

## GROUP QUANTIZATION ON CONFIGURATION SPACE \*

*Miguel Navarro<sup>†1</sup>, Víctor Aldaya<sup>2,3</sup> and Manuel Calixto<sup>2</sup>*

1. The Blackett Laboratory, Imperial College, Prince Consort Road, London SW7 2BZ; United Kingdom.
2. Instituto Carlos I de Física Teórica y Computacional, Facultad de Ciencias, Universidad de Granada, Campus de Fuentenueva, 18002, Granada, Spain.
3. IFIC, Centro Mixto Universidad de Valencia-CSIC, Burjassot 46100-Valencia, Spain.

### Abstract

New features of a previously introduced Group Approach to Quantization are presented. We show that the construction of the symmetry group associated with the system to be quantized (the “quantizing group”) does not require, in general, the explicit construction of the phase space of the system, i.e., does not require the actual knowledge of the general solution of the classical equations of motion: in many relevant cases an implicit construction of the group can be given, directly, on configuration space. As an application we construct the symmetry group for the conformally invariant massless scalar and electromagnetic fields and the scalar and Dirac fields evolving in a symmetric curved space-time or interacting with symmetric classical electromagnetic fields. Further generalizations of the present procedure are also discussed and in particular the conditions under which non-abelian (mainly Kac-Moody) groups can be included.

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<sup>†</sup>On leave of absence from [2] and [3].

# 1 Introduction.

The conventional perturbative methods of quantization does not work properly with several relevant field theories. In addition to this, even in the case of theories for which a perturbative approach is possible, there is some information which cannot be obtained by perturbative techniques because of its global nature. It is, therefore, necessary to look for other non-perturbative methods to extract this information from these quantum theories. A quantization method which might be specially suitable to perform this task is the Group Approach to Quantization (GAQ) formalism.

The GAQ formalism was introduced several years ago (see, e.g. [1, 3] and references therein) as an improved version of the Geometric Quantization and the Kirillov coadjoint orbit methods of quantization [11, 7]. One of the major aims in the construction of the algorithm was the possibility of arriving at the quantum solutions of a given physical system without explicitly solving the corresponding classical equation of motion, thus allowing for a quantum system for which a classical limit is not properly defined or the classical equations do not have a well defined general solution. However, the required understanding of the basic symmetry group  $G$  is so accurate that, in many cases, the effort to find it could be nearly equivalent to solving the problem. In this sense, GAQ is perhaps more useful as a tool for associating exactly solved (quantum systems) with already classified Lie groups than a method to quantize physical systems originally defined by a Lagrangian.

This paper is intended to be a first step in the opposite direction, i.e. towards the construction of quantum groups, closely related to actual Lagrangians (or, equivalently, equations of motions), and depending on fields given in configuration space, that is to say, formal groups of sections of a given fiber bundle on space-time directly attached to the physical system. We shall call this form of the formalism *configuration-space image* of the formalism. As basic result, previous to this goal, it will be shown that the construction of the quantum group does not require, in principle, the previous step of going to the phase space of the system, i.e., does not require to solve explicitly the classical equations of motion.

The configuration-image of the formalism will provide us in addition a clearer view of the exact nature of the quantum group and will make it clear the relationship of this formalism with the Lagrangian and canonical formalisms.

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\*In this paper we shall refer to this group as a quantum group. The reader should not confuse this notion with another notion of “quantum group” which frequently appears in the literature and which, in fact, does not correspond to a group nor a quantum system either.

We shall also show some of the advantages of expressing the formalism in configuration-space image by constructing the quantum group for several non-trivial fields in a natural and straightforward way: the conformally invariant massless escalar and electromagnetic fields, and the scalar field in a symmetric curved space-time or interacting with arbitrary symmetric classical electromagnetic fields (Section 3).

This image of the formalism, as well as the examples here presented, can also serve as a guide for further generalizations. We will discuss on some of them and will argue that a direct generalization for non-abelian Kac-Moody groups requires the equations of motion to be of first order. We shall show by means of two examples that this is not the case for other types of non-abelian groups (Section 4).

In this paper we shall not get involved with the subtleties of the proper quantization procedure. The interested reader may found them in some of the quoted references.

## 2 The harmonic oscillator and the Klein-Gordon field.

In this section we shall make use of two simples examples, the harmonic oscillator and the Klein-Gordon field, to present the basic features of the formalism when the quantum group can be written in terms of fields in configuration espace.

