A multidisciplinary teaching tool: Solving chaotic systems by electrical analogy

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Subject:

Innovation experiences based on the use of TIC.
New scenarios of teaching and learning

Abstract

Ordinary differential equations represent the mathematical models of a great variety of problems in Science and Engineering which means that two different problems are equivalent from mathematical point of view if they are formulated by the same governing equations; a subject that is forgotten and even not perceived by most of students. Within this field of problems are those concerning with chaotic systems of an only variable belonging to the modern theory of chaos. As a multidisciplinary tool of teaching and learning, the subject of this communication is to design network models, or circuits, whose governing equation are formally equivalent to that of chaotic system, allowing its dynamic simulation easily in suitable codes of free use. Thanks to the lineal and non-lineal electrical components contained in the libraries of these codes, very few and intuitive programing rules are required for the design. So, we have a multidisciplinary tool that allows the students of first course of Graduate in Engineering and Sciences to solve this kind of systems, whatever is the order of equations, grade or type of non-linearity. An application is presented to illustrate the proposed subject.

Keywords: Multidisciplinary teaching, Chaos, Network method, Dynamic simulation

Resumen

Las ecuaciones diferenciales ordinarias representan modelos matemáticos de una gran variedad de problemas de ciencia e ingeniería, lo que significa que dos problemas diferentes desde el punto de vista físico o matemático son equivalentes si se formulan con el mismo conjunto de ecuaciones, un tema olvidado e incluso que pasa desapercibido para la mayoría de estudiantes. Dentro de este campo de disciplinas está la teoría del caos o de los sistemas caóticos sencillos de una variable. A modo de una herramienta multidisciplinar de enseñanza, el objeto de esta comunicación es el aprendizaje y enseñanza del diseño de modelos en red (circuitos) cuya ecuación de gobierno es formalmente equivalente a la del sistema caótico, permitiendo su simulación dinámica directa en códigos apropiados de libre uso. Gracias a los componentes lineales y no-lineales contenidos en las librerías de estos códigos, las reglas para la programación del modelo son muy pocas e

intuitivas, dotando así a los estudiantes de primeros cursos de ciencias e ingeniería de una herramienta multidisciplinar potente para resolver problemas caóticos, cualquiera que sea el orden, grado o tipo de no-linealidad contenida en la ecuación de gobierno. Se presenta una aplicación para ilustrar el objetivo propuesto.

Palabras Claves: Enseñanza multidisciplinar, Caos, Método de redes, Simulación dinámica

Introduction

Ordinary differential equations are the mathematical formulation of a great variety of problems in different fields of science and engineering, for example in mechanical engineering or applied physics [1]. In the first courses of graduate studies, students learn to solve certain types of ordinary differential equations after establishing their formal classification, both by means or analytic or semi-analytic methods, in disciplines belonging to the mathematical area, while other more complex equations required numerical methods to be solved, a subject also integrated in the same area of knowledge but out of the scope of first courses.

The subject of this communication is double. On the one hand, to use the electric analogy for the design of circuits (which we named network models) whose governing equation is formally equivalent to that of the mathematical model of interest – equation that has no restriction as regards the order, grade or kind of non-linearity. On the second hand, to run these models in a suitable, free software of circuit simulation [2] in order to obtain a numerical solution. Taking into account these subjects, the student has in their hands a multidisciplinary and pòwerful tool that allows him the solution of simple (anharmonic pendulum, dumped oscillator, skydiver equation...) and not so easy problems (Herzt oscillator, atractor separated by a repeller, Duffing equation...).

Fisrtly, the proposed analogy requires a formal equivalence between the dependent variable, an arbitrary physic quantity, and that of the circuit (electric voltage or current). Time is the common independent variable both in the physical processes as well as in the network model. For example, the displacement of a particle in a mechanical problem can be related either to the voltage or to the current variable; in the first case the instantaneous velocity is defined as the derivative function of the electric voltage (a quantity easily measured or implemented in the network model by means of a basic auxiliary circuit) while in the second case the velocity is given by the instantaneous changes of the electric current (a quantity also easily implemented by an auxiliar circuit). Another example, the instantaneous change of mass of a certain body can be related to the electric current providing that the total change of mass along a finite time is related to the electric charge, a new quantity obtained by direct integratiojn of the current.

