component of a dual metric by considering the overlap between states that infinitesimally differ in the renormalization scale of cMERA. However, a more refined proposal would require to define local areas in the bulk of the tensor network in terms of the differential entanglement generated along the cMERA flow, and see if these local areas can be mapped into minimal areas in AdS spacetimes.

Continuing this line of thought, in this paper we have obtained the entanglement entropy for the half-space of a free scalar theory in a Gaussian cMERA tensor network as a function of the local density of disentanglers, the variational parameter defining the tensor network. This result, which was conjectured in [10]

J. J. Fernández-Melgarejo
Departamento de Física
Universidad de Murcia
Campus de Espinardo
Murcia 30100, Spain
E-mail: melgarejo@um.es

J. Molina-Vilaplana
Universidad Politécnica de Cartagena
Calle Dr. Fleming, S/N, Cartagena, Murcia 30202
E-mail: javi.molina@upct.es

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based on an estimation of the entropy in the discrete version of MERA, spurs us to broaden our analysis when other (non-)Gaussian cMERA circuits with additional disentanglers are considered. In particular, we observe that the entanglement entropy must be computed in cMERA through the Fisher information metric g_{uu}, which is the avatar of the bond dimension in the discrete version of the MERA circuit.

In addition, our result explicitly shows how the infinitesimal change in the entropy can be cast in terms of the differential contribution to the area of a minimal surface in a dual AdS geometry. As a result, the dual geometry is defined in terms of the variational parameters of the tensor network through a computation of the entanglement entropy in a QFT state.

The paper is structured as follows. In Section 2 we review the obtaining of the entanglement entropy of half space in quantum field theory. Then, after briefly introducing the Gaussian cMERA formalism in Section 3, in Section 4 we study the entanglement entropy in cMERA. In particular, we provide an expression for the entropy as a function of the variational parameter. In Section 5 we elaborate on the relation of this expression with the Ryu-Takayanagi formula and establish an explicit relation between the AdS metric and the cMERA variational parameter. Finally, we discuss our results and explain our conclusions in Section 6.

2. Entanglement Entropy of Half-Space in QFT

A standard method for the calculation of the entanglement entropy in a field theory is the *replica trick*. To illustrate this, and following,[15] let us consider a quantum field $\psi(X)$ in a $(d+1)$-dimensional spacetime and choose the Cartesian coordinates $X^\mu = \{\tau, x, x^i_\perp\}$ with $i = 1, .., d-1$, where τ is Euclidean time, such that a surface $A_\perp$ is defined by the condition $x = 0$ and $x^i_\perp, i = 1, .., d-1$ are the coordinates on $A_\perp$.[1]

Here we consider the wavefunction for the vacuum state, which is built by performing the path integral over the lower half of the total Euclidean spacetime ($\tau \leq 0$) such that the quantum field satisfies the boundary condition $\psi(\tau = 0, x, x_\perp) = \psi_0(x, x_\perp)$

$$\Psi[\psi_0(x, x_\perp)] = \int_{\psi(X)|_{\tau=0} = \psi_0(x, x_\perp)} D\psi \, e^{-W[\psi]}, \tag{3}$$

where $W[\psi]$ is the action of the field. The (co-dimension 2) surface $A_\perp$ separates the hypersurface $\tau = 0$ into two parts: $x < 0, (A_c)$ and $x > 0, (A)$. Thus the path integral boundary data $\psi(x, x_\perp)$ are split into

$$\psi(x, x_\perp) = \begin{cases} \psi_-(x, x_\perp) = \psi_0(x, x_\perp), & x < 0, \\ \psi_+(x, x_\perp) = \psi_0(x, x_\perp), & x > 0. \end{cases}$$

The reduced density matrix describing the subregion A ($x > 0$) of the vacuum state is then obtained by tracing over the set of boundary fields ψ_- located in the complementary region A_c. In the Euclidean path integral, this corresponds to integrating out ψ_- over the entire spacetime, but with a cut from negative infinity to $A_\perp$ along the $\tau = 0$ surface (i.e., along $x < 0$). We must

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[1] In this case, $A_\perp$ is a plane and the Cartesian coordinate x is orthogonal to $A_\perp$.

therefore impose boundary conditions for the remaining field ψ_+ as this cut is approached from above (ψ_+^1) and below (ψ_+^2). Hence we have:

$$\rho_A(\psi_+^1, \psi_+^2) = \int D\psi_- \Psi(\psi_+^1, \psi_-) \Psi(\psi_+^2, \psi_-). \tag{4}$$