The group law proposed in ref. [1] for the one-dimensional harmonic oscillator [the extended one-dimensional Galilei group, the quantum group for the non-relativistic free particle can be obtained as the limiting case  $\omega \rightarrow 0$ ] was:

$$\begin{aligned}
 A'' &= A + A' \cos \omega B + \left(\frac{V'}{\omega}\right) \sin \omega B \\
 V'' &= V + V' \cos \omega B - \omega A' \sin \omega B \\
 B'' &= B + B' \\
 \zeta'' &= \zeta \zeta' \exp \frac{i}{2} \left[ V(A' \cos \omega B + \frac{V'}{\omega} \sin \omega B) - A(V' \cos \omega B - \omega A' \sin \omega B) \right]
 \end{aligned} \tag{1}$$

Nevertheless, appart from the fact that the algebra of this group is isomorphic to the algebra of the basic observables of the harmonic oscillator not much has been explained on the manner in which this group is actually associated with the harmonic oscillator. Let us consider, however, the H.O. from the lagrangian point of view. The action is:

$$S_{H.O.} = \frac{1}{2}m \int dt [\dot{q}^2 - \omega^2 q^2] \quad (2)$$

The equations of motion are, therefore,

$$\left[ \frac{d^2}{dt^2} + \omega^2 \right] q = 0 \quad (3)$$

with general solution:

$$\begin{aligned} q(t) &= q_0 \cos \omega t + \frac{\dot{q}_0}{\omega} \sin \omega t \\ \Rightarrow \dot{q}(t) &= -\dot{q}_0 \sin \omega t + \omega q_0 \cos \omega t \end{aligned} \quad (4)$$

Hence, by means of the identification  $A \equiv q_0$ ,  $V \equiv \dot{q}_0$ ,  $t \equiv B$ ; and associating a time evolution to the coordinates of the phase space in the natural manner, the group law in eq. (1) can be written in the form:

$$\begin{aligned} A'' &= A + A'(B) \\ V'' &= V + V'(B) \\ B'' &= B + B' \\ \zeta'' &= \zeta \zeta' \exp \frac{i}{2} m [V A'(B) - A V'(B)] \end{aligned} \quad (5)$$

Now, taking into account that  $\dot{q} \equiv \dot{A} = V$ , this group law can straightforwardly be written on configuration space:

$$\begin{aligned} B'' &= B + B' \\ q''(t) &= q(t) + q'(t + B) \\ \zeta'' &= \zeta \zeta' \exp \frac{i}{2} m [\dot{q}(t) q'(t + B) - q(t) \dot{q}'(t + B)]_{|t=t_0} \end{aligned} \quad (6)$$

Instead of discussing here what can we learn from the simple manipulations above and the results obtained from them, it is preferable to consider first the case of a field, the Klein-Gordon field for instance. The action and equations of motion are:

$$S_{KG} = \frac{1}{2} \int d^4x [\partial^\mu \phi \partial_\mu \phi - m^2 \phi^2] \quad (7)$$

$$\Rightarrow [\square + m^2] \phi = 0 \quad (8)$$

The analogy with the harmonic oscillator can serve us as a guide to propose a quantum group for the fields directly on configuration space (in order to simplify the discussion we will not consider here Lorentz transformations, but only the symmetries associated with space-time translations). Let us try with the following composition law:

$$\begin{aligned}
a'' &= a + a' & a \in R^4 \\
\phi''(x) &= \phi(x) + \phi'(x + a) \\
\zeta'' &= \zeta' \zeta \exp \frac{i}{2} \xi_{KG}(g', g) \\
&\equiv \zeta' \zeta \exp \frac{i}{2} \int_{\Sigma} d\sigma_{\mu}(x) \mathcal{J}_{KG}^{\mu}(g, g')(x)
\end{aligned} \tag{9}$$

where

$$\mathcal{J}_{KG}^{\mu}(g, g')(x) = \partial^{\mu} \phi(x) \phi'(x + a) - \phi(x) \partial^{\mu} \phi'(x + a) \tag{10}$$

and  $\Sigma$  is any space-like hypersurface.

Now, let us expand the fields  $\phi$  that are solution of the equation of motion (8) in Fourier modes by means of

$$\phi(x) = \int \frac{d^3k}{2k^0} \left( \Phi(k) e^{-ikx} + \Phi^+(k) e^{ikx} \right) \tag{11}$$

It is not difficult to see that we get for the Fourier modes  $\Phi(k)$  the composition law postulated in ref. [2]. This implies that the composition law above actually corresponds to a group law. It is, nonetheless, easy to see that we do not have to make use of the general solution (11) to show that eqs. (9,10) defines a group law. In fact, the requirement that the fields in eq. (10) be solutions of the equation of motion is enough to show that

a) the quantity  $\xi(g', g)$  fulfils the cocycle property:

$$\xi(g'', g') + \xi(g'' * g', g) = \xi(g'', g' * g) + \xi(g', g) \tag{12}$$

and that

b) the current  $\mathcal{J}^{\mu}$  is conserved

$$\partial_{\mu} \mathcal{J}^{\mu} = 0 \tag{13}$$

In fact, in all the cases studied in this paper there is a double implication a)  $\Leftrightarrow$  b): to find out a divergenceless current made up of the fields solution of the equations of motion has always proved to be sufficient for the quantity

$\xi$ , constructed on it, to fulfil the cocycle property. We do not know whether this is the general case.