So, the succesive derivative terms of the dependent variable can have physical meaning and, in consequence, be of interest for the user; however, their derivation does not require new equivalences between these terms and the electric quantities of the network model since these are implicit in the first relation between physicasl and network systems. In fact, the derivative terms – appearing or not in the governing equation – are obtained by mathematical manipulation of the output simulation data or by the implementation of new auxiliary circuits in the model.

The design of the model makes use of:

i) Constitutive equations that define the connection between voltage and current in lineal basic – electric – components such as resistors, coils and capacitors [3],

ii) Controlled – voltage and current – sources that allow the implementationn of any kind of non-linearity contained in the governing equation of the physical problem [4].

Constitutive quations are generally given by differential or their correspondent integral equations of first order that, applying them succesively, allow to represent a term of any order within the equation . So, from the expression

$$i_{c,1} = C_1 \left(\frac{dv_{c,1}}{dt}\right) \tag{1}$$

that relates the current through a capacitor ($i_{c,1}$) with the difference of voltage at its ends ($v_{c,1}$), it is inmediate to define the derivative term

$$\left(\frac{di_{c,1}}{dt}\right) = C_1 \left(\frac{d^2 v_{c,1}}{dt^2}\right) \tag{2}$$

following the steps: (i) to change the current $i_{c,1}$ into a voltage quantity of the same value with a special source named ‘current-controlled voltage source‘; (2) to apply the ends of this source to a new capacitor C_2. The current through this capacitor is given by

$$i_{c,2} = C_2 \left(\frac{dv_{c,2}}{dt}\right) = C_2 \left(\frac{di_{c,1}}{dt}\right) = C_1 C_2 \left(\frac{d^2 v_{c,1}}{dt^2}\right) \tag{3}$$

Repeating this process, providing a capacitance unity to C_1, the succesive derivative terrms of the current $i_{c,1}$,

$$\left(\frac{d^2 i_{c,1}}{dt^2}\right) = \left(\frac{d^3 v_{c,1}}{dt^3}\right), \left(\frac{d^3 i_{c,1}}{dt^3}\right) = \left(\frac{d^4 v_{c,1}}{dt^4}\right) \ldots$$

can be derived. The existance of four types of controlled sources (‘controlled-current voltage-source‘, ‘controlled-voltage voltage-source‘, ‘controlled-current current-source‘ and ‘controlled-voltage current-source‘) together with the possibility of specifying tha control action of these sources by software, as a function of the dependent variable, extend the design of the model to any form of the governing equation that, generally, lucks of analytical solution.

Based on the above, the basic knowledge the student requires for the design of the models – notions of a differential equation, non-lineal concept, Ohm´s law and Kirchooff´s theorems and constitutive equations of resistors, capacitors and coils - are given him in disciplines of first courses; so that in this level he is able to use this as powerful and useful tool for the solution of ordnary diferential equations.

With all, the proposed work intends to reach an added aim, the multidisciplinary conception of the application of similar differential equations to different fields of science and engineering.

Frequently, the student learns the way of solving these equations with established algorithms and protocols without to know the meaning of what the equation represents. Note that a differential equation is not more than the expression, in mathematical simbology, of the balance of a particular physical process (movement of a particle, radiatice decay of a nuclear body, population evolution...), balance that can be alternatively implemented by a suitable electric circuit. In this sense, the relations between equations and electric models allows him to connect two disciplines and enhance the use of interdisciplinariety to study and solve more complex problems.