Computing the von Neumann entropy $S_A = -\text{Tr}\rho_A \log \rho_A$ from this formal object is an extremely difficult task for all but the very simple systems. The solution is given by the replica trick. The trace of the n-th power of the density matrix (4) is given by the Euclidean path integral over fields defined on an n-sheeted covering of the cut geometry associated to ρ_A. Taking polar coordinates (r, ϕ) in the (τ, x) plane, the cut corresponds to values $\phi = 2\pi k, k = 1, 2, .., n$. In building the n-sheeted cover, we glue sheets along the cut in such a way that the fields are smoothly continued from $\psi_+^{1,2}|_k$ to $\psi_+^{1,2}|_{k+1}$. The resulting space is a cone C_n, with angular deficit $2\pi(1-n)$ at $A_\perp$. The partition function for the fields over this n-fold, which is denoted by $Z[C_n]$ and then $\text{Tr}\rho_A^n = Z[C_n]$. Assuming that one can consider an analytic continuation to non-integer values of n, we have

$$S_A = -\text{Tr}\rho_A \log \rho_A = -(\lambda \partial_\lambda - 1) \log \text{Tr}\rho_A^\lambda|_{\lambda=1}. \tag{5}$$

Hence, introducing the effective action $W[\lambda] = -\log Z[C_\lambda]$ for fields on an Euclidean spacetime with a conical singularity at $A_\perp$, the cone C_λ is defined, in polar coordinates, by making $\phi = \phi + 2\pi\lambda$. Then taking the limit in which $(1 - \lambda) \ll 1$, the entanglement entropy is given by the replica trick as

$$S_A = (\lambda \partial_\lambda - 1) W(\lambda)|_{\lambda=1}. \tag{6}$$

It is the action $W(\lambda)$ the function to be calculated. It can be shown that for a bosonic field whose partition function is $Z = \det^{-1/2} \mathcal{D}$, with $\mathcal{D}$ a differential operator, this action can be written as

$$W = -\frac{1}{2} \int_{\epsilon^2}^\infty \frac{ds}{s^2} \text{Tr} K(s), \tag{7}$$

with $K(s, X, X')$ the heat kernel satisfying

$$(\partial_s + \mathcal{D}) K(s, X, X') = 0, \qquad K(0, X, X') = \delta(X - X'). \tag{8}$$

The heat kernel $K_\lambda(s, X, X')$ is obtained by applying the Sommerfeld formula[16]

$$K_\lambda(s, X, X') = K(s, X, X') + \Delta_\lambda(s, X, X'), \tag{9}$$

where Δ_λ ensures the $2\pi\lambda$ periodicity (see [15] for further details).

Entanglement Entropy in Free Field Theory

It can be proven that, for the operator $\mathcal{D} = -\nabla^2 + m^2$, one can obtain $K_\lambda(s, X, X')$ and then calculate $W(\lambda)$. The final result is

$$\text{Tr} K_\lambda(s) = \frac{1}{(4\pi s)^{d/2}} \left(\lambda V + 2\pi \frac{1 - \lambda^2}{6\lambda} s|A_\perp| \right), \tag{10}$$

where V is the spacetime volume and $|A_\perp| = \int d^{d-2}x$ is the area of the surface $A_\perp$. From the the Euclidean path integral for the

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fields in a free theory, after properly normalizing, we have

$$W = \frac{1}{2} \log \det \mathcal{D} = \frac{1}{2} \mathrm{Tr} \, \log(-\nabla^2 + m^2), \tag{11}$$

where $\mathcal{D}^{-1} \equiv G(x, y) = \langle \psi(x)\psi(y) \rangle$. With this, $\mathrm{Tr} K_{A_\perp}(s)$, which is given by

$$\mathrm{Tr} K_{A_\perp}(s) = \frac{|A_\perp|}{(4\pi s)^{\frac{d-2}{2}}} \exp(-m^2 s), \tag{12}$$

can be interpreted as the trace of the heat kernel of $\mathcal{D}(A_\perp)$, where $\mathcal{D}(A_\perp)$ is the differential operator over the codimension-2 plane $A_\perp$:

$$\log \det \mathcal{D} = -\int_{\epsilon^2}^{\infty} \frac{ds}{s} \mathrm{Tr} K_{A_\perp}(s). \tag{13}$$

Thus, upon a straightforward identification we obtain

$$S_A = -\frac{1}{12} \log \det \mathcal{D}(A_\perp) = -\frac{1}{12} \mathrm{Tr}_{A_\perp} \log \mathcal{D}. \tag{14}$$

Based on this expression, in Sections 4 and 5 we will establish a relation between cMERA and the Ryu-Takayanagi formula.

3. A cMERA Primer

cMERA[9,10] amounts to a real space renormalization group procedure on the quantum state that builds, through a Hamiltonian evolution in scale, scale dependent wavefunctionals $\Psi[\phi, u]$ given by,

$$\Psi[\phi, u] = \langle \phi|\Psi_u \rangle = \langle \phi| \mathcal{P} \, e^{-i\int_{u_{\mathrm{IR}}}^{u} (K(u')+L) \, du'} |\Omega\rangle. \tag{15}$$

Here u parametrizes the scale of the renormalization and $\mathcal{P}$ is the u-ordering operator. L represents the dilatation operator and $K(u)$ is the generator of evolution in scale, the so-called "entangler" operator. The scale parameter u is taken to be in the interval $[u_{\mathrm{IR}}, u_{\mathrm{UV}}] = (-\infty, 0]$. u_{UV} is the scale at the UV cut off ϵ, and the corresponding momentum space UV cut off is $\Lambda = 1/\epsilon$. u_{IR} is the scale in the IR limit.