We can, therefore, state the following features of the formalism when expressed in configuration space:

1. The basic group (the group to be centrally extended) is a group irrespective whether or not the fields involved fulfil the equations of motions.

2. The central extension involves the integral of a divergenceless current  $\mathcal{J}^\mu$  over an hypersurface  $\Sigma$ . This current is divergenceless only over the fields that obey the equations of motion.

Therefore the centrally extended group, the quantum group, *involves only the fields that are solutions of the equations of motion*. It is constructed upon the *phase space* of the system.

In this phase space different coordinates can, of course, be chosen. The choice of the Fourier modes  $\Phi(k)$  leads us to the composition law in ref. [2]. Another choice is the familiar one of fields and time-like derivatives of the fields in the hypersurface  $\Sigma$ . We can write down the group law in these coordinates by making use of the following propagation property:

$$\phi(y) = \int_{\Sigma} d\sigma_{\mu}(x) [\partial^{\mu}\Delta(y-x)\phi(x) - \Delta(y-x)\partial^{\mu}\phi(x)] \quad (14)$$

where the propagator  $\Delta$  obeys the equation of motion [10]:

$$[\square + m^2]\Delta = 0 \quad (15)$$

3. The classical equations of motion, *but not their general solution*, are required to show that the quantum group is, in fact, a group.

4. The equation of motion are not determined uniquely by the group: eq. (13) implies eq. (8) for *some*  $m$  but not for a particular  $m$ . Therefore eq. (13) *almost* implies eq. (8) but there is not a complete implication. The groups in configuration space for the harmonic oscillator and the free particle are the same whereas the equations of motion are not. It is in terms of phase-space coordinates when the difference between these systems explicitly appears in the group.

### 3 Applications.

In this section we will construct the quantum group for several physically interesting systems as an application of the formalism presented above: the conformally-invariant massless scalar and electromagnetic fields, the scalar and Dirac field in a symmetric curved space-time and these same fields interacting with symmetric electromagnetic fields.

[Unlike in the previous section, where the semidirect action of the space-time symmetries were given in a rather unnatural way - a way adapted to the Schrödinger representation - we shall write in the sequel the quantum groups in the natural way; the other expression can be obtained by a simple change of coordinates.]

### 3.1 Conformally invariant fields.

The conformal group, whose composition law have not, up to now, been given in a closed form, is made up of compositions of the following actuations on the space-time:

- a) Space-time translations:  $(ux)^\alpha = x^\alpha + a^\alpha$
- b) Lorentz transformations:  $(ux)^\alpha = \Lambda_\mu^\alpha x^\mu$
- c) Dilatations:  $(ux)^\alpha = e^\lambda x^\alpha$
- d) Special conformal transformations:  $(ux)^\alpha = \frac{x^\alpha + c^\alpha x^2}{1 + 2cx + c^2 x^2}$

#### 3.1.1 The massless scalar field.

The group law (in four dimensions, in which case the conformal dimension of a scalar field is  $l = -1$ ) is:

$$\begin{aligned} u'' &= u' * u && \text{Conformal (sub)group} \\ \phi''(x) &= \phi'(x) + \Omega^{-1}(u'^{-1}, x)\phi(u'^{-1}(x)) \end{aligned} \quad (16)$$

$$\begin{aligned} \zeta'' &= \zeta' \zeta \exp \frac{i}{2} \xi_{MS}(g', g) \\ &= \zeta \zeta' \exp \frac{i}{2} \int_\Sigma d\sigma_\mu(x) \mathcal{J}_{MS}^\mu \end{aligned} \quad (17)$$

where

$$\mathcal{J}_{MS}^\mu = \phi'(x) \partial^\mu [\Omega^{-1}(u'^{-1}, x)\phi(u'^{-1}(x))] - \partial^\mu \phi'(x) [\Omega^{-1}(u'^{-1}, x)\phi(u'^{-1}(x))] \quad (18)$$

and the function  $\Omega$  is given by:

$$\Omega(u, x) = \begin{cases} 1 + 2cx + c^2 x^2 & \text{for special conformal transformation,} \\ e^{-\lambda} & \text{for dilatations,} \\ 1 & \text{for the Poincaré subgroup.} \end{cases}$$

The function  $\Omega$  for a general conformal transformation can be obtained by making use of its property

$$\Omega(u, x)\Omega(u', ux) = \Omega(u'u, x) \quad (19)$$

which is required for eq. (16) to define a group.