The ordinary differential equations

The general scheme of the type of differential equation we intend to solve by the network method is the following:

$$\left(\frac{d^n y}{dt^n}\right) + f_{n-1}\left(\frac{d^{n-1}y}{dt^{n-1}}\right)^{q_{n-1}} + f_{n-2}\left(\frac{d^{n-2}y}{dt^{n-2}}\right)^{q_{n-2}} + \cdots + f_1\left(\frac{dy}{dt}\right)^{q_1} + f_0 = 0 \qquad (4)$$

where y and t are the dependent and independent variables, respectively, and n the order of the equation; the larger integer value of the exponential coefficients, $q_i(1 \le i \le n-1)$, defines the grade of the equation; finally, $f_i(0 \le i \le n-1)$ are arbitrary functions whose arguments can be t, y or any of the y derivatives (y', y'', y'''…). In this way, equation (4) ranges all the possible spectrum of existing differential equations. As examples we will write the followings:

$\left(\frac{d^2 y}{dt^2}\right) + a_o\left(\frac{dy}{dt}\right)^2 - g = 0$ Sky-diver equation

$\left(\frac{d^2 y}{dt^2}\right) + a_o \mathrm{sen}(y) = 0$ Pendulum equation(non-harmonic oscillator)

$\left(\frac{d^2 y}{dt^2}\right) + \alpha\left(\frac{dy}{dt}\right) + \beta y = 0$ Dumped oscillator

$\left(\frac{d^2 y}{dt^2}\right) + 0.1\left(\frac{dy}{dt}\right) + 20 \mathrm{sen}(y) = 0$ Typical non-lineal oscillator

$\left(\frac{d^2 y}{dt^2}\right) + 2\beta(1 - \alpha y^2 + \gamma y^4)\left(\frac{dy}{dt}\right) + y = 0$ Attractor separated by repeller

$\left(\frac{d^2 y}{dt^2}\right) + a_o y + b_o y^3 = 0$ Duffing equation

The particular solution of ordinary differential equations requires, besides, a number of initial conditions.

Design of the network model

The first rule for the design is to assume each addend of the equation as an electric current that balances algebraically with the rest of the addends in a common 'principal' node where the solution y(t) – voltage of this node – is electrically adjusts in order to satisfy the imposed balance. Generally, the addends are implemented as controlled current-sources whose output is defined by programming, using a simple language, as functions of any argument (y, t, dy/dt, d^2y/dt^2…) whose lecture is read at the suitable nodes of auxiliary circuits or at the proper main node.

The successive derivative terms are implemented by elemental auxiliary circuits that contain lineal electric components as well as controlled-sources. Finally, initial conditions are implemented fixing the initial charge or voltage at the capacitors.

Auxiliary circuits. Design of the successive derivative terms.

The first step for the design of the model is to implement the auxiliary circuits that allow to obtaining the successive derivative terms dy/dt, d^2y/dt^2, d^3y/dt^3… Since each term requires information referred to the terms of less order, the auxiliary circuits are implemented on a given order, from the first term, dy/dt, to the last, d^ny/dt^n, regardless the term is or not contained in the equation.