The state $|\Psi_\Lambda\rangle \equiv |\Psi_{\mathrm{UV}}\rangle$ is the state in the UV limit and it may be the ground state of a quantum field theory. The state $|\Omega\rangle$ is defined to have no entanglement between spatial regions. $|\Omega\rangle$ is invariant with respect to spatial dilatations, so that $e^{-iLu}|\Omega\rangle = |\Omega\rangle$ or, equivalently $L|\Omega\rangle = 0$.

For a free bosonic theory, $|\Omega\rangle$ is defined by

$$\left(\sqrt{M} \, (\phi(\mathbf{k}) - \bar{\phi}) + \frac{i}{\sqrt{M}} \, \pi(\mathbf{k}) \right) |\Omega\rangle = 0, \tag{16}$$

for all momenta $\mathbf{k}$, where $M = \sqrt{\Lambda^2 + m^2}$ with m the mass of the particles in the free theory and $\bar{\phi} \equiv \langle\Omega|\phi(x)|\Omega\rangle$. This state satisfies $\langle\Omega|\phi(\mathbf{p})\phi(\mathbf{q})|\Omega\rangle = \frac{1}{2M} \, \delta^d(\mathbf{p}+\mathbf{q})$ and $\langle\Omega|\pi(\mathbf{p})\pi(\mathbf{q})|\Omega\rangle = \frac{M}{2} \, \delta^d(\mathbf{p}+\mathbf{q})$.

The nonrelativistic dilatation operator L, that can be understood as as the "free" piece of the cMERA Hamiltonian evolution

in scale, does not depend on the scale u but only by the scaling dimensions of the fields [2].

On the other hand, the entangler operator $K(u)$, contains all the variational parameters to be optimized, creating entanglement between field modes with momenta $|\mathbf{k}| < \Lambda$, where Λ is the cutoff mentioned above. The entangler is considered as the "interacting" part of the cMERA Hamiltonian. From this point of view, the unitary operator in Eq. (15)

$$U(u_1, u_2) \equiv \mathcal{P} \exp\left[-i \int_{u_2}^{u_1} du \, (K(u) + L) \right] \tag{17}$$

is understood as a Hamiltonian evolution with $K(u) + L$. Usually it is useful to define cMERA in the "interaction picture" through the unitary transformation $|\Phi_u\rangle = e^{iLu}|\Psi_u\rangle$ in which the u-evolution is determined by the unitary operator $\mathcal{U}(u_1, u_2) = \mathcal{P} e^{-i\int_{u_2}^{u_1} du \, \hat{K}(u)}$.

Gaussian cMERA

For free scalar theories in $(d + 1)$ dimensions, $K(u)$ is given by the quadratic operator[3][9,10]

$$K(u) = \frac{1}{2} \int_{pq} g(p; u) \left[\phi(\mathbf{p})\pi(\mathbf{q}) + \pi(\mathbf{p})\phi(\mathbf{q}) \right] \bar{\delta}(\mathbf{p} + \mathbf{q}), \tag{19}$$

where $\int_p \equiv \int (2\pi)^{-d} d^d\mathbf{p}$. The conjugate momentum of the field $\phi(\mathbf{p})$ is $\pi(\mathbf{p})$, such that $[\phi(\mathbf{p}), \pi(\mathbf{q})] = i\bar{\delta}(\mathbf{p} + \mathbf{q})$, with $\bar{\delta}(\mathbf{p}) \equiv (2\pi)^d \delta(\mathbf{p})$. The function $g(p; u)$ in (19) is the only variational parameter to be optimized in the cMERA circuit. This function factorizes as

$$g(p; u) = g(u) \cdot \Gamma(p/\Lambda), \tag{20}$$

where $\Gamma(x)$ is a cut off function which, in general, will be assumed $\Gamma(x) \equiv \Theta(1 - |x|)$ with $\Theta(x)$ is the Heaviside step function. $g(u)$ is a real-valued function known as *density of disentanglers* and $\Gamma(p/\Lambda)$ implements a high frequency cutoff such that $\int_p \equiv \int_0^\Lambda d^d\mathbf{p}$.[9,10] The sharp cutoff function, which is assumed by default along this paper, ensures that $K(u)$ acts locally in a region of size $\epsilon \simeq \Lambda^{-1}$.

The optimized cMERA ansatz for the relativistic free massive scalar theory can be obtained as follows[9,10]: The expectation value of the Hamiltonian of the theory w.r.t. the cMERA state $|\Psi_\Lambda\rangle$ is calculated in terms of the variational function $f(k, u_{\mathrm{IR}})$

$$f(k, u_{\mathrm{IR}}) = \int_0^{u_{\mathrm{IR}}} g(ke^{-u}; u) \, du = \int_0^{-\log \Lambda/k} g(u) \, du. \tag{21}$$

The optimization process yields

$$f(k, u_{\mathrm{IR}}) = \frac{1}{4} \log \frac{k^2 + \mu^2}{M^2}, \quad g(u) = \frac{1}{2} \frac{\Lambda^2 e^{2u}}{(\Lambda^2 e^{2u} + \mu^2)}, \tag{22}$$