It is not difficult to show that if  $\phi(x)$  and  $\phi'(x)$  are solutions of the equation of motion

$$\partial^\mu \partial_\mu \phi(x) = 0 \quad (20)$$

so is  $\phi''(x)$ . It is also straightforward to show, by making use of eqs. (19- 20), that the current  $\mathcal{J}^\mu$  is divergenceless and fulfils the cocycle property (12).

### 3.1.2 The electromagnetic field.

The group law is given by:

$$\begin{aligned} u'' &= u' * u && \text{Conformal (sub)group} \\ A''_\mu(x) &= A'_\mu(x) + \frac{\partial u'^{-1\alpha}}{\partial x^\mu} A_\alpha(u'^{-1}x) && (21) \\ &\equiv A'_\mu(x) + (S(u'^{-1})A)_\mu(x) \end{aligned}$$

where  $S$  is the representation of the conformal group that acts on the electromagnetic vector field. This actuation is the natural one and means that the potential vector has null conformal weight. This actuation induces the following one on the tensor field  $F_{\mu\nu}$ :

$$F''_{\mu\nu}(x) = F'_{\mu\nu}(x) + \frac{\partial u'^{-1\alpha}}{\partial x^\mu} \frac{\partial u'^{-1\beta}}{\partial x^\nu} F_{\alpha\beta}(u'^{-1}x) \quad (22)$$

$$\equiv F'_{\mu\nu}(x) + (S(u'^{-1})F)_{\mu\nu}(x) \quad (23)$$

It is easy to show that this actuation leaves invariant Maxwell's action

$$S_M = \int d^4x F_{\mu\nu} F^{\mu\nu} \quad (24)$$

and, therefore, leaves invariant Maxwell's equations.

The central extension is given by:

$$\begin{aligned} \zeta'' &= \zeta' \zeta \exp \frac{i}{2} \xi_M(g', g) \\ &= \zeta' \zeta \exp \frac{i}{2} \int_\Sigma d\sigma_\mu(x) \mathcal{J}_M^\mu(g', g)(x) \end{aligned} \quad (25)$$

with divergenceless current

$$\mathcal{J}_M^\mu(g', g)(x) = F'^{\mu\nu}(x)(S(u'^{-1})A)_\nu(x) - A'_\nu(x)(S(u'^{-1})F)^{\mu\nu}(x) \quad (26)$$

If we restrict ourselves to the symmetry group of space-time translations the group is written:

$$\begin{aligned} a'' &= a + a' \\ A''_\mu(x) &= A'_\mu(x) + A_\mu(x - a') \\ \zeta'' &= \zeta \zeta' \exp \frac{i}{2} \int_\Sigma d\sigma_\mu(x) \mathcal{J}_M^\mu(g', g)(x) \end{aligned} \quad (27)$$

with a current

$$\mathcal{J}_M^\mu(g', g)(x) = F'^{\mu\nu}(x)A_\nu(x - a') - A'_\nu(x)F^{\mu\nu}(x - a') \quad (28)$$

Here we can see again, in this last example, the issue already notized above: The current  $\mathcal{J}_M^\mu$  is divergenceless and fulfils the cocycle property if Maxwell equations are obeyed, but a Proca-like equation  $\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0$ , with a non-null mass  $m$ , is also allowed. This is, however, no longer the case when the full conformal group is considered.

### 3.2 Matter fields in a symmetric curved space-time.

In this section we will present the quantum group for matter fields, the Klein-Gordon and Dirac fields, evolving in a symmetric, but on the other hand arbitrary, curved space-time with metric  $g_{\mu\nu}$ .

Let us suppose that the set of transformations  $v$  is a group of isometries of the metric (see for instance refs. [8, 5]). Then we have

$$g_{\mu\nu}(vx) \frac{\partial(vx)^\mu}{\partial x^\alpha} \frac{\partial(vx)^\nu}{\partial x^\beta} = g_{\alpha\beta}(x) \quad (29)$$

and, in the same way,

$$g^{\mu\nu}(vx) = g^{\alpha\beta}(x) \frac{\partial(vx)^\mu}{\partial x^\alpha} \frac{\partial(vx)^\nu}{\partial x^\beta} \quad (30)$$