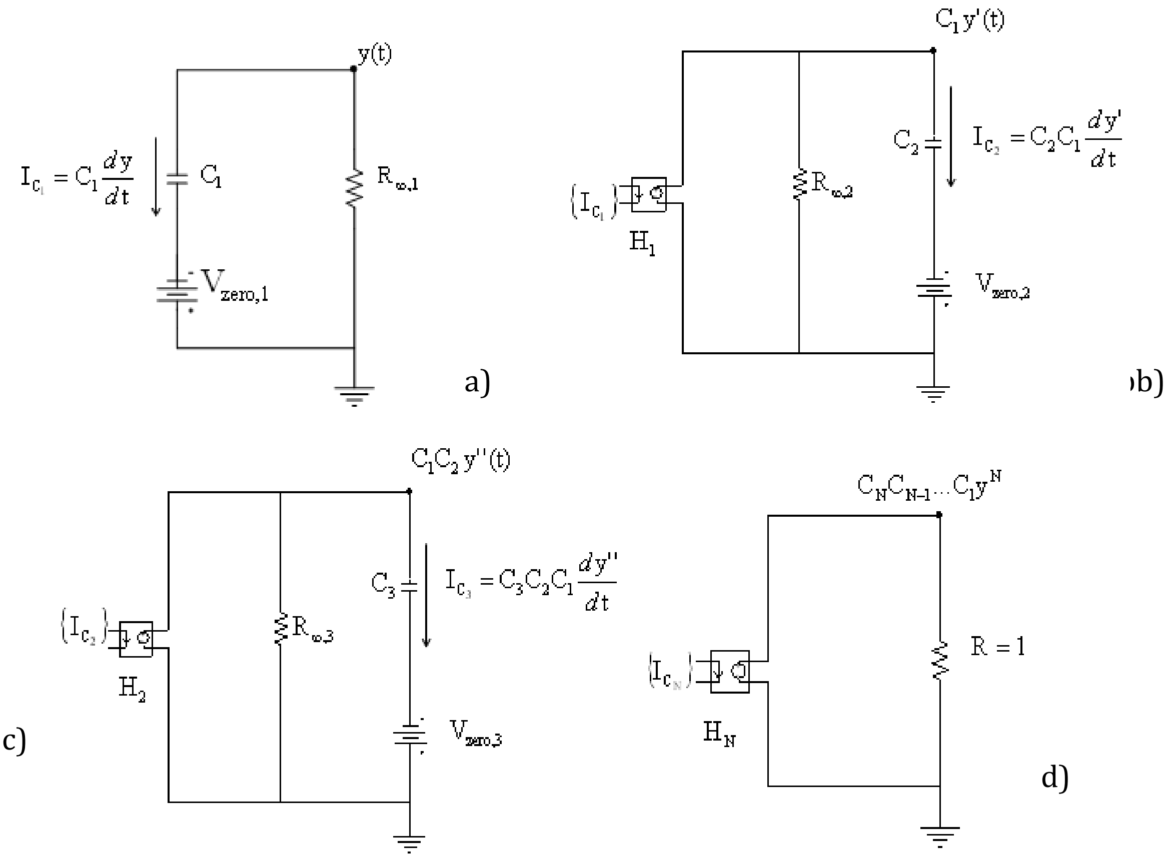


Figure 1.Auxiliary circuits to implement the successive derivative terms
dy/dt (a y b), d^2y/dt^2 (c) y d^3y/dt^3 (d)

Starting from the potential solution y(t) that is found at the main node, the first derivative, dy/dt, is the current through the capacitor C_1 of capacitance unity applying the voltage y(t) at its ends, since $i_{C1} = C_1(dv_c/dt) = dy(t)/dt$, Figure 1a.

Derivatives functions, d^2y/dt^2 y d^3y/dt^3, are obtained by similar circuits, Figures 1b and 1c. In this figures, current-controlled voltage-source changes the currents i_{C1} and i_{C2} to voltages of the same values and apply them to the respective capacitors C_2 and C_3, whose currents are the derivatives d^2y/dt^2 and d^3y/dt^3. Derivative terms of higher order are implemented by new auxiliary but similar circuits formed by controlled voltage-source and capacitors. Finally, the resistors that appear in the circuit are imposed by the simulation code (Pspice, [4]) in order to fit continuity criteria to the circuit. The last derivative term, a resistor of resistance unity is fixed at the ends of the last source, Figure 1d.

Initial conditions of the problem,

$$y_{(t=0)} = y_o,\ dy/dt_{(t=0)} = y_o',\ d^2y/dt^2_{(t=0)} = y_o'' \ldots, \qquad (5)$$

are implemented by fixing these initial voltages to the associated capacitors of each auxiliary circuit.

Main circuit. The solution y(t)

As mentioned, there are four controlled-sources in the libraries of the standard circuit simulation codes such as Pspice, Orcad and others. These special and non-lineal components allow to implement any kind of conversion between voltage and current and vice-versa as well as to implement any mathematical expression of the terms in the differential equation that contains dependencies with voltages at any node of the model or with currents at any component. In all these sources, the output is specified with simple and intuitive rules by programming.

The main circuit is directly related with the topology of the governing equation under study and, as a consequence, contains as many branches as terms in such equation. Each branch contains a controlled-voltage current-source that provides the current that go into the node establishing the required balance; the quantities that control these sources are voltages read directly at the convenient nodes of the auxiliary circuits: dy/dt is read at the ends of source H_1, d^2y/dt^2 at the ends of H_2…, and the solution y(t) at the node of the main circuit.