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[2] $L = -\frac{1}{2} \int \, dx[\pi(\mathbf{x})(\mathbf{x} \cdot \nabla \phi(\mathbf{x})) + (\mathbf{x} \cdot \nabla \phi(\mathbf{x}))\pi(\mathbf{x}) + \frac{d}{2}(\phi(\mathbf{x})\pi(\mathbf{x}) + \pi(\mathbf{x})\phi(\mathbf{x}))]$.
[3] In the interaction picture the entangler operator reads as

$$\hat{K}(u) = \frac{1}{2} \int_{pq} g(pe^{-u}; u) \left[\phi(\mathbf{p})\pi(\mathbf{q}) + \pi(\mathbf{p})\phi(\mathbf{q}) \right] \bar{\delta}(\mathbf{p} + \mathbf{q}). \tag{18}$$

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where μ is a variational mass parameter that in the free case equals the bare mass m of the theory.

Remarkably, in [17], a cMERA circuit based on the quadratic entangler (19) was used to study the self-interacting ϕ^4 scalar theory. This model has a mass gap and flows to a free theory in the IR, where the IR ground state is exactly a Gaussian wavefunctional. Similar to the free case, by minimizing the expectation value of the Hamiltonian with respect to the ansatz wavefunctional one obtains (22) but in this case μ is the modified mass of propagating free quasi-particles given by the *gap* equation

$$\mu^2 = m^2 + \frac{\lambda}{2} \left(\bar{\phi}^2 + G(0) \right) \equiv m^2 + \frac{\lambda}{2} \Delta,$$

$$G(0) = \frac{1}{2} \int_k \frac{1}{\sqrt{k^2 + \mu^2}}, \tag{23}$$

where λ is the coupling constant. The cMERA wavefunctional thus obtained is a vacuum state for a free theory with mass given by (23). The optimized ansatz captures all 1-loop 2-point correlation functions, as well as the resummation of all cactus-like diagrams.

4. Entanglement Entropy in Gaussian cMERA

Here, we consider the entanglement entropy of half space in free scalar theory in $(d + 1)$-dimensions. In this case the entangling surface is $A_\perp = \mathbb{R}^{d-1}$ and its area will be denoted by $|A_\perp|$. According to the heat kernel result (14), the entanglement entropy of the half space can be written as [18, 19]

$$S_A = \frac{|A_\perp|}{6} \int d^{d-1}\mathbf{k}_\perp \, \log\langle \Psi_\Lambda | \phi(\mathbf{k}_\perp)\phi(-\mathbf{k}_\perp) | \Psi_\Lambda \rangle + \text{const}, \tag{24}$$

where const represents a (UV dependent) quantity independent of mass and $|\Psi_\Lambda\rangle$ is the ground state of the field theory under consideration defined at some fixed cutoff Λ. The integration is carried out over the $(d - 1)$ transverse momenta in $A_\perp$. Upon this assumption, from now on we will simplify the notation $\int d^{d-1}\mathbf{k}_\perp \to \int d^{d-1}k$.

The Gaussian cMERA circuit introduced in the previous section is exactly solvable for the free scalar theory and thus, the UV cMERA approximation to the ground state of the theory $|\Psi_\Lambda\rangle$, is exact in this case. Upon these conditions, we compute the entanglement entropy of the half-space by renormalizing the 2-point correlator in (24) with cMERA. First we note that,

$$\langle \Psi_\Lambda | \phi(\mathbf{k})\phi(-\mathbf{k}) | \Psi_\Lambda \rangle = \frac{1}{2M} e^{-2f(k, u_{\text{IR}})} \delta^d(0). \tag{25}$$

Therefore, using (22) we express the entropy as

$$S_A = \frac{|A_\perp|}{6} \int d^{d-1}k \, \log\left(\frac{e^{-2f(k, u_{\text{IR}})}}{2M} \right) + \widetilde{\text{const}}, \tag{26}$$

where $\widetilde{\text{const}}$ is a new UV dependent quantity independent of mass.

When applying the Gaussian cMERA to the self interacting $\lambda\phi^4$ scalar theory,[17] one may expand for weak λ to recover the

half space entropy for the theory at 1-loop,[18]

$$S_A = -\frac{|A_\perp|}{12} \int d^{d-1}k \left[\log\left(k^2 + m^2\right) + \frac{\lambda}{2} \frac{\Delta}{(k^2 + m^2)} \right]$$
$$+ \widetilde{\text{const}} + \mathcal{O}(\lambda^2). \tag{27}$$

This is precisely a consequence of the 1-loop exactitude of the Gaussian ansatz.