### 3.2.1 The scalar field.

The equations of motion for a scalar field evolving in this background metric are (see for instance ref. [4, 6]):

$$\left[ \square(x) + \alpha R(x) + m^2 \right] \phi(x) = 0 \quad (31)$$

with

$$\square(x)\phi(x) \equiv \frac{1}{\sqrt{g(x)}} \partial_\mu \left( \sqrt{g(x)} g^{\mu\nu}(x) \partial_\nu \right) \phi(x) \quad (32)$$

The group law that would describe the quantum dynamics of this system is given by:

$$\begin{aligned} v'' &= v' * v \\ \phi''(x) &= \phi'(x) + \phi(v'^{-1}(x)) \end{aligned} \quad (33)$$

$$\zeta'' = \zeta \zeta' \exp \frac{i}{2} \int_\Sigma d\sigma_\mu(x) \mathcal{J}_{SCS}^\mu \quad (34)$$

with

$$\mathcal{J}_{SCS}^\mu = \sqrt{g(x)} g^{\mu\nu}(x) \left[ \phi'(x) \partial_\nu \left[ \phi(v'^{-1}(x)) \right] - \partial_\nu \phi'(x) \phi(v'^{-1}(x)) \right] \quad (35)$$

(Notice that eq. (29) implies  $\square(x) = \square(vx)$  and  $R(vx) = R(x)$  (see also ref. [5]).)

### 3.2.2 The Dirac field.

Let now  $\Psi$  be a Dirac field with equations of motion (see e.g. ref. [8, 4]):

$$\left[ i\hat{\gamma}^\mu \left( \partial_\mu + \frac{1}{2} i\Gamma_{\mu}^a{}^b \Sigma_{ab} \right) - m \right] \Psi = 0 \quad (36)$$

where

$$g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b \quad (37)$$

$$\hat{\gamma}^\mu = \gamma^a e_a^\mu \quad , \quad \Sigma_{ab} = \frac{1}{4} i[\gamma_a, \gamma_b] \quad (38)$$

$$\Gamma_{ab}^c = e_\nu^c e_a^\mu \left( \partial_\mu e_b^\nu + e_b^\lambda \Gamma_{\mu\lambda}^\nu \right) \quad (39)$$

The transformations  $v$  being isometries implies that there exist a set of local Lorentz transformations  $\Lambda(v, x)$  such that:

$$e^a{}_{\mu}(vx) \frac{\partial(vx)^\mu}{\partial x^\lambda} = \Lambda(v, x)^a{}_c e^c{}_{\lambda}(x) \quad (40)$$

On this grounds it can be shown that the following set of transformations is a (super)group:

$$\begin{aligned} v'' &= v' * v \\ \Psi''(x) &= \Psi'(x) + \rho(\Lambda(v'^{-1}, x))\Psi(v'^{-1}(x)) \end{aligned} \quad (41)$$

$$\begin{aligned} \bar{\Psi}''(x) &= \bar{\Psi}'(x) + \bar{\Psi}(v'^{-1}(x))\rho(\Lambda(v'^{-1}, x))^{-1} \\ \zeta'' &= \zeta'\zeta \exp \frac{i}{2} \xi_{DCS}(g', g) \\ &= \zeta'\zeta \exp \frac{i}{2} \int_{\Sigma} d\sigma_{\mu}(x) \mathcal{J}_{DCS}^{\mu}(g', g)(x) \end{aligned} \quad (42)$$

with

$$\begin{aligned} \mathcal{J}_{DCS}^{\mu}(g', g)(x) &= i \left[ \bar{\Psi}'(x) \hat{\gamma}^{\mu}(x) \rho(\Lambda(v'^{-1}, x)) \Psi(v'^{-1}x) \right. \\ &\quad \left. - \bar{\Psi}(v'^{-1}x) \rho(\Lambda(v'^{-1}, x))^{-1} \hat{\gamma}^{\mu}(x) \Psi'(x) \right] \end{aligned} \quad (43)$$

and  $\rho$  the usual spin representation of the Poincaré group which verifies:

$$\rho(\Lambda)^{-1} \gamma^a \rho(\Lambda) = \Lambda^a{}_b \gamma^b \quad (44)$$

### 3.3 Matter fields coupled to symmetric electromagnetic fields.

In this subsection we will present the quantum group for matter fields, the Klein-Gordon and Dirac fields, coupled to a symmetric, but on the other hand arbitrary, electromagnetic field.

The espace-time in this section will be flat with a Minkowskian metric  $\eta_{\mu\nu}$ . The set of transformation  $v$  will be any subgroup of the Poincare group leaving invariant the electromagnetic field:  $A_{\mu}(vx) = A_{\mu}(x)$ .