Since the specification of the controlled-sources of the main circuit is carried out by programming using simple rules, it is immediate to implement any mathematical function of a term in the governing equation. To this end, it is enough to write the given expression substituting the derivative terms that appear as arguments by the voltage at the associated nodes.

In the following section the network model of a system formed by two attractors separated by repeller is described step by step, in order to illustrate all the aspects related to the design of the main and auxiliary circuits.

It is convenient to mention at this point that most of the circuit simulation codes such as Pspice, the one chosen in this work, allow create the file of the model not only as a text file but also as a schematic file using a symbolic standard electric language quite familiar even to the students of first courses in science and engineering. In addition, these codes provide their own output graphic ambient, quite powerful and illustrative, that allows to showing the dynamic solution of the problem as well as other post-processed representations.

Application: Attractors separated by repeller

The governing equation of this oscillator is

$$\left(\frac{d^2y}{dt^2}\right) + 2\beta(1 - \alpha y^2 + \gamma y^4)\left(\frac{dy}{dt}\right) + y = 0 \quad (6)$$

This is a non-lineal equation of second order and first grade, without analytical solution [5]. α, β and γ are physical parameters of the system whose values provide solutions in the form of particular orbits y = y(t). Expression (6) is a particular case of Van der Pol's equation [6,7]

$$\left(\frac{d^2y}{dt^2}\right) + \mu(1 - \alpha y^2)\left(\frac{dy}{dt}\right) + y = 0 \quad (7)$$

which represents auto-oscillation systems that emerge in electronics generators of the vacuum tube type.

The network model is formed by three branches related to the three terms in the equation. The first, implemented by the controlled source G_1, drives the current d^2y/dt^2 while the second, implemented by G_2, drives the current $2\beta(1-\alpha y^2+\gamma y^4)(dy/dt)$; finally, the third term, implemented by G_3, drives the current y(t), just the solution of the equation once the balances adjust each other, Figure 2a. The sense of the current in each source is coherent with the algebraic sign in the equation in order to carry out the balance correctly. Nodes of the model related to the control variables of the sources are point out between brackets close to each source in the figure.

The first auxiliary circuit, Figure 2b, is formed by the voltage-controlled voltage-source E_1 and the capacitor C_1 of capacitance unity. Input and output voltages of E_1 is that of node 1, y(t), producing as the current through the capacitor the value of the derivative dy/dt. This values is transported to the second auxiliary circuit formed by the current-controlled voltage-source H_1 and the capacitor C_2 of capacitance unity. The current through this capacitor is, in this way, d^2y/dt^2.

Finally, to change d^2y/dt^2 to a voltage to control the first source of the main circuit, G_1, the auxiliary circuit formed by H_2 (voltage-controlled current-source) and R (resistor of resistance unity) is implemented. With all, the following association between nodes of the model and solutions of the problem can be established:

Nodo 1 (circuito principal)	↔	y(t)
Nodo 10 (circuito auxiliar)	↔	y(t)
Nodo 20 (circuito auxiliar)	↔	dy/dt
Nodo 21 (circuito auxiliar)	↔	d^2y/dt^2

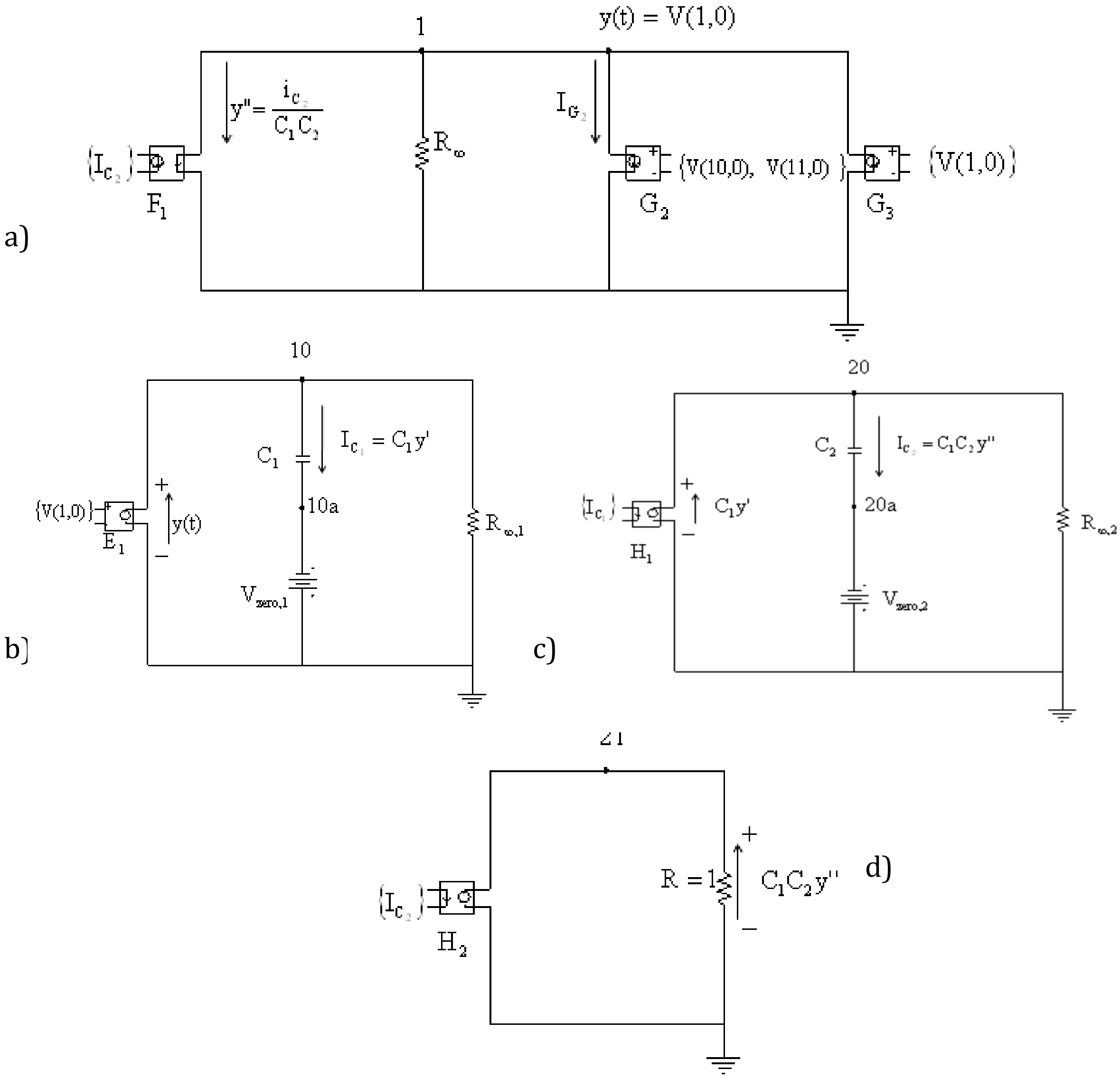


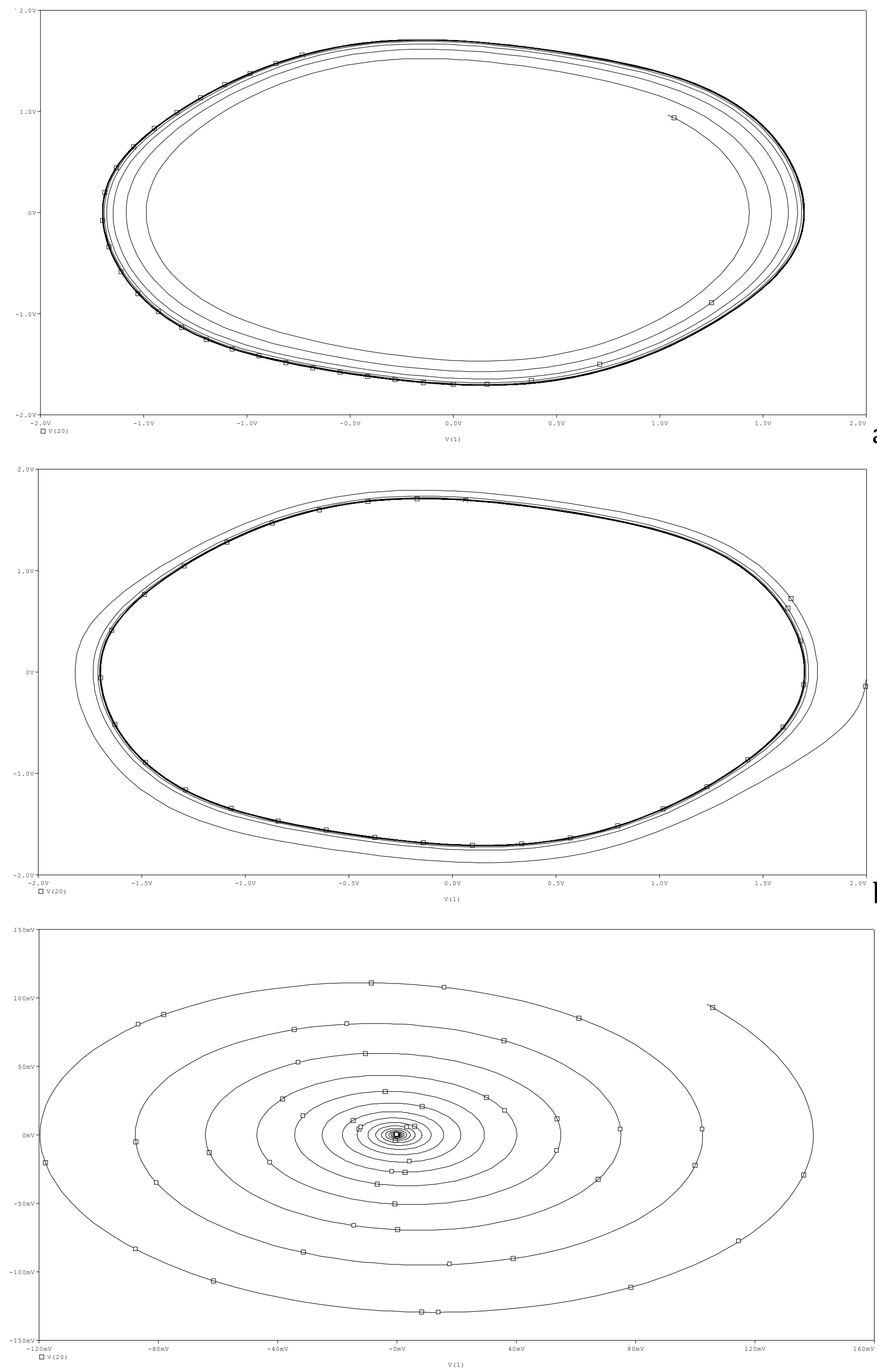
Figure 2.Network model of the equation $y'' + 2\beta(1-\alpha y^2+\gamma y^4)y' + y = 0$.
a: main circuit, b-d: auxiliary circuits

The simulation of the network model in Pspice[4] provides the solutions, in the form of phase diagrams (y versus *d*y/*d*t), show in the following figures:

Figure 3a for the initial conditions $y_{(t=0)} = dy/dt_{(t=0)} = 1$,

Figura3b for the initial conditions $y_{(t=0)} = 2$, $dy/dt_{(t=0)} = 0$

These curves are attractors that converge towards the same orbit, the first, from the exterior and the second from the interior of the limit orbit. For the conditions $y_{(t=0)} = dy/dt_{(t=0)} = 0.1$, Figure 3c, the solution is an orbit that converges towards the origin $y = dy/dt = 0$. These results are in according with numerical solutions obtained by classical methods [6]



a)

b)

c)

Figure 3.Phase diagramsof the Van der Pol´s oscillator.
Initial conditions: y´=y=1 (a); y´= 0, y=2 (b) and y´=y=0.1 (c)

Summary

Constitutive laws in terms of derivative expressions of the lineal electric components (resistors, coils and capacitors) together with Kirchoff theorems related to the conservation of the electric charge and unicity of the electric voltage, have allowed the design of network models capable to solve ordinary differential equations regardless the

order, grade and type of non-linearity. This provides the student of first courses of graduate sciences and engineering carriers with an interdisciplinary teaching and learning tool that help him in task related to this field of knowledge.

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