Let us now write the half space entropy in cMERA in terms of the tensor network bulk variational parameters. To do this we note that

$$\log\langle \Psi_\Lambda | \phi(\mathbf{k})\phi(-\mathbf{k}) | \Psi_\Lambda \rangle = -2f(k, u_{\text{IR}}) + \text{const}'. \tag{28}$$

Consequently, we have

$$S_A = -\frac{|A_\perp|}{3} \int d^{d-1}k f(k, u_{\text{IR}}) + \widetilde{\text{const}}'. \tag{29}$$

Now, according to the definition of $f(k, u_{\text{IR}})$, we have

$$f(k, u_{\text{IR}}) = \int_0^{u_{\text{IR}}} du \, g(ke^{-u}, u), \qquad g(k, u) \equiv g(u) \cdot \Gamma(k/\Lambda). \tag{30}$$

Considering these quantities and taking into account that we are integrating over momenta $0 \le k \le \Lambda$, we find

$$S_A = \frac{|A_\perp|}{3} \int d^{d-1}k \int_{u_{\text{IR}}}^0 du \, g(ke^{-u}, u) + \widetilde{\text{const}}'$$
$$= \frac{|A_\perp|}{3} \int_{u_{\text{IR}}}^0 du \, g(u) \int d^{d-1}k \, k^{d-2} \, \Gamma(ke^{-u}/\Lambda) + \widetilde{\text{const}}'. \tag{31}$$

We note that this expression is also valid for other choices of the cutoff function $\Gamma(k/\Lambda)$. Namely one can rewrite the last expression as

$$S_A = \frac{|A_\perp|}{3} \int_{u_{\text{IR}}}^0 du \, g(u) \cdot \Sigma(u) + \widetilde{\text{const}}',$$

$$\Sigma(u) \equiv \int d^{d-1}k \, k^{d-2} \, \Gamma(ke^{-u}/\Lambda). \tag{32}$$

The cut off functions, which act as alternative UV regularization schemes of the cMERA formalism, will determine $\Sigma(u)$. In **Table 1**, it is shown the resulting $\Sigma(u)$ for some cutoff functions that have been proposed in the literature.[9,20] We observe that the result is similar to the one obtained through sharp cut off function up to some numerical factors. One might interpret that these numerical factors can be absorbed into a redefinition of the UV cutoff ϵ. Consequently, and for convenience, in the rest of this section we will consider the sharp cutoff $\Gamma(x) = \Theta(1 - x)$. In this case the entanglement entropy results

$$S_A = \frac{1}{3} \frac{S^{(d-1)}}{(d-1)} |A_\perp| \Lambda^{d-1} \int_{u_{\text{IR}}}^0 du \, g(u) \, e^{u(d-1)} + \widetilde{\text{const}}', \tag{33}$$

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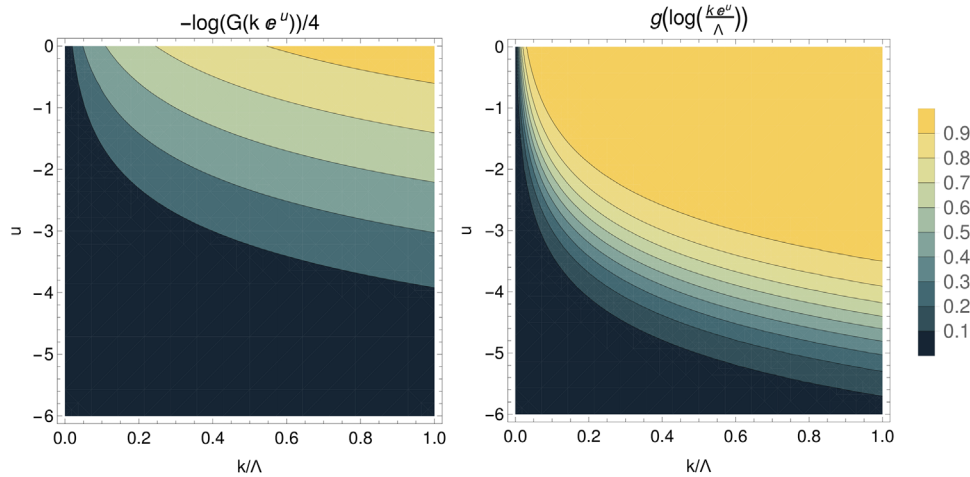der Physik

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Figure 1. Left: Rate of entanglement entropy per unit momentum (*c.f.* Eq. (34)). Right: Rate of entanglement entropy per unit scale (*c.f.* Eq. (36)). Parameters: $d = 1$, $\Lambda = 100$, $m = \frac{\Lambda}{100}$.

Table 1. For various cutoff functions $\Gamma(k/\Lambda)$, their corresponding momentum integral $\Sigma(u)$ defined in (32) and the ratio between the radial AdS metric component and the variational parameter, $\sqrt{G_{uu}}/g(u)$ are shown. $S^{(d)}$ is the area of a unit d-sphere, $S^{(d)} \equiv \frac{2\pi^{d/2}}{\Gamma(d/2)}$.