#### 3.3.1 The Klein-Gordon field.

Let the scalar field  $\phi$  obey the equation of motion

$$(D_{\mu} D^{\mu} + m^2) \phi = 0 \quad (45)$$

with  $D_\mu \equiv \partial_\mu - iA_\mu$ .

Then the following composition law defines a group:

$$v'' = v' * v \quad (46)$$

$$\phi''(x) = \phi'(x) + \phi(v'^{-1}(x)) \quad (47)$$

$$\begin{aligned} \zeta'' &= \zeta' \zeta \exp \frac{i}{2} \xi_{SAF}(g', g) \\ &= \zeta' \zeta \exp \frac{i}{2} \int_\Sigma d\sigma_\mu(x) \mathcal{J}_{SAF}^\mu(g', g)(x) \end{aligned} \quad (48)$$

with

$$\begin{aligned} \mathcal{J}_{SAF}^\mu(g', g)(x) &= D^\mu \phi(v'^{-1}) \phi^{*'}(x) - \phi(v'^{-1}) (D^\mu \phi)^{*'}(x) \\ &\quad + \left( D^\mu \phi(v'^{-1}) \right)^* \phi'(x) - \phi(v'^{-1})^* (D^\mu \phi)'(x) \end{aligned} \quad (49)$$

### 3.3.2 The Dirac field.

Let now  $\Psi$  be a Dirac field with equations of motion:

$$(i\gamma^\mu D_\mu - m) \Psi = 0 \quad (50)$$

Then the following set of transformations is a group:

$$\begin{aligned} v'' &= v' * v \\ \Psi''(x) &= \Psi'(x) + \rho(\Lambda(v'^{-1})) \Psi(v'^{-1}(x)) \end{aligned} \quad (51)$$

$$\begin{aligned} \bar{\Psi}''(x) &= \bar{\Psi}'(x) + \bar{\Psi}(v'^{-1}(x)) \rho(\Lambda(v'^{-1}))^{-1} \\ \zeta'' &= \zeta' \zeta \exp \frac{i}{2} \xi_{DAF}(g', g) \\ &= \zeta' \zeta \exp \frac{i}{2} \int_\Sigma d\sigma_\mu(x) \mathcal{J}_{DAF}^\mu(g', g)(x) \end{aligned} \quad (52)$$

with

$$\begin{aligned} \mathcal{J}_{DAF}^\mu(g', g)(x) &= i \left[ \bar{\Psi}'(x) \gamma^\mu \rho(\Lambda(v'^{-1})) \Psi(v'^{-1}x) \right. \\ &\quad \left. - \bar{\Psi}(v'^{-1}x) \rho(\Lambda(v'^{-1}))^{-1} \gamma^\mu \Psi'(x) \right] \end{aligned} \quad (53)$$

### 3.4 Comments.

We should point out here that the requirement of being symmetric for the classical fields considered above, such as the space-time metric or the electromagnetic field in subsection 3.2 and 3.3 respectively, is not very restrictive a requirement: many interesting systems, such as the one of fields evolving in a Schwarzschild black-hole background [4, 12], are not excluded by these constraint. An analysis in depth of some of these systems is in progress and will appear elsewhere.

## 4 Generalizations. The Virasoro group and the Schwarzian derivative.

Let us consider from another point of view the manner in which we arrived at the group law for the harmonic oscillator in Sec. 2, eq. (1) [Similar consideration can be made on the group laws for the other fields mentioned above.] We started with an abelian Kac-Moody group (the group of loops on  $R$ ) [composed in a semidirect way with the temporal translations which are the zero modes of the Virasoro group on the real line]. From this group we extracted the subgroup of the functions that obey certain differential equations (the equations of motion). The subgroup obtained this way can be extended, the principal ingredient for the extension being a divergenceless current.

This point of view leads us immediately to a possible generalization: in place of an abelian Kac-Moody group let us consider more general, non-abelian Kac-Moody groups. But, what are the differential equations (equations of motion) to be applied to the basic group? In another words: what differential equation is there such that if  $g(t)$  and  $h(t)$  are solutions of this equation also is  $g(t) * h(t)$  for a general non-abelian Kac-Moody group?. (This is a sort of symmetry that goes beyond those usually considered in the literature.) For the abelian Kac-Moody group the solution is obvious: any linear differential equation is such that if  $h(t)$ ,  $g(t)$  are solutions so is  $h(t)+g(t)$ . From this simple property we can construct the quantum group for the harmonic oscillator, the non-relativistic free-particle, the Klein-Gordon and Maxwell fields... But, what differential equation fulfils this property for, say, a  $SU(2)$ -Kac-Moody group?

There is strong indications that no equation of order greater than one with this property exists for any non-abelian Kac-Moody group. In fact, let us consider the Lie algebra of these Kac-Moody groups. For any abelian Kac-Moody group, the basic Lie algebra is:

$$\{\varphi_a(x), \varphi_b(y)\} = 0 \quad (54)$$

From this algebra we have to extract the subalgebra made up of the tangent fields that obey the (linearized) equations of motion and further extend it. In the abelian case this can be done with the result

$$\{\varphi_a(x), \varphi_b(y)\} = \Delta_{ab}(x - y) \quad (55)$$

where  $\Delta_{ab}$  is the propagator similar to that of eqs. (14-15). [If  $\Delta(x - y)$  is the propagator for the Klein-Gordon field, the propagator for the Dirac field is  $\Delta_D(x - y) = (i\gamma^\mu \partial_\mu + m) \Delta(x - y)$  and for the electromagnetic (or Proca) field is  $\Delta_{M\mu\nu}(x - y) = -\eta_{\mu\nu} \Delta(x - y)$ . For the harmonic oscillator it is  $\Delta_{HO}(B) = \frac{1}{\omega} \sin \omega B$  and for the non-relativistic free particle  $\Delta_{FP}(B) = B$ .] This central extension is consistent with the (linearized) equations of motion for the tangent fields.

For non-abelian fields we should start from the basic Lie algebra:

$$\{T^a(x), T^b(y)\} = f_c^{ab} T^c(x) \delta(x - y) \quad (56)$$

The Lie bracket of two elements of the Lie algebra

$$X = \int d x f_a(x) T^a(x), \quad Y = \int d x g_a(x) T^a(x) \quad (57)$$

is given by

$$[X, Y] = \int d x f_a(x) g_b(x) f_c^{ab} T^c(x) \quad (58)$$

Therefore the Lie algebra (56) can equivalently be written in terms of the coefficient functions  $f$  as follows:

$$[f, g]_c(x) = f_a(x) g_b(x) f_c^{ab} \quad (59)$$

For any equation of motion that we impose now on the group elements, the induced linearized equations of motion for the elements of the Lie algebra will be, of course, linear. But, for non-null structure constants  $f_c^{ab}$ , eq. (59) implies that no subalgebra can be defined by linear equations of order greater than one.

This result is a sort of *no go* theorem which implies that the construction above for abelian bosonic groups (fields), with equations of motion of second order, cannot be *directly* extended to non-abelian Kac-Moody groups. This

extension would require expressing the equations of motion in a typically Hamiltonian form. Only in the case of abelian Kac-Moody groups the splitting of coordinate-momentum can be made without breaking the group structure.

Now, we will show that this obstruction is not present in other kind of non-abelian groups. Let us consider, as an example, the Schwarzian derivative  $S(f)$ :

$$S(f) = \partial_t \left\{ \frac{\partial_t^2 f}{\partial_t f} \right\} - \frac{1}{2} \left( \frac{\partial_t^2 f}{\partial_t f} \right)^2 = \frac{\partial_t^3 f}{\partial_t f} - \frac{3}{2} \left( \frac{\partial_t^2 f}{\partial_t f} \right)^2 \quad (60)$$

This operator fulfils the so called Cayley property:

$$S(fog) = S(f)og(\dot{g})^2 + S(g) \quad (61)$$

where  $o$  stands for composition of functions. Therefore we will have that

$$S(f) = 0 \quad , \quad S(g) = 0 \quad \implies \quad S(fog) = 0. \quad (62)$$

Therefore, for the Virasoro group (the group of loops defined in a one dimensional manifold with the composition of function as its composition law), the vanishing of the Schwarzian derivative is a differential equation with the property we were looking for. (We will say that the Schwarzian derivative is *closed* under the Virasoro group.) Since the Virasoro group *is not* a Kac-Moody group, it does not fulfil the hipotesis of the no-go theorem above, and we see that it does fulfil its tesis neither.

The algebra of the loop group on  $R$  is the Lie algebra of all vector fields  $v(t)\partial_t$  with Lie bracket:

$$\{u, v\}(t) = -(\dot{u}(t)v(t) - u(t)\dot{v}(t)) \quad (63)$$

If the functions of the group satisfy  $S(f) = 0$ , by taking variation and taking into account that the identity function is  $f(t) = t$ , we arrive at the following (of course linear) equation for the functions in the Lie algebra of the group

$$\partial_t^3 v(t) = 0. \quad (64)$$

It is straightforward to show b applying the lineal operator  $\partial_t^3$  to both sides of eq. (63) that the set of functions which satisfy eq. (64) close into a subalgebra.

The general solution of equation (64) is a general linear combination of  $1, t, t^2$  which generates the general solution of the equation  $S(f) = 0$ :

$$f(t) = \frac{at + b}{ct + d}, \quad \text{with} \quad ad - bc = 1 \quad (65)$$

Therefore the subgroup of the Virasoro group generated by functions obeying  $S(f) = 0$  is the  $SL(2, R)$  group.

#### 4.1 The Virasoro-Kac-Moody Group and the induced 2D gravity in the light-cone gauge.

A construction similar to the one presented above for the Virasoro group can also be made for the *Virasoro-Kac-Moody group*. [Do not get confused with the Kac-Moody-Virasoro groups which are nothing other but the natural semidirect product of the Virasoro group by a Kac-Moody group]. The Virasoro-Kac-Moody group is the set of functions

$$\begin{aligned} f : R &\longrightarrow \text{Virasoro} \\ \sigma &\longrightarrow f(\sigma) \end{aligned} \quad (66)$$

with composition law  $(f * g)(\sigma)(\tau) = (f(\sigma)og(\sigma))(\tau)$ . The elements of this group are, therefore, parametrized by two coordinates  $\sigma, \tau$  and can be considered as functions of two variables,  $f(\sigma, \tau)$ .

The vanishing of the Schwarzian derivative

$$\{f(\sigma, \tau), \tau\} = 0 \quad (67)$$

is, of course, a *closed* equation for this group, thus defining a subgroup, the  $SL(2, R)$ -Kac-Moody group in this case.

The identifications  $\sigma \equiv x^+ = t+x, \tau \equiv x^- = t-x$  transform eq. (67) into the equation of motion,  $\{f, x^-\} = 0$ , of the induced 2D gravity in the light-cone gauge [9]. Moreover, in this gauge, the symplectic form is the canonical one of the  $SL(2, R)$ -Kac-Moody (sub)group. All this would indicate that the quantum theory of this system should be described in terms of irreducible, unitary representations of the  $SL(2, R)$ -Kac-Moody group. However, it is well known that there are no unitary, standard highest-weight representations with a nonzero central charge. In fact, a more rigorous analysis of this theory based on a local form of the action shows that its true reduced phase space is quite different from that one and cannot be obtained as a constrained version of the former [9]. Therefore, the role played by the  $SL(2, R)$ -Kac-Moody symmetry is in any case different from the one expected from the analysis of the non-local theory in the light-cone gauge.

## 5 Discussion.

We have presented the basic features of the GAQ formalism when the quantum group can be written in terms of fields given on configuration space. The quantum group for several physically meaningful systems have been given, albeit most of them in an implicit form.

But one might wonder what is the utility of writing down the quantum group in this implicit form. The key point in considering the quantum group is that it collects all the information about both the classical and the quantum theories. Of course we can do the most with it when we know the general solution of the equations of motion, i.e. the phase space of the theory (and, in fact, there are many important examples in which this is the case) but we believe (and this issue will be investigated in forthcoming publications) that, even when the general solution is not known, there can still be a lot of information of the quantum theory that we can possibly extract from the quantum group.

An interesting case occurs when we only know *some* of the classical solutions of the theory. With the GAQ formalism we would be able to quantize these solutions provided that we were able to find out pairs of solutions that are coordinate- momentum conjugate of each other. (In the GAQ formalism, the cocycle may be viewed as a sort of “symplectic product” which measures the extent to which a pair of classical solutions are coordinate-momentum conjugate of each other). This sort of *minisuperspace approach* would provide us with a first draft of the quantum theory and deserves a study of their own.

In addition to this, and in some sense, the phase space of any classical theory is roughly known: it is characterized by the fields and time derivative of the fields in a Cauchy hypersurface and evolving in accordance with the classical equations of motion. Putting aside topological or, in general, global issues of the phase space, the GAQ formalism equipped with this rough description of the phase space must necessarily work better than the familiar Canonical Quantization do.

Furthermore, the configuration-space image of the GAQ formalism appears as a well-suited formalism to deal with gauge theories. Indeed, gauge symmetries, far from being a mere useful tool to solve a previously given theory, determine themselves the theory and this philosophy is quite the same that inspires the GAQ formalism. However, we have proved in Sec. 4 a sort of no-go theorem which delimites the type of equations of motion of a (non-linear) system with a non-abelian Kac-Moody symmetry: these equations must be kept in a Hamiltonian-like, first-order form, since the restriction to the pure coordinate space satisfying second-order equations would destroy

the group structure. This situation resembles that of WZW models [13] where the Kac-Moody symmetry comes out in a natural way when written in a set of coordinates (light-cone coordinates) where the Lagrangian is not regular, making this way the difference between the Modified and Ordinary Hamilton Principle.

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