Higher-$U(2,2)$-spin fields and higher-dimensional $\mathcal{W}$-gravities: quantum AdS space and radiation phenomena

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Abstract

A physical and geometrical interpretation of previously introduced tensor operator algebras of $U(2,2)$ in terms of algebras of higher-conformal-spin quantum fields on the anti-de Sitter space AdS_5 is provided. These are higher-dimensional $\mathcal{W}$-like algebras and constitute a potential gauge guide principle towards the formulation of induced conformal gravities (Wess-Zumino-Witten-like models) in realistic dimensions. Some remarks on quantum (Moyal) deformations are given and potentially tractable versions of non-commutative AdS spaces are also sketched. The role of conformal symmetry in the microscopic description of Unruh's and Hawking's radiation effects is discussed.

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1 Introduction

A consistent and feasible quantum theory of the gravitational field is still lacking, but we know of some crucial tests that such a candidate theory ought to pass. Mainly, it should account for radiation phenomena, like black hole evaporation, and it should exhibit a critical behaviour at short distances (Planck scale), where the intrinsic structure of space-time and the physics we know should radically change. Actually, both the Bekenstein-Hawking formula for black hole entropy $S = \frac{k_B \Sigma}{4G\hbar}$ —and temperature (59)— and Planck length $\lambda\!\!\!^{-} = \sqrt{\hbar G/c^3}$ provide a remarkable mixture of: quantum mechanics (Planck's constant $\hbar$), gravity (Newton's constant G), relativity (velocity of light c), thermodynamics (Boltzmann's constant k_B), and geometry (the horizon area Σ).

A statistical mechanical explanation of black hole thermodynamics in terms of counting of microscopic states has been recently given in [1]. According to this reference, there is strong evidence that conformal field theories provide a universal (independent of details of the particular quantum gravity model) description of low-energy black hole entropy, which is only fixed by symmetry arguments. The Virasoro algebra turns out to be the relevant subalgebra of surface deformations of the horizon of an arbitrary black hole and constitutes the general gauge (diffeomorphism) principle that governs the density of states.

Although surface deformations appear as a constraint algebra, under which one might expect all the physical states on the horizon to be singlets, quantum anomalies and boundary conditions introduce central charges and change this picture, thus causing gauge/diffeomorphism modes to become physical along the horizon.* In this way, the calculation of thermodynamical quantities, linked to the statistical mechanical problem of counting microscopic states, is reduced to the study of the representation theory and central charges of a relevant symmetry algebra.

Unruh effect [5] (vacuum radiation in uniformly accelerated frames) is another interesting physical phenomenon linked to the previous one. A statistical mechanical description (from first principles) of it has also been given in [6] (see also Section 3.3 below) and related to the dynamical breakdown of part of the conformal symmetry: the special conformal transformations (usually interpreted as transitions to a uniformly relativistic accelerated frame), in the context of conformally invariant quantum field theory. The Unruh effect can be considered as a "first order effect" that gravity has on quantum field theory, in the sense that transitions to uniformly accelerated frames are just enough to account for it. To account for higher-order effects one should consider more general diffeomorphism algebras, but the infinite-dimensional character of conformal symmetry seems to be an exclusive patrimony of two-dimensional physics, where the Virasoro algebra provides the main gauge guide principle. This statement is not rigorously true, and the present article is intended to provide higher-dimensional analogies of the infinite two-dimensional conformal symmetry, which could be useful as potential gauge guiding principles towards the formulation of gravity models in realistic dimensions.

Actually, the so called w-algebras constitute a generalization of the Virasoro symmetry, and two-dimensional w-gravity models, generalizing Polyakov's induced gravity, have been formulated (see next Section). They turn out to be constrained Wess-Zumino-Witten models. The

*This seems to be an important and general feature of quantum gauge theories as opposite to their classical counterparts. For example, see Refs. [2, 3, 4] for a cohomological (*Higgs-less*) generation of mass in Yang-Mills theories through non-trivial representations of the gauge group; in this proposal, gauge modes become also physical and the corresponding extra field degrees of freedom are transferred to the vector potentials (longitudinal components) to form massive vector bosons.

algebras w have also a space origin as (area preserving) diffeomorphisms and Poisson algebras of functions on symplectic manifolds (e.g. cylinder). There is a group-theoretic structure underlying their quantum (Moyal [7]) deformations (collectively denoted by $\mathcal{W}$), according to which $\mathcal{W}$ algebras are just particular members of a one-parameter family $\mathcal{L}_\rho(sl(2,\mathbb{R}))$ —in the notation of the present paper— of non-isomorphic infinite-dimensional Lie-algebras of $SL(2,\mathbb{R})$ tensor operators (see later). The connection with the theory of higher-spin gauge fields in (1+1)- and (2+1)-dimensional anti-de Sitter space AdS [8] —homogeneous spaces of $SO(1,2) \sim SL(2,\mathbb{R})$ and $SO(2,2) \sim SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, respectively— is then apparent in this group-theoretical context. The AdS spaces are arousing an increasing interest as asymptotic background spaces in (super)gravity theories, essentially sparked off by Maldacena's conjecture [9], which establishes a correspondence of holographic type between field theories on AdS and conformal field theories on the boundary (locally Minkowski). The AdS space plays also an important role in the above mentioned attempts to understand the microscopic source of black hole entropy.

This scenario constitutes a suitable approach for our purposes. Indeed, the (3+1)-dimensional generalization of the previous constructions is just straight-forward when considering the infinite-dimensional Lie algebras $\mathcal{L}_{\vec{\rho}}(so(4,2))$ of $SO(4,2)$-tensor operators (where $\vec{\rho}$ is now a three-dimensional vector). They can be regarded as infinite enlargements of the (finite) conformal symmetry $SO(4,2)$ in 3+1 dimensions which incorporate the subalgebra diff(4) of diffeomorphisms [the higher-dimensional analogue of the Virasoro algebra diff(S^1)] of the four-dimensional space-time manifold (locally Minkowski), in addition to interacting fields with all $SO(4,2)$-spins. This fact makes $\mathcal{L}_{\vec{\rho}}(so(4,2))$ a potential gauge guide and a main ingredient towards the formulation of higher-dimensional (Wess-Zumino-Witten-like) gravity models.

The classification and labelling of tensor operators of Lie groups other than $SL(2,\mathbb{R})$ and $SU(2)$ is not an easy task in general. In Refs. [10, 11], the author provides a quite appropriated basis of tensor operators for $\mathcal{L}_{\vec{\rho}}(u(N_+,N_-)), \vec{\rho} = (\rho_1,\ldots,\rho_N)$, $N \equiv N_+ + N_-$, and calculates the structure constants for the particular case of the boson realization of quantum associative operator algebras on algebraic (Kähler) manifolds $F_{N_+N_-} = U(N_+,N_-)/U(1)^N$, also called *flag* manifolds (see Refs. [12, 13] for a formal definition of flag manifolds). We are just interested in the particular case of $U(2,2) \simeq (SO(4,2) \times U(1))/\mathbb{Z}_4$. Tensor labelling coincides here with the standard pattern for wave-functions in the carrier space of unirreps of $U(N)$ (see later).

In this article we look at this abstract, purely algebraic, construction from a more physical, field-theoretic, point of view. In particular, we shall show how to recover $\mathcal{L}_{\vec{\rho}}(\mathcal{G})$ from quantum field theory on homogeneous spaces $M = G/P$, formulated as a second quantization on the group G. The particular cases of $G = SO(1,2) \sim SU(1,1)$ and $G = SO(4,2) \sim SU(2,2)$ will give us the (generalized) w-structures as algebras of higher-conformal-spin quantum fields on anti-de Sitter spaces. One can interpret $\mathcal{L}_{\vec{\rho}}(\mathcal{G})$ as the quantum analogue of the Poisson and symplectic (volume-preserving) diffeomorphism algebras, $C^\infty(F)$ and sdiff(F), of functions ψ (higher-G-spin fields) on coadjoint orbits F of G, with a given associative and non-commutative $\star$-product. The Planck length $\lambda\!\!\!^-$ enters here as a deformation parameter and it could be basically interpreted as an upper-resolution bound to any position measurement in any quantum gravity model. The ideas of Non-Commutative Geometry (NCG) apply perfectly in this picture, providing "granular" descriptions of the underlying 'quantum' space: the non-commutative analogue of M (see [14] for similar concepts). The classical (commutative) case is recovered in the limit $\lambda\!\!\!^- \to 0$ (large scales) and $\rho \to \infty$ (high density of points), so that 'volume elements' remain finite. In this limit, the classical geometry is recovered as:

$$\lim_{\lambda\to 0,\rho\to\infty} \mathcal{L}_\rho(\mathcal{G}) \simeq C^\infty(F) \subset \mathrm{sdiff}(F), \tag{1}$$

where F is the (generalized) phase-space associated with the space-time M. We shall discuss later all these structures in more detail.

The organization of the paper is as follows. The next Section is devoted to a brief excursion through w algebras, their quantum deformations $\mathcal{W}$ and their underlying group-theoretic nature. Moyal $\star$-products are discussed in Sec. 2.1, in connection with the particular case of the non-commutative torus. In Sec. 3, a general approach to the quantization of fields in homogeneous spaces $M = G/P$ from a second quantization on a group G is exposed. The simple cases of de Sitter $\mathrm{dS}_2 \simeq SO(2,1)/SO(1,1)$ and anti-de Sitter $\mathrm{AdS}_2 \simeq SO(1,2)/SO(1,1)$ spaces in 1+1 dimensions are explicitly laid out in Sec. 3.2, leaving the (4+1)-dimensional case $\mathrm{AdS}_5 \simeq SO(4,2)/SO(4,1)$ as an Appendix, where the unitary irreducible representations of $U(2,2)$ are explicitly given in terms of holomorphic wave functions on flag (Kähler) manifolds $F_{2,2} = U(2,2)/U(1)^4$. The interesting physical effects of vacuum radiation phenomena are discussed in Sec. 3.3 in the context of a quantum field theory on AdS_2, representing the left-(or right-)moving sector of a conformal field theory in 1+1 dimensions; the spectrum of outgoing particles is exactly calculated and is proven to be a generalization of the black body (Planckian) spectrum, this recovered as a given limit. The embedding of quantum field operators and space-time symmetries in a larger structure (higher-dimensional $\mathcal{W}$ algebras) containing the difeomorphisms of the space-time manifold (higher-dimensional analogue of the Virasoro algebra) and all fields with arbitrary (generalized) spin is given in Sec. 4. A geometrical interpretation in terms of Poisson (and symplectic diffeomorphism) algebras $C^\infty(F)$ of functions ψ on algebraic manifolds F (coadjoint orbits) of a group G and its quantum deformations $\mathcal{L}_{\vec{\rho}}(G)$ is also discussed. These quantum deformations can be seen as non-commutative $C^\star$-algebras with an associative and non-commutative $\star$-product. This prepares us to define what we mean by quantum (non-commutative) AdS space in this context. The last Section is devoted to conclusions and outlook.

2 $\mathcal{W}$ algebras and 2D quantum space

In the last decade, a large body of literature has been devoted to the study of the so-called $\mathcal{W}$-algebras. These algebras were first introduced as higher-spin extensions of the Virasoro algebra [15] through the operator product expansion of the stress-energy tensor and primary fields in two-dimensional conformal field theory. $\mathcal{W}$-algebras have been widely used in two-dimensional physics, mainly in condensed matter (quantum Hall effect), integrable models (Kurteweg-de Vries, Toda), phase transitions in two dimensions, stringy black holes and, at a more fundamental level, as the underlying gauge symmetry of two-dimensional gravity models generalizing the Virasoro gauged symmetry in the light-cone discovered by Polyakov [16, 17] by adding spin > 2 currents (see e.g. [18] and [19, 20] for a review).

Only when all (infinite) conformal spins $s \geq 2$ are considered, the algebra (denoted by w_∞) is proven to be of Lie type; moreover, currents of spin $s = 1$ can also be included, thus leading to the Lie algebra $w_{1+\infty}$, which plays a determinant role in the classification of all universality classes of incompressible quantum fluids and the identification of the quantum numbers of the excitations in the quantum Hall effect [21].

As already said, the algebras w prove to have a space-time origin as (symplectic) diffeomorphysms algebras and Poisson algebras of functions ψ on symplectic manifolds. For example, $w_{1+\infty}$ is related to the algebra of diffeomorphisms of the cylinder. In fact, let us choose the next set of classical functions of the bosonic (harmonic oscillator) variables $a = \frac{1}{\sqrt{2}}(q + ip)$, $\bar{a} = \frac{1}{\sqrt{2}}(q - ip)$ (we are using mass and frequency $m = 1 = \omega$, for simplicity):

$$L^I_{|n|} \equiv \frac{1}{2} a^{2|n|}(a\bar{a})^{I-|n|}, \;\; L^J_{-|m|} \equiv \frac{1}{2}\bar{a}^{2|m|}(a\bar{a})^{J-|m|}, \tag{2}$$

where $n, m \in \mathbb{Z}/2; I, J \in \mathbb{Z}^+/2$. A straightforward calculation from the basic Poisson bracket $\{a, \bar{a}\} = i$ provides the following formal Poisson algebra:

$$\{L^I_m, L^J_n\} = -i[Jm - In]L^{I+J-1}_{m+n}\,, \tag{3}$$

of functions L on a two-dimensional phase space (see [22, 23]). The (conformal-spin-2) generators $L_n \equiv L^1_n$ close the Virasoro algebra without central extension,

$$\{L_m, L_n\} = i(n - m)L_{m+n}, \tag{4}$$

and the (conformal-spin-1) generators $\phi_m \equiv L^0_m$ close the non-extended Abelian Kac-Moody algebra of "string-modes",

$$\{\phi_m, \phi_n\} = 0. \tag{5}$$

In general, the higher-spin fields L^I_n have conformal-spin $s = I + 1$ and conformal-dimension n (the eigenvalue of L^1_0).

Induced actions for these "w-gravities" have been written (see for example [18]), which turn out to be constrained Wess-Zumino-Witten models [24], as happens with standard induced gravity. The quantization procedure *deforms* the classical algebra w to the quantum algebra $\mathcal{W}$ due to the presence of anomalies —deformations of Moyal type of Poisson and symplectic-diffeomorphism algebras caused essentially by normal order ambiguities (see Sec. 2.1). Also, generalizing the $SL(2,\mathbb{R})$ Kac-Moody hidden symmetry of Polyakov's induced gravity, there are $SL(\infty,\mathbb{R})$ and $GL(\infty,\mathbb{R})$ Kac-Moody hidden symmetries for $\mathcal{W}_\infty$ and $\mathcal{W}_{1+\infty}$ gravities, respectively [25].