$\Gamma(k/\Lambda)$	$\Sigma(u)$	$\frac{4G_N}{\gamma}\sqrt{G_{uu}}/g(u)$
$\Theta(1 - \frac{ke^{-u}}{\Lambda})$	$\frac{S^{(d-1)}}{d-1}\Lambda^{d-1}e^{u(d-1)}$	1
$\frac{1}{2}\exp(-\frac{1}{\sigma}\frac{k^2 e^{-2u}}{\Lambda^2})$	$\frac{S^{(d-1)}}{4}\sigma^{\frac{d-1}{2}}\Gamma(\frac{d-1}{2})\Lambda^{d-1}e^{u(d-1)}$	$\frac{1}{4}\sigma^{\frac{d-1}{2}}\Gamma(\frac{d-1}{2})$
$\frac{1}{2}\frac{\Lambda^2}{\Lambda^2 + k^2 e^{-2u}}$	$\frac{S^{(d-1)}}{4}\pi\Lambda^{d-1}e^{u(d-1)}$ $(d=2)$	$\frac{\pi}{4}$

with $S^{(d)}$ the area of a d-sphere, $S^{(d)} \equiv \frac{2\pi^{d/2}}{\Gamma(d/2)}$. Thus the final result can be written as

$$S_A = \frac{1}{3}\frac{S^{(d-1)}}{(d-1)}\frac{|A_\perp|}{\epsilon^{d-1}}\int_{u_{\mathrm{IR}}}^0 du\, g(u)\, e^{u(d-1)} + \widetilde{\mathrm{const}}'. \tag{34}$$

where $\epsilon \equiv 1/\Lambda$ can be identified with the lattice constant.

Let us compare this expression with the half space entanglement entropy S_A in the discrete MERA for a $(d+1)$-dimensional quantum system on a lattice which was obtained in [10]. In this case the entanglement entropy is given by

$$S_A \propto L^{d-1}\sum_{u=-\infty}^0 n(u)\cdot 2^{(d-1)u}, \tag{35}$$

where L^{d-1} is the number of lattice points on the boundary of A and $n(u)$ is the strength of the bonds at the layer specified by the non-positive integer u. $n(u)$ is typically given by the logarithm of the bond dimension at layer u. The comparison between these expressions shows a manifestly similar structure and suggests that $g(u)$ in cMERA might be interpreted as a local *bond dimension*. We will elaborate on this in Secs. 5 and 6.

So far, we have obtained the same quantity, S_A, as an integral over momenta, Eq. (24) and over the scale, Eq. (34). Upon considering the integrand of the former expression, we define the entanglement entropy at the scale u, $S_A(u)$, as

$$S_A(u) \equiv \frac{|A_\perp|}{6}\int_0^{\Lambda e^u} d^{d-1}k_\perp\, \log\langle\Psi_\Lambda|\phi(\mathbf{k}_\perp)\phi(-\mathbf{k}_\perp)|\Psi_\Lambda\rangle + \mathrm{const}$$

$$= \frac{|A_\perp|}{6}\int_0^\Lambda d^{d-1}k_\perp\, e^{u(d-1)}\log\langle\Psi_\Lambda|\phi(\mathbf{k}_\perp e^u)\phi(-\mathbf{k}_\perp e^u)|\Psi_\Lambda\rangle$$

$$+ \mathrm{const}. \tag{36}$$

To gain some insight into the entropy production rate as a function of the momentum and as a function of the scale, let us study the integrands of (34) and (36) [4], as both procedures must give the same result. In **Figure 1** we plot, for $d = 1$, the log of the 2-point correlator $G(k_\perp e^u) \equiv \langle\Psi_\Lambda|\phi(k_\perp e^u)\phi(-k_\perp e^u)|\Psi_\Lambda\rangle$ from (36) and the variational parameter $g(\log(ke^u/\Lambda))$ from (34). For the latter, a change of variable is needed to consistently compare the two definite integrals. We observe that, qualitatively, both of them exhibit an area law behavior: highest momenta, i.e. short distance correlations, provide the highest contributions to the entropy. In contrast, while in the left plot higher momenta are gradually incorporated as $u \to 0$, in the right plot all momenta are contributing at relatively low u and only the renormalization scale determines the strength of the entanglement.

In this respect, the variational solution in (22) shows that in the conformal case ($m = 0$), $g(u)$ is constant and all length scales contribute equally to (34). In the massive case, the bond dimension in cMERA is $g(u) \approx 1/2$ for $u_w \ll \log\mu/\Lambda$ and exponentially decays to zero for $u_w > \log\mu/\Lambda$. From the bond dimension interpretation, the scale u_w marks a sort of IR wall *end of* the tensor network due to the mass gap. This wall effectgively shortens the

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[4] Let us note that, because we are introducing some momentum cutoffs, const in (36) is just a finite UV cut off dependent quantity.

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integration domain in (34) from $[-\infty, 0]$ to $[u_w, 0]$. Interestingly, in the Gaussian cMERA, as the interaction dress up the mass of the free quasiparticles through (23), the tensor network IR wall moves towards $u_w \to 0$ as the interaction strength grows. This phenomena decreases the value of the entanglement entropy w.r.t the free case at least in the regime of validity of the Gaussian approximation to the interacting case.[19,21]

5. cMERA and Holography

In order to establish an explicit connection between cMERA and holography, it would be desirable to define local areas in the bulk in terms of the differential entanglement generated along the tensor network, as it was suggested in [22]. That is to say, the infinitesimal contribution to the area of an hypothesized minimal surface homologous to the half space, must be defined to be proportional to the change in entropy. From (34), the differential entropy production rate is given by

$$\frac{dS_A}{du} = \gamma \, \frac{|A_\perp|}{\epsilon^{d-1}} \, g(u) \, e^{u(d-1)}, \tag{37}$$

where $\gamma \equiv \frac{1}{3} \frac{S^{(d-1)}}{(d-1)}$. With this, the proposal amounts to recovering a dual tensor network bulk metric as an inverse problem in terms of this differential entropy.[5]

With this, we are in a position to establish, at a proof of concept level, a relation between the AdS geometry and the cMERA tensor network through the Ryu-Takayanagi formula. To this end, let us consider a cMERA of a large number N of interacting scalar fields with $O(N)$ symmetry. Here we assume that the leading contribution to the bond dimension is given in terms of a Gaussian cMERA and the quantum corrections to this bond dimension are $1/N$.[23] In this case the cMERA entangler can be safely approximated by N copies of the entangler for a free scalar theory. With this, it is straightforward to conclude that the half space entropy amounts to

$$S_A = N \gamma \, \frac{|A_\perp|}{\epsilon^{d-1}} \int_{u_{IR}}^{0} du \, g(u) \, e^{u(d-1)} + \widetilde{\text{const}}'. \tag{39}$$

For convenience, we write (39) as

$$dS_A = N \gamma \cdot dA_{TN}, \tag{40}$$

where

$$dA_{TN} \equiv \frac{|A_\perp|}{\epsilon^{d-1}} \, g(u) \, e^{u(d-1)} \, du. \tag{41}$$

We understand dA_{TN} as an infinitesimal area surface in the bulk of the tensor network. To see this, let us assume that an ansatz for a geometric description of the tensor network is given as the

spatial slice of an asymptotically $(d+2)$-dimensional AdS metric with a radial coordinate labeled by the cMERA parameter u:

$$ds^2 = G_{uu} \, du^2 + \frac{e^{2u}}{\epsilon^2} \, d\mathbf{x}_d^2 + G_{tt} dt^2. \tag{42}$$

Here, the AdS radius has been fixed to be unity for simplicity and $G_{uu} \to$ constant as $u \to 0$. On the other hand, the Ryu-Takayanagi formula for the half space in this geometry implies that[10]

$$d \, S_A = \frac{1}{4 G_N^{(d+2)}} \cdot dA_{RT}, \tag{43}$$

where

$$dA_{RT} \equiv \frac{|A_\perp|}{\epsilon^{d-1}} \, \sqrt{G_{uu}} \, e^{u(d-1)} \, du \tag{44}$$

is an infinitesimal area in the bulk geometry.

When comparing (41) and (44) we obtain that the bulk geometry may be described in terms of the variational parameters of the tensor network. This definition arises from a computation of the entanglement entropy in cMERA following a QFT prescription and comparing the result with the holographic calculation. As a result, the entropic RG flow generated by cMERA can be consistently encoded in terms of an AdS geometry as far as the radial component of the metric is related to the variational parameter of the tensor network as

$$\frac{1}{4 G_N} \sqrt{G_{uu}} = \gamma \, g(u). \tag{45}$$

Upon identifying $\gamma = \frac{1}{4 G_N}$, we give the ratio $\sqrt{G_{uu}}/g(u)$ for other alternative cutoff functions $\Gamma(k/\Lambda)$ in the last column of Table 1. It is straightforward to note that different cMERA UV regularization schemes implemented by cutoff functions $\Gamma(k e^{-u}/\Lambda)$ do not affect the dual bulk metric up to order one numerical factors that, as mentioned above, can be reabsorbed into a redefinition of the UV cutoff. In addition, as noted in [10], for any cutoff function, changing it by $\Gamma(k \eta(e^{-u})/\Lambda)$ just amounts to a coordinate transformation along the AdS radial direction of the form $e^{-u} \to \eta(e^{-u})$.

At this point, it is interesting to interpret the IR wall scale of the tensor network commented above in terms of the differential area surfaces. Following the previous discussion, in the conformal limit where $g(u)$ is a constant in the range $[-\infty, 0]$, the bulk tensor network surface for a half space A_{TN} is interpreted as a holographic surface which hangs straight down into the bulk. On the other hand, for the massive case, the A_{TN} deeps straightly into the bulk up the IR wall u_w where it ends. From this, by means of our inverse method one might conclude that the metric G_{uu} inherits the IR hard wall feature of the bond dimension $g(u)$.

The results given above, at the proof of concept level, make precise a previous heuristic relating the scale dependent bond dimension in the discrete setting of MERA to a notion of scale dependent entanglement generation and to a notion of bulk area element of a putative holographic description. It is obvious that further generalizations must be considered. First, it is interesting to see if our conjecture for the structure of the cMERA bond dimension holds when explicitly considering a large number of fields and strong interactions. This may be

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[5] We note that for the free scalar, $0 \leq g(u) \leq 1$, and thus (37) fulfills

$$\frac{dS_A}{du} \leq \gamma \, \frac{|A_\perp|}{\epsilon^{d-1}} \, e^{u(d-1)}, \tag{38}$$

an upper bound obtained in [9], using very general conditions of the cMERA Hamiltonian evolution in scale.

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hopefully addressed with a new kind of cMERA circuits that nonperturbatively include non-quadratic entangler operators that give account for interactions.[14] Second, our matching between differential entanglement along cMERA and minimal Ryu-Taakayanagi surfaces has been established for the half space where the RT surface deeps vertically into the bulk. It is a matter of further investigation to see if/how this matching occurs for other region shapes. Using the real time approach[24] it has been numerically shown that the Gaussian cMERA matches the correct scaling of entanglement entropy of intervals of variable size for scalar and fermionic theories in $(1 + 1)$ dimensions.[25] Finding geometrical-like analytic expressions for this scaling in terms of $g(u)$ is worth to be investigated instead of the challenge it poses to find these expressions by finding analytical expressions for the eigenvalues of the covariance matrix.[24]

Fisher Information Metric

In [10], in an attempt to make the connection between cMERA and the AdS/CFT more precise, authors thought in terms of quantum distances in order to obtain dual geometrical descriptions of the tensor network. For this purpose, authors considered two cMERA states infinitesimally displaced in u. The Hilbert-Schmidt distance between them, which measures how different these states are, was given by

$$D_{\mathrm{HS}}(\Psi(u), \Psi(u + du)) \equiv 1 - |\langle \Psi(u)|\Psi(u + du)\rangle|^2 = g_{uu}(u)\, du^2,$$
(46)

where $g_{uu}(u)$ is the so called cMERA Fisher information metric. This is more conveniently defined as

$$g_{uu}(u)du^2 = \mathcal{N}^{-1}\left(1 - |\langle \Phi(u)|\Phi(u + du)\rangle|^2\right),$$

$$|\Phi(u)\rangle \equiv P e^{-i \int_{u_{\mathrm{IR}}}^{u} \hat{K}(s)ds} |\Omega\rangle,$$
(47)

where $\mathcal{N}$ is a normalization constant. For the particular case of the Gaussian entangler (19), the metric results $g_{uu} = g(u)^2$.

Our results above provide a more concise relationship between the Fisher metric and a dual geometrical description of the cMERA circuit. Indeed, because g_{uu} measures the density of disentanglers, the entanglement entropy obtained from the formula (34) can be naturally interpreted as the summation of the entanglers that cut the curve that divides the system at a certain scale u. In other words, the entanglement entropy for a cMERA circuit is given by

$$S_A = \gamma \frac{|A_\perp|}{\epsilon^{d-1}} \int_{u_{\mathrm{IR}}}^0 du \, \sqrt{g_{uu}(u)}\, e^{u(d-1)} + \widetilde{\mathrm{const}}'.$$
(48)

where the Fisher information factor $\sqrt{g_{uu}}$, is the integral measure of the "curved" tensor network bulk space along the entanglement renormalization direction. In this sense, our formulation establishes a precise connection between the Fisher information metric and putative holographic descriptions of the cMERA tensor network.

In view of these results, we note that in case of enlarging the number of entangler operators in a cMERA circuit,

$$\hat{K}(u) = \hat{K}_1(g_1(u); u) + \cdots + \hat{K}_n(g_n(u); u),$$
(49)

with $\hat{K}_i$ an entangler containing a generic dependence on the scale parameters and a variational parameter $g_i(u)$, the Fisher metric g_{uu} will be a quadratic function of the variational parameters $g_{uu} = \sum_{ij} \alpha_{ij} g_i(u)g_j(u)$, where α_{ij} are real valued coefficients associated to the expectation values of the multiple (products of) disentanglers. This suggests that adding more entanglers, and consequently having more terms in this sum, is the analog of increasing the bond dimension in the discrete MERA circuit. In this respect, some recent non-Gaussian formulations of cMERA for interacting field theories, incorporate additional non quadratic entanglers (with their respective variational parameters), giving rise to genuinely non-Gaussian effects.[13,14,17] Consequently, we conjecture that, for those circuits, the entanglement entropy can be computed through (48). The results so obtained would be compared with recent results in the literature[19,26,27] dealing with entanglement entropy in interacting theories.

6. Conclusions

We have obtained the entanglement entropy for the half-space of a free scalar theory in a Gaussian cMERA tensor network. The resulting entropy can be written as a function of the local bond dimension, the variational parameter defining the tensor network. Our results explicitly show that the differential change in the entropy along the cMERA flow can be cast in terms of the differential contribution to the area of a minimal surface of a dual AdS geometry. As a result, the dual geometry is defined in terms of the local bond dimension of cMERA. Namely, the entanglement entropy can be computed through the Fisher information metric g_{uu}, which we justify, amounts to the avatar of the bond dimension in the discrete version of the MERA circuit. The Fisher-metric acts precisely as the integral measure of the "curved" tensor network bulk space along the scale direction. Our proof of concept result makes precise previous heuristics relating the scale dependent bond dimension in the discrete setting to a notion of scale dependent entanglement generation and to a notion of bulk area element of a putative holographic description. Finally, our results spurs us to broaden this analysis to more general (non-)Gaussian cMERA circuits and regions with a more general shape.

Conflict of Interest

The authors have declared no conflict of interest.

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