The group-theoretic structure underlying these $\mathcal{W}$ algebras was elucidated in [26], where $\mathcal{W}_\infty$ and $\mathcal{W}_{1+\infty}$ appeared to be distinct members ($\rho = 0$ and $\rho = -1/4$ cases, respectively) of a one-parameter family $\mathcal{L}_\rho(sl(2,\mathbb{R}))$ of non-isomorphic [27] infinite-dimensional Lie-algebras of $SL(2,\mathbb{R}) \simeq SU(1,1)/\mathbb{Z}_2$ tensor operators

$$\hat{L}^I_{\pm|m|} \sim \underbrace{\Big[\hat{L}_\mp, \Big[\hat{L}_\mp, \dots \Big[\hat{L}_\mp,}_{I-|m| \text{ times}} \;(\hat{L}_\pm)^I\Big]\dots\Big]\Big] = (\mathrm{ad}_{\hat{L}_\mp})^{I-|m|}(\hat{L}_\pm)^I, \tag{6}$$

where the $su(1,1)$ Lie-algebra generators $\hat{L}, \hat{L}_0$ fulfill the standard commutation relations

$$\Big[\hat{L}_\pm, \hat{L}_0\Big] = \pm\hbar\hat{L}_\pm\,, \qquad \Big[\hat{L}_+, \hat{L}_-\Big] = 2\hbar\hat{L}_0. \tag{7}$$

In more formal language, $\mathcal{L}_\rho(su(1,1))$ is the factor algebra $\mathcal{L}_\rho(su(1,1)) = \mathcal{U}(su(1,1))/\mathcal{I}_\rho$ of the universal enveloping algebra $\mathcal{U}(su(1,1))$ by the ideal $\mathcal{I}_\rho = (\hat{C} - \hbar^2\rho)\mathcal{U}(su(1,1))$ generated by the Casimir operator $\hat{C} = (\hat{L}_0)^2 - \frac{1}{2}(\hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+)$ of $su(1,1)$ (ρ denotes an arbitrary complex number). This simply means that we substitute the Casimir operator $\hat{C}$ by the constant $\hbar^2\rho$ whenever it appears in the commutator (structure constants). The structure constants for $\mathcal{L}_\rho(su(2))$ and $\mathcal{L}_\rho(su(1,1))$ are well known for the Racah-Wigner basis of tensor operators [28], and they can be written in terms of Clebsch-Gordan and (generalized) $6j$-symbols [29, 26, 30]. Another interesting feature of $\mathcal{L}_\rho(su(2))$ —or its non-compact version $SU(1,1)$— is that, when ρ coincides with the eigenvalue of $\hat{C}$ in an irrep D_j of $SU(2)$, that is $\rho = j(j+1)$, there exists and ideal χ in $\mathcal{L}_\rho(su(2))$ such that the quotient $\mathcal{L}_\rho(su(2))/\chi \simeq sl(2j+1, C)$ or $su(2j+1)$ —by taking a compact real form of the complex Lie algebra [31]. That is, for $\rho = j(j+1)$ the infinite-dimensional algebra $\mathcal{L}_\rho(su(2))$ *collapses* to a finite-dimensional one (a matrix algebra). This fact was used in [29] to approximate $\lim_{\substack{\rho\to\infty \\ \hbar\to 0}} \mathcal{L}_\rho(su(2)) \simeq \mathrm{sdiff}(S^2)$ by $su(N)|_{N\to\infty}$ ("large number of colors") in the relativistic spherical membrane. A physical interpretation of this "collapse phenomenon" could be given in the context of Non-Commutative Geometry, which provides a finite, "fuzzy or cellular" description of the (non-commutative) space [14, 32]. A simple example will be discussed in Section 2.1 in connection with the *non-commutative torus*.

Noncommutative geometry is said to be a 'pointless' geometry, where the notion of a pure state ψ (more precisely, its 'covariant symbol' $\hat{\psi}$) on the manifold $M = G/P$ replaces that of a point. In this sense, the parameter ρ above could be conceived as a 'density of (quantum) points' (e.g., the dimension of the corresponding representation). Actually, just as the standard differential geometry of $M = S^2$ can be described by using the (commutative) algebra $C^\infty(M)$ of smooth complex functions ψ on M (spherical harmonics), a noncommutative geometry for M can take $\mathcal{L}_\rho(\mathcal{G})$ (seen as an associative algebra with a noncommutative $\star$-product) as the starting point. For the critical values $\rho = d$ (the dimension of an irrep of $G = SU(2)$), the 'quantum sphere' has a finite number of quantum points per unit volume, as the infinite-dimensional algebra $\mathcal{L}_\rho(\mathcal{G})$ collapses to a (finite) complex matrix algebra. The classical geometry on M is then recovered in the particular limit (1) —i.e. high density of quantum points and small quantum cells: the Planck area $\lambda\!\!\!^{-2}$— upon which, the commutator $[\cdot,\cdot]$ becomes the usual Poisson bracket $\{\cdot,\cdot\}$ on the sphere. It is worth-mentioning that the whole idea was noticed long time ago by Dirac [33], who realized the possibility of describing phase-space physics in terms of the quantum analogue of the algebra of functions (the covariant symbols) and the absence of localization expressed by the Heisenberg uncertainty principle. As happens in phase-space, the fundamental scale (Planck length) $\lambda\!\!\!^{-}$ should establish an upper-resolution bound to any position measurement in any quantum gravity model.

Let us illustrate briefly this "collapse" phenomenon with the simple example of the $\mathcal{W}$ algebra $\mathcal{L}_\rho(su(1,1))$ in connection with the quantum AdS_2 space-time and the non-commutative torus.

2.1 Noncommutative Torus

Let us consider the following new set of classical functions of the bosonic variables $a(\bar{a}) = \frac{2\pi}{\ell}(x_1 \pm i x_2)$:

$$L_{\vec{n}} \equiv e^{\frac{2\pi i}{\ell}\vec{n}\cdot\vec{x}} = \sum_{I=0}^{\infty}\sum_{l=0}^{I} 2(-1)^I \frac{(n_1+in_2)^{I+l}}{2^{I+l}(I+l)!}\frac{(n_1-in_2)^{I-l}}{2^{I-l}(I-l)!}\; L_l^I, \tag{8}$$

obtained from (2), where $\vec{x} = (x_1, x_2)$ is a pair of real coordinates (modulo ℓ) and $\vec{n} = (n_1, n_2)$ is a pair of integer numbers. We identify (8) with the set $C^\infty(T^2)$ of smooth functions on a two-dimensional torus, which is embedded in the set of functions (2) on a cylinder [note that the lower-index l of L_l^I in (8) is restricted by $0 \leq l \leq I$, whereas it can take any (half-)integer value in (2)]. We shall restrict ourselves to this subset of the whole w-algebra (3) where the above-mentioned "collapse" phenomenon will be more apparent.

The ordinary product of functions $L_{\vec{m}} \cdot L_{\vec{n}} = L_{\vec{m}+\vec{n}}$ defines $C^\infty(T^2)$ as a commutative algebra. We can assign a (Hamiltonian) vector field $H_{\vec{m}} \equiv \{L_{\vec{m}}, \cdot\}_{\rm P}$ to any function $L_{\vec{m}}$, where:

$$\{L_{\vec{m}}, L_{\vec{n}}\}_{\rm P} = P^1(L_{\vec{m}}, L_{\vec{n}}) = \Upsilon_{jk} \frac{\partial L_{\vec{m}}}{\partial x_j} \frac{\partial L_{\vec{n}}}{\partial x_k} = \frac{4\pi^2}{\ell^2} \vec{n} \times \vec{m}\, L_{\vec{m}+\vec{n}}\,, \tag{9}$$

denotes the Poisson bracket and $\Upsilon_{2\times 2} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the symplectic form on the torus. The vector fields $H_{\vec{m}}$ constitute a base of symplectic diffeomorphisms sdiff(T^2), that is, they preserve the area element $dx_1 \wedge dx_2$ of the torus.

The quantum analogue of $C^\infty(T^2)$ can be captured from a classical construction by extending the Poisson bracket (9) to its deformed version: the Moyal bracket,

$$\{L_{\vec{m}}, L_{\vec{n}}\}_{\rm M} = L_{\vec{m}} \star L_{\vec{n}} - L_{\vec{n}} \star L_{\vec{m}} = \sum_{r=0}^{\infty} \frac{2}{(2r+1)!} \left(\frac{\lambda^2}{2\pi i}\right)^{2r+1} P^{2r+1}(L_{\vec{m}}, L_{\vec{n}}), \tag{10}$$

where

$$L_{\vec{m}} \star L_{\vec{n}} \equiv \exp(\frac{\lambda^2}{2\pi i} P)(L_{\vec{m}}, L_{\vec{n}}) = e^{2\pi i \frac{\lambda^2}{\ell^2} \vec{m}\times\vec{n}} L_{\vec{m}+\vec{n}} \tag{11}$$

is an invariant, associative and noncommutative $\star$-product, and by P^r we mean

$$\begin{aligned} P^r(L, L') &\equiv \Upsilon_{j_1 k_1} \ldots \Upsilon_{j_r k_r} \frac{\partial^r L}{\partial x_{j_1} \ldots \partial x_{j_r}} \frac{\partial^r L'}{\partial x_{k_1} \ldots \partial x_{k_r}} \\ &= \sum_{l=0}^{r} (-1)^l \begin{pmatrix} r \\ l \end{pmatrix} [\partial_{x_1}^{r-l} \partial_{x_2}^{l} L][\partial_{x_2}^{r-l} \partial_{x_1}^{l} L'] \end{aligned} \tag{12}$$

where $P^0(L, L') \equiv L \cdot L'$ denotes the ordinary (commutative) product of functions.

The $\star$-product (11) also admits an integral representation:

$$(L \star L')(\vec{x}) = \frac{i\ell^2}{4\pi\lambda^6} \int_0^\ell d\vec{x}' d\vec{x}'' e^{-\frac{2\pi i}{\lambda^2}|\vec{x}\vec{x}'\vec{x}''|} L(\vec{x}') L'(\vec{x}'') \tag{13}$$

with

$$|\vec{x}\vec{x}'\vec{x}''| \equiv \vec{x} \times \vec{x}' + \vec{x}' \times \vec{x}'' + \vec{x}'' \times \vec{x}\,, \tag{14}$$

which is interesting when one wants to extend the $\star$-product to other symplectic manifolds like coadjoint orbits of certain groups (see Sec. 4.1).

The Moyal bracket (10) reminds us a commutator between operators

$$\{L_{\vec{m}}, L_{\vec{n}}\}_{\rm M} = \left[\hat{L}_{\vec{m}}, \hat{L}_{\vec{n}}\right] \equiv \hat{L}_{\vec{m}} \star \hat{L}_{\vec{n}} - \hat{L}_{\vec{n}} \star \hat{L}_{\vec{m}} = 2i \sin\left(2\pi \frac{\lambda^2}{\ell^2} \vec{m} \times \vec{n}\right) \hat{L}_{\vec{m}+\vec{n}}\,. \tag{15}$$

Thus, this $\star$-product equips the set (8) with a noncommutative $C^\star$-algebra structure, which we shall denote by $C^\star_\rho(T^2)$. Just as the standard geometry of T^2 can be described by using the algebra $C^\infty(T^2)$ of smooth complex functions (8) on T^2 with the ordinary (commutative) product, a noncommutative geometry for T^2 can be described by using its "quantum" analogue $C^\star_\rho(T^2)$. Noncommutative geometry offers a broader spectrum of possibilities. In fact, let us see how the definition of "quantum torus" is richer than that of the (standard) "classical torus", which eventually comes up as a particular limiting case of the former one.

Note that, when the surface of the torus ℓ^2 contains an integer number q of times the *minimal cell* $\lambda\!\!\!^-{}^2$ (that is, $\ell^2 = q\lambda\!\!\!^-{}^2$), the infinite-dimensional algebra (15) collapses to a finite-dimensional matrix algebra: the Lie algebra of the unitary group $U(q/2)$ for q even or $SU(q)\times U(1)$ for q odd (see [34]). In fact, taking the quotient in (15) by the equivalence relation $\hat{L}_{\vec{m}+q\vec{a}} \sim \hat{L}_{\vec{m}}$, $\forall \vec{a} \in \mathbb{Z}\times\mathbb{Z}$, it can be seen that the following identification $\hat{L}_{\vec{m}} = \sum_k e^{\frac{2\pi i}{q}m_1 k} X_{k,k+m_2}$ implies a change of basis in the step-operator Lie-algebra (61) of $U(q)$.

Thinking of $\rho = \frac{\ell^2}{\lambda\!\!\!^-{}^2}$ as a 'density of quantum points', we can conclude that: for the critical values $\rho_c = q \in \mathbb{Z}$, the Lie algebra (15) is *finite*; that is, the quantum analogue of the torus has a 'finite number q of quantum points'. It is in this sense that we talk about a 'cellular structure of space'. Actually, given the formal basic commutator $[x_1, x_2] = -i\lambda\!\!\!^-{}^2/\pi$ between "position operators" on the torus, this cellular structure is a consequence of the absence of localization expressed by the Heisenberg uncertainty relation $\Delta x_1 \Delta x_2 \geq \lambda\!\!\!^-{}^2/(2\pi)$ (see Figure 1).

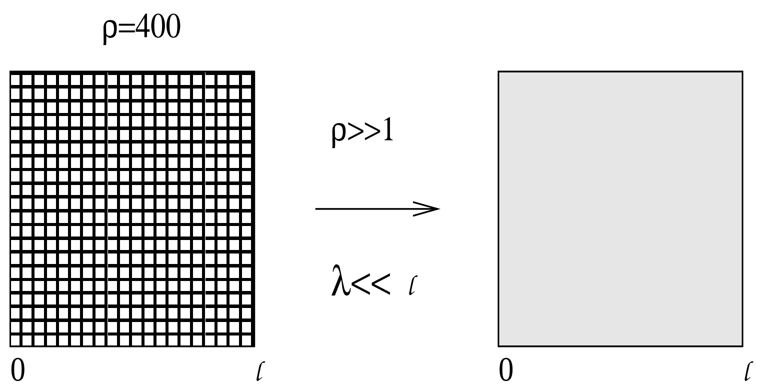

Figure 1: In extreme quantum conditions $\ell \sim \lambda\!\!\!^-$ (size of the space comparable to the Planck length), the cellular structure of space could be conspicuous.

In the (classical) limit of large number of points $\rho \to \infty$ and $\lambda\!\!\!^- \to 0$ (such that the size $\ell^2 = \rho\lambda\!\!\!^-{}^2$ of the torus remains finite) we recover the original (commutative) geometry on the torus. For example, it is easy to see that

$$\lim_{\rho\to\infty,\,\lambda\!\!\!^-\to 0} \frac{i\pi}{\lambda\!\!\!^-{}^2}\left[\hat{L}_{\vec{m}}, \hat{L}_{\vec{n}}\right] = \frac{4\pi^2}{\ell^2}\vec{n}\times\vec{m}\, L_{\vec{m}+\vec{n}} = P^1(L_{\vec{m}}, L_{\vec{n}}) \tag{16}$$

coincides with the (classical) Poisson bracket (9) of functions in $C^\infty(T^2)$. In particular, the last equality states that the Poisson algebra (9) formally coincides with the Lie algebra of the group of infinite unitary matrices $U(\infty)$ [34]. This is just a facet of the general problem of approximating infinite-dimensional Lie algebras of symplectic diffeomorphisms on homogeneous manifolds by large-N matrix algebras.

This simple example gives us a taste of what should happen in higher dimensions and less trivial quantum manifolds. Although the general subject of Noncommutative Geometry is rather developed (see e.g. [32, 14]), there is still a technical difficulty partially due to the lack of tractable (yet non-trivial) noncommutative versions of curved spaces. Let us proceed, firstly, by showing how to obtain quantum fields on a homogeneous space $M = G/P$ and, secondly, how to embed those fields in a noncommutative structure ("higher-dimensional $\mathcal{W}$-algebra") living in the universal enveloping algebra of G.

3 QFT in Homogeneous Spaces from a Second Quantization on a Group

The sequel to the previous constructions will consist in generalizing the concept of $\mathcal{W}$-algebra to realistic dimensions. Once the group-theoretic nature of $\mathcal{W}$-algebras is elucidated, there is no conceptual difficulty in jumping from 1+1 to 3+1 dimensions. Indeed, we should just substitute $G = SU(1,1)$ by $G = SU(2,2)$ (the conformal group $SO(4,2)$ in 3+1 dimensions, except for discrete symmetries) and consider now $\mathcal{L}_{\vec{\rho}}(su(2,2))$ as a higher-dimensional generalization of the $\mathcal{W}$-algebras. However, there is a technical difficulty; the generalization of these constructions to general unitary groups proves to be quite unwieldy, and a canonical classification of $U(N)$-tensor operators has, so far, been proven to exist only for $U(2)$ and $U(3)$ (see [28] and references therein). As already mentioned, tensor labelling is provided in these cases by the Gel'fand-Weyl pattern for vectors in the carrier space of the irreps of $U(N)$. A step forward in this direction is given in the letter [10], where the author provides a quite appropriate basis of operators for $\mathcal{L}_{\vec{\rho}}(u(N_+, N_-)), \vec{\rho} = (\rho_1, \dots, \rho_N)$, $N \equiv N_+ + N_-$, and calculates the structure constants, for the particular case of the boson realization of quantum associative operator algebras on algebraic manifolds $F_{N_+ N_-} = U(N_+, N_-)/U(1)^N$ (*flag* manifolds). This is a particular, yet important, case inside the N-parameter family $\mathcal{L}_{\vec{\rho}}(u(N_+, N_-))$, where the standard Moyal bracket [7] provides the primary quantum deformations (i.e. an invariant associative $\star$-product). The general case is still under study, although Moyal deformation still captures much of the essence of the full quantum algebra, as happens to deformations of $\mathcal{W}$ algebras [35].

The aim of this article will not be to provide an exhaustive study of quantum deformations $\mathcal{L}_{\vec{\rho}}(u(2,2))$ but, rather, to give a physical and geometrical interpretation of the classical limit $\mathcal{L}_\infty(G)$ for $G = U(1,1)$ and $G = U(2,2)$ in terms of higher-G-spin algebras of fields ψ on anti-de Sitter spaces in 1+1 and 4+1 dimensions, respectively. For this, let us outline how to formulate a quantum field theory in a homogeneous space $M = G/P$ as a second quantization on a group G. Some comments on quantum deformations will be given in Section 4.1.

3.1 Second Quantization on a Group

In the standard formulation, the Quantum Mechanics of a particle in a curved space M is explicitly constructed in the configuration space (space image), making use of the intrinsic differentiable structure of M (generally, a pseudo-Riemann, globally hyperbolic manifold with metric $g^{\mu\nu}$). However, when M is symmetrically enough or, more precisely, when M is somehow embedded in a symmetry group G, this case offers more possibilities. In fact, we can always use the large amount of geometrical and algebraic resources that a group offer to us: left- and right-invariant vector fields, Casimir operators, invariant forms (e.g. volume elements), symplectic structures on the coadjoint orbits, results of the representation theory, etc; in particular, the problem of calculating the general solution to the equations of motion

$$(g^{\mu\nu}\nabla_\mu\nabla_\nu + m^2 + \xi R)\phi(x) = 0 \tag{17}$$

(∇_μ denotes the covariant derivative) is reduced to finding the unitary and irreducible representations with a fixed value of the Casimir operator $\hat{C} \sim \frac{\partial^2}{\partial t^2} + K$ on a given support space of constant curvature given by the eigenvalue (k) of $\hat{C}$ (for example, $\hat{C} = \partial_\mu\partial^\mu$ for the Poincaré group). Such is the case of de Sitter and anti-de Sitter spaces in $d+1$ dimensions:

$$\mathrm{dS}_{1+d} \simeq SO(d+1,1)/SO(d,1)\,, \quad \mathrm{AdS}_{1+d} \simeq SO(d,2)/SO(d,1)\,, \tag{18}$$

which (mainly AdS) are arousing an increasing interest as asymptotic background spaces in (super)gravity theories, as already said in the Introduction. Let us outline a general method for constructing a quantum field theory in a homogeneous space M of a group G.

Let $\mathcal{H}_s(G) = \{\psi^{(s)}_k\}_{k\in I}$ (the lower-index k represents quantum numbers that lie on a set I for a given G-spin label s) be an irreducible representation of G obtained from the regular representation, that is, the space of arbitrary complex functions $\psi : G \to \mathbb{C}$, $\psi(g) \in \mathbb{C}$, $g \in G$). Given the left action of G on itself $L_{g'}(g) = g' \bullet g$ (where $\bullet$ denotes the composition law), the finite action ρ of G on $\mathcal{H}_s(G)$

$$\psi'(g) \equiv (\rho(g')\psi)(g) = \psi(L^{-1}_{g'}(g)) = \psi(g'^{-1} \bullet g) \tag{19}$$

defines a unitary representation with respect to the scalar product:

$$\langle\psi|\psi'\rangle \equiv \int_G d^L g\, \bar{\psi}(g)\psi'(g)\,, \tag{20}$$

where $d^L g = \theta^{L1} \wedge \overbrace{\dots}^{r=\dim(G)} \wedge\theta^{Lr}$ denotes the left-invariant integration measure, which is constructed as an exterior product of left-invariant one-forms $\theta^{L\beta} = \Lambda^\beta_\alpha(g)dg^\alpha$, and $\{g^\alpha\}^r_{\alpha=1}$ is a system of coordinates on G. The matrices $\Lambda^\beta_\alpha(g)$ are calculated by duality $\theta^{L\beta}(X^L_\alpha) = \delta^\beta_\alpha$ with the left-invariant vector fields

$$X^L_\alpha \equiv L^\beta_\alpha(g)\frac{\partial}{\partial g^\beta}\,, \quad L^\beta_\alpha(g) = \left.\frac{\partial(g\bullet g')^\beta}{\partial g'^\alpha}\right|_{g'=e}\,, \tag{21}$$

where e denotes the identity element; that is, $\Lambda^\beta_\alpha = (L^{-1})^\beta_\alpha$. In general, the finiteness of this measure is not guaranteed; however, this definition will be enough for our purposes.

For simplicity, we shall restrict ourselves to real quantum fields

$$\hat{\phi}^{(s)}(g) = \hat{\phi}^{(s)}_-(g) + \hat{\phi}^{(s)}_+(g) = \sum\!\!\!\!\!\!\int_{k\in I} \hat{a}^{(s)}_k \psi^{(s)}_k(g) + \hat{a}^{(s)\dagger}_k \bar{\psi}^{(s)}_k(g), \tag{22}$$

where $\hat{a}^{(s)}_k, \hat{a}^{(s)\dagger}_k$ denote the Fourier coefficients of the expansion of $\hat{\phi}^{(s)}$ in the basis $\{\psi^{(s)}_k\}_{k\in I}$. The interested reader can consult Refs. [6, 36] for a more technical exposition of the subject in the context of a Group Approach to Quantization formalism [37].

The infinitesimal, second-quantized, counterpart of the finite action (19) can be written as:

$$\begin{aligned} \left[\hat{X}_\alpha, \hat{\phi}^{(s)}(g)\right] &= \hbar X^R_\alpha \hat{\phi}^{(s)}(g)\,, \\ \left[\hat{X}_\alpha, \hat{a}^{(s)}_l\right] &= \hbar \sum\!\!\!\!\!\!\int_k \left.\frac{\partial\rho^{(s)}_{lk}(g)}{\partial g^\alpha}\right|_{g=e} \hat{a}^{(s)}_k\,, \end{aligned} \tag{23}$$

where $\rho^{(s)}_{lk}(g) \equiv \langle\psi^{(s)}_l|\rho(g)|\psi^{(s)}_k\rangle$ are the matrix elements of ρ in the basis of $\mathcal{H}_s(G)$. The operators $\hat{X}_\alpha$ are the second-quantized version of the infinitesimal (right-invariant) generators

$$X^R_\alpha \equiv R^\beta_\alpha(g)\frac{\partial}{\partial g^\beta}\,, \quad R^\beta_\alpha(g) = \left.\frac{\partial(g'\bullet g)^\beta}{\partial g'^\alpha}\right|_{g'=e} \tag{24}$$

of the finite left action (19), which fulfill the commutation relations:

$$\left[\hat{X}_\alpha, \hat{X}_\beta\right] = \hbar C^\gamma_{\alpha\beta}\hat{X}_\gamma\,, \quad C^\gamma_{\alpha\beta} = (R^\sigma_\alpha\partial_\sigma R^\kappa_\beta - R^\sigma_\beta\partial_\sigma R^\kappa_\alpha)(R^{-1})^\gamma_\kappa\,, \tag{25}$$

where $C^\gamma_{\alpha\beta}$ denote the structure constants of the Lie-algebra of G. The commutator between fields in the Fourier space and the configuration space is:

$$\begin{aligned} \left[\hat{a}^{(s)}_k, \hat{a}^{(s)\dagger}_l\right] &= \delta_{kl}\hat{1}\,, \\ \left[\hat{\phi}^{(s)}_-(g), \hat{\phi}^{(s)}_+(g')\right] &= \Delta^{(s)}(g,g')\hat{1}\,, \end{aligned} \tag{26}$$

where

$$\Delta^{(s)}(g,g') \equiv \langle g|g'\rangle = \sum\!\!\!\!\!\!\int_k \psi^{(s)}_k(g)\bar{\psi}^{(s)}_k(g') \tag{27}$$

denotes the propagator in the configuration space. In the last equality, a closure relation $1 = \sum\!\!\!\!\!\!\int_k \left|\psi^{(s)}_k\right\rangle\left\langle\psi^{(s)}_k\right|$ has been introduced. The Hilbert space of the second-quantized theory is the representation space of the (infinite-dimensional) algebra of quantum fields $\hat{\phi}$ and space-time symmetries G with commutation relations (23,25,26). It can be constructed as the orbit of the creation operators acting on the vacuum:

$$|N(k_1),...,N(k_j)\rangle \equiv \frac{(\hat{a}^{(s)\dagger}_{k_1})^{N(k_1)}...(\hat{a}^{(s)\dagger}_{k_j})^{N(k_j)}}{(N(k_1)!...N(k_j)!)^{1/2}}|0\rangle\,, \tag{28}$$

where $N(k_j)$ means "number of particles with quantum numbers k_j". The vacuum $|0\rangle$ is characterized by the conditions:

$$\hat{a}^{(s)}_k|0\rangle = 0 = \hat{X}_\alpha|0\rangle\,, \;\; \forall k \in I,\, \alpha = 1,\ldots,\dim(G), \tag{29}$$

that is, it is annihilated by the destruction field operators $\hat{a}^{(s)}_k$ and it is invariant under the action of the basic symmetry group G —i.e. it looks the same to any freely falling observer anywhere in the homogeneous (curved) space $M \simeq G/P$. It can be seen that the operators $\hat{X}_\alpha$ (e.g. the Hamiltonian) are written in terms of the (basic) field operators as:

$$\hat{X}_\alpha = \hbar\int_G d^Lg\,\hat{\phi}^{(s)}_+(g)X^R_\alpha\hat{\phi}^{(s)}_-(g)\,. \tag{30}$$

We shall see how the Lie algebra of the second-quantization group $G^{(2)} = G\times_s(H-W)$ [the semidirect product of the basic symmetry group G times the infinite-dimensional Heisemberg-Weyl group of fields ϕ, that is, the solution manifold to the field equations (17)] defined by the commutation relations (23,25,26) is just a piece of a richer structure that incorporates the whole algebra of diffeomorphisms of M, diff(M), —as an enlargement of G— together with all the representations $\mathcal{H}_s(G)$ ("higher-G-spin fields") of G, in a consistent manner. We shall denote such algebraic structures by $\mathcal{L}_\infty(\mathcal{G})$, which are a particular (classical) limit of more general tensor operator algebras $\mathcal{L}_{\vec{\rho}}(\mathcal{G})$ (see e.g [26, 30]) already mentioned in Sec. 2. Higher-spin algebras [8], which are said to be a guide principle to the (still unknown) "M-theory", are also of this kind. Not much is known in the literature about the explicit form of these algebras (e.g.: labelling of basic generators, structure constants, etc), except for the simple cases of $G = SU(2)$ and $G = SL(2,\mathbb{R}) \simeq SU(1,1)/\mathbb{Z}_2$, for which $w_\infty \simeq \mathcal{L}_\infty(sl(2,\mathbb{R}))$.

Before entering more explicitly in this "promotion" of quantum-filed-theory algebras to higher structures containing "gravity modes", let us work out the simple example of quantum field theory on $\text{AdS}_2 = SO(1,2)/SO(1,1)$ as a second quantization on $G = SO(1,2) \simeq SL(2,\mathbb{R})$. The 4+1-dimensional case $\text{AdS}_5 = SO(4,2)/SO(4,1)$ is left as an Appendix A, where explicit expressions for holomorphic unitary irreducible representations of $SU(2,2)$ and propagators are given. Vacuum radiation phenomena is also discussed as a physical application of QFT in AdS_2.

3.2 QFT in $AdS_2 = SO(1,2)/SO(1,1)$

In restricting to $1+1D$, we find an apparent ambiguity in the distinction between AdS_2 and dS_2 in (18), as $SO(1,2) \simeq SO(2,1)$. We shall see that AdS_2 and dS_2 correspond to diferent orbits of the same group $SU(1,1) \sim SO(1,2)$ (except for discrete transformations) or, in other words, diferent choices of time generator (see later).

A system of coordinates for

$$SU(1,1) = \left\{ g = \begin{pmatrix} u_1 & \bar{u}_2 \\ u_2 & \bar{u}_1 \end{pmatrix}, u_i, \bar{u}_i \in \mathbb{C}/\det(g) = |u_1|^2 - |u_2|^2 = 1 \right\}, \quad (31)$$

can be the following:†

$$\begin{aligned} g^{(1)} = h \equiv \frac{u_1}{|u_1|}, \;\; g^{(2)} = z \equiv \frac{u_2}{u_1}, \;\; g^{(3)} = \bar{z} \equiv \frac{\bar{u}_2}{\bar{u}_1}, \\ h \in U(1), \quad z, \bar{z} \in D_1, \end{aligned} \quad (32)$$

where D_1 denotes the open unit disk in the complex plane $\mathbb{C}$.

The group law $g'' = g' \bullet g$ in these coordinates adopts the form:

$$\begin{aligned} h'' &= \frac{u_1''}{|u_1''|} = h\frac{\sqrt{h'^2 + z\bar{z}'}}{\sqrt{1 + h'^2 z'\bar{z}}}, \\ z'' &= \frac{u_2''}{u_1''} = \frac{z'h'^2 + z}{h'^2 + z\bar{z}'}, \\ \bar{z}'' &= \frac{\bar{z}_2''}{\bar{z}_1''} = \frac{\bar{z}h'^2 + \bar{z}'}{1 + h'^2 z'\bar{z}}. \end{aligned} \quad (33)$$

The matrices L_α^β and R_α^β in (21,24) corresponding to the left- and right-invariant vector fields are (row α, column β):

$$(L_\alpha^\beta) = \begin{pmatrix} h & 0 & 0 \\ h^{-1}\bar{z}/2 & h^{-2}(1 - z\bar{z}) & 0 \\ -h^3 z/2 & 0 & h^2(1 - z\bar{z}) \end{pmatrix}, \quad (R_\alpha^\beta) = \begin{pmatrix} h & -2z & 2\bar{z} \\ -h\bar{z}/2 & 1 & -\bar{z}^2 \\ hz/2 & -z^2 & 1 \end{pmatrix}. \quad (34)$$

†see also Appendix A for the 4+1-dimensional $SU(2,2)$ case

From here, the structure constants (25) and commutation relations of the $su(1,1)$ Lie algebra are:

$$\left[\hat{X}_h, \hat{X}_z\right] = 2\hbar\hat{X}_z, \ \left[\hat{X}_h, \hat{X}_{\bar{z}}\right] = -2\hbar\hat{X}_{\bar{z}}, \ \left[\hat{X}_z, \hat{X}_{\bar{z}}\right] = \hbar\hat{X}_h, \tag{35}$$

which coincide with the well known $[L_m, L_n] = (m-n)L_{m+n}$ after the identification $L_1 \equiv \hat{X}_{\bar{z}}, L_{-1} \equiv -\hat{X}_z, L_0 \equiv \frac{1}{2}\hat{X}_h$. The Casimir operator in this basis of generators is given by:

$$\hat{C} = \hat{X}_h^2 + 2\hat{X}_z^2\hat{X}_{\bar{z}}^2 + 2\hat{X}_{\bar{z}}^2\hat{X}_z^2 \,. \tag{36}$$

In order to construct the Hilbert space of wave functions on AdS_2 and dS_2, let us start from the regular representation, that is, complex functions $\psi : SU(1,1) \to \mathbb{C}$ defined on the whole group $SU(1,1)$. The AdS_2 and dS_2 spaces correspond to two different integral manifolds characterized by the vector fields (perpendicular to the manifold) ‡

$$\mathrm{AdS}_2: \ X^L_{p_1} \equiv X^L_z \,, \ \ \mathrm{dS}_2: \ X^L_{p_2} \equiv X^L_z + X^L_{\bar{z}} + iX^L_h \,, \tag{37}$$

and related to two possible choices of time:

$$X^L_{t_1} \equiv \frac{\omega}{2}X^L_h \,, \ \ X^L_{t_2} \equiv \frac{i\omega}{2}(X^L_z - X^L_{\bar{z}}) \,, \tag{38}$$

compact $t_1 \sim \frac{2i}{\omega}\ln h$ for AdS_2 and non compact $t_2 \sim \frac{2i}{\omega}(z-\bar{z})$ for dS_2 (ω is a parameter with frequency dimensions). The vector fields $X^L_{p_j}$ constitute *(complex) polarizations* of coadjoint orbits [38] in the language of Geometric Quantization (see, for instance, [39])

Thus, wave functions do not depend on the coordinates p_j "perpendicular" to the manifold, which means

$$X^L_{p_1}\psi_{\mathrm{AdS}} = 0 \,, \ \ X^L_{p_2}\psi_{\mathrm{dS}} = 0 \,. \tag{39}$$

The equations of motion

$$X^L_{t_1}\psi_{\mathrm{AdS}} = \omega s_1 \psi_{\mathrm{AdS}} \,, \ \ X^L_{t_2}\psi_{\mathrm{dS}} = \omega s_2 \psi_{\mathrm{dS}} \tag{40}$$

[the $SU(1,1)$-spin labels $s_1, s_2 \in \mathbb{Z}/2$ are related to the *zero-point energy* —or the *curvature* of space [36]— and characterize the corresponding representation], together with (39), yield the solution:

$$\psi^{(s_1)}_{\mathrm{AdS}} = h^{2s_1}(1-z\bar{z})^{s_1}\,\varphi_1(\bar{z}) \,, \ \ \psi^{(s_2)}_{\mathrm{dS}} = \frac{(1-z\bar{z})^{s_2}}{(zh^2-i)^{s_2}(\bar{z}h^{-2}+i)^{s_2}}\,\varphi_2(\gamma) \,, \tag{41}$$

where $\gamma \equiv h^{-2}\frac{zh^2-i}{\bar{z}h^{-2}+i}$ corresponds to the spatial (compact) $S^1 \subset \mathrm{dS}_2$ coordinate, and

$$\varphi_1 \equiv \sum_{n=0}^{\infty} a^{(s_1)}_n \bar{z}^n, \quad \varphi_2 \equiv \sum_{m=-\infty}^{\infty} b^{(s_2)}_m \gamma^m \tag{42}$$

is an arbitrary, analytic function of $\bar{z}$ and γ, respectively. Note that the (holomorphic) AdS_2 representation has a vacuum, whereas the spectrum in dS_2 has no lower-bound.

Let us restrict hereafter, for simplicity, to the AdS_2 case. Given the natural integration measure

$$\begin{aligned} d^L g &\equiv \frac{-i}{(2\pi)^2}\theta^{L(h)} \wedge \theta^{L(z)} \wedge \theta^{L(\bar{z})} = \frac{-i}{(2\pi)^2}\det(L^\beta_\alpha)^{-1} dh \wedge d\Re(z) \wedge d\Im(z) \\ &= \frac{-i}{(2\pi)^2}\frac{1}{(1-z\bar{z})^2}h^{-1}dh \wedge d\Re(z) \wedge d\Im(z) \,, \end{aligned} \tag{43}$$

‡We shall consider holomorphic representations for the AdS case

the scalar product of basic functions $\tilde{\psi}_n^{(s)} \equiv h^{2s}(1-z\bar{z})^s\bar{z}^n$,

$$\langle\tilde{\psi}_m^{(s)}|\tilde{\psi}_n^{(s)}\rangle = \frac{\Gamma(m+1)\Gamma(2s-1)}{\Gamma(2s+m)}\delta_{m,n} \equiv N_m^{(s)}\delta_{m,n}\,, \tag{44}$$

is finite for "conformal-spin" $s > 1/2$; here Γ denotes the standard gamma function ($\Gamma(q) = (q-1)!$, when $q \in \mathbb{N}$). Thus, $B(\mathcal{H}_s) = \{\psi_n^{(s)} \equiv \frac{1}{\sqrt{N_m^{(s)}}}\tilde{\psi}_n^{(s)}\}$ is an orthonormal basis of the Hilbert space $\mathcal{H}_s(SU(1,1))$. The matrix elements $\rho_{mn}^{(s)}(g) \equiv \langle\psi_m^{(s)}|\rho(g)|\psi_n^{(s)}\rangle$ of the representation (19) in this basis have the following form:

$$\begin{aligned}\rho_{mn}^{(s)}(g) \quad = \quad & \sqrt{\frac{N_m^{(s)}}{N_n^{(s)}}} \sum_{q=\max(0,m-n)}^{m} \begin{pmatrix} n \\ m-q \end{pmatrix} \begin{pmatrix} 2s+n+q-1 \\ q \end{pmatrix} \\ & \times(-1)^{n-m+q}h^{-2s-2n}z^q\bar{z}^{n-m+q}(1-z\bar{z})^s,\end{aligned} \tag{45}$$

and are the main ingredient to construct the quantum field theory in AdS$_2$ (second quantization on $SU(1,1)$). Given the expansion in modes of a real field with conformal-spin s,

$$\hat{\phi}^{(s)} = \sum_{n=0}^{\infty} \hat{a}_n^{(s)}\psi_n^{(s)} + \hat{a}_n^{(s)\dagger}\bar{\psi}_n^{(s)}\,, \tag{46}$$

the general commutation relations (23) adopt the particular form

$$\begin{aligned}\left[\hat{X}_h, \hat{a}_n^{(s)}\right] &= -2\hbar(s+n)\hat{a}_n^{(s)}\,, \\ \left[\hat{X}_z, \hat{a}_n^{(s)}\right] &= \hbar\sqrt{n(2s+n-1)}\,\hat{a}_{n-1}^{(s)}\,, \\ \left[\hat{X}_{\bar{z}}, \hat{a}_n^{(s)}\right] &= -\hbar\sqrt{(n+1)(2s+n)}\,\hat{a}_{n+1}^{(s)}\,,\end{aligned} \tag{47}$$

together with the corresponding conjugated expressions ($\hat{X}_h^\dagger = \hat{X}_h$, $\hat{X}_z^\dagger = -\hat{X}_{\bar{z}}$).

For completeness, we shall give the explicit expression of the propagator (27) in the present holomorphic picture of AdS$_2$, which is easily calculated as:

$$\Delta^{(s)}(g,g') = \sum_{n=0}^{\infty} \bar{\psi}_n^{(s)}(g')\psi_n^{(s)}(g) = (2s-1)(hh'^{-1})^{2s}\frac{(1-z'\bar{z}')^s(1-z\bar{z})^s}{(1-z'\bar{z})^{2s}}\,. \tag{48}$$

Before passing on to the embedding of the second-quantization group $G^{(2)}$ [with general Lie-algebra commutation relations (23,25,26) and, more particularly, (47,35)] in a larger structure containing the diffeomorphisms of the manifold and higher-spin fields, let us consider briefly an interesting thermodynamic result in AdS.

3.3 Vacuum radiation

It is straightforward to see from (30) that the *total energy* operator $\hat{E}_{\rm tot} \equiv \frac{\hbar\omega}{2}\hat{X}_h$ is written in terms of the creation and annihilation operators as:

$$\hat{E}_{\rm tot} = \hat{E}_0 + \hat{E} = E_0\hat{N} + \sum_{n=1}^{\infty} E_n\hat{a}_n^{(s)\dagger}\hat{a}_n^{(s)} \tag{49}$$

where $E_0 \equiv \hbar\omega s$ is the zero-point energy, $E_n \equiv \hbar\omega n$ and $\hat{N} \equiv \sum_{n=0}^{\infty} \hat{a}_n^{(s)\dagger} \hat{a}_n^{(s)}$ represents the operator *number of particles*.

In addition to the ordinary vacuum $|0\rangle$, there are other states with null renormalized energy $\hat{E} = \hat{E}_{\rm tot} - \hat{E}_0$. For example, coherent states of *zero modes*:

$$|0\rangle_\vartheta \equiv e^{-\frac{1}{2}|\vartheta|^2} e^{\vartheta \hat{a}_0^{(s)\dagger}} |0\rangle \,, \tag{50}$$

which verify that: $\hat{a}_{n\geq 1}^{(s)} |0\rangle_\vartheta = 0$ and $\hat{a}_0^{(s)} |0\rangle_\vartheta = \vartheta |0\rangle_\vartheta$, that is, they are eigenstates of the annihilation operator $\hat{a}_0^{(s)}$. Although they behave as vacua from an energetic point of view, these states exhibit a thermal response under space-time transformations (let us say, accelerations). Indeed, according to the general expression (45), the finite action

$$\hat{a}_0^{(s)\dagger} \to \hat{b}_0^{(s)\dagger} = \sum_{n=0}^{\infty} \bar{\rho}_{0n}^{(s)}(h=1, \bar{z}=0, z) \hat{a}_n^{(s)\dagger} = \sum_{n=0}^{\infty} (-1)^n \sqrt{\frac{N_0^{(s)}}{N_n^{(s)}}} z^n \hat{a}_n^{(s)\dagger} \,, \tag{51}$$

generated by the 'creation' operator $\hat{X}_z = -L_{-1}$, leads to the following transformation of the vacuum (50) (for simplicity, we shall restrict ourselves to the case $\vartheta = 1$)

$$|0\rangle_1 \to |\Psi(z)\rangle_1 \equiv e^{-\frac{1}{2}} e^{\hat{b}_0^{(s)\dagger}} |0\rangle = \sum_{q=0}^{\infty} z^q \sum_{\substack{m_1, \dots, m_q : \\ \sum_{n=1}^q n m_n = q}} \prod_{n=0}^{q} \frac{r_n^{m_n}}{m_n!} \prod_{n=0}^{q} (\hat{a}_n^{(s)\dagger})^{m_n} |0\rangle \,, \tag{52}$$

where $r_n \equiv (-1)^n \sqrt{\frac{N_0^{(s)}}{N_n^{(s)}}}$ and $m_0 \equiv 0$. This state has been identified in [6] as a (Weyl-invariant) vacuum seen from an accelerated frame (see later), in the context of a 1+1D conformally invariant QFT. This comparison is justified because, in 1+1D, the (finite) conformal group $SO(2,2) \simeq SU(1,1) \times SU(1,1)$ (except for discrete symmetries) splits into left- and right-moving modes, so that we are just dealing with "one direction" when working in AdS$_2$. The transition to an accelerated frame is realized by special conformal transformations: L_{-1} and $\bar{L}_{-1}$ (see [6] for more details).

The relative probability of observing a state with total energy $E_q = \hbar\omega q$ in the accelerated vacuum $|\Psi(z)\rangle_1$ is

$$\begin{aligned} P_q &= \Lambda(E_q)(|z|^2)^q \,, \\ \Lambda(E_q) &\equiv \sum_{\substack{m_1, \dots, m_q : \\ \sum_{n=1}^q n m_n = q}} \prod_{n=0}^{q} \frac{r_n^{2m_n}}{m_n!} \,. \end{aligned} \tag{53}$$

It is interesting to see how we can associate a *thermal bath* with this distribution function by noticing that $\Lambda(E_q)$ behaves as a relative weight proportional to the number of states with energy $E_q = \hbar\omega q$; the factor $(|z|^2)^q$ fits this weight properly to a temperature as

$$(|z|^2)^q = e^{q \log |z|^2} = e^{-\frac{E_q}{k_B T}} \,, \quad \text{where} \quad T \equiv -\frac{\hbar\omega}{k_B \log |z|^2} = \frac{\hbar a}{2\pi c k_B} \tag{54}$$

is the temperature associated with a given acceleration $a \equiv \frac{2\pi\omega c}{\log|z|^2}$, and k_B, c denote the Boltzmann constant and speed of light, respectively. This simple, but profound, result was first considered by Unruh [5].

After some intermediate calculations, the expected value of the total energy $\hat{E}$ in the accelerated vacuum $|\Psi(z)\rangle_1$ proves to be:

$$\frac{{}_1\langle\Psi(z)|\hat{E}|\Psi(z)\rangle_1}{{}_1\langle\Psi(z)|\Psi(z)\rangle_1} = s\hbar\omega\frac{|z|^2}{(1-|z|^2)^{2s+1}}\,, \tag{55}$$

which coincides with the *mean energy per mode* of the Bose-Einstein statistic, for $s = 0$, when we substitute (54) inside (55). In D spatial dimensions, the number of states with frequency ω is proportional to ω^{D-1}. Thus, the spectral distribution of the radiation of the accelerated vacuum $|\Psi(z)\rangle_1$ for $D = 3$ is given by the formula

$$u_s(x) = \epsilon_0\frac{x^3e^{-x}}{(1-e^{-x})^{2s+1}}\,, \tag{56}$$

where $x \equiv \frac{\hbar\omega}{k_BT_0}$ and ϵ_0 is a constant, for a fixed temperature T_0, with dimensions of energy per unit of volume. Figure 2 represents the spectral distribution $u_s(x)$ for diferent values of s (related to physical magnitudes like the zero-point energy and the curvature of the space). For $s \to 0$, we recover the Planckian (black body) spectrum.[§] For $s > 0$, we have a deformation of the Planckian spectrum; the value $s = (D-1)/2$ corresponds to a *critical* situation: over this value, the theory exhibits an "infrared catastrophe". The physical meaning of this divergence is unclear to the author. A possible explanation could be that higher-spin fields exhibit anomalies and require more careful treatment. The reason for this belief lies in the fact that quantization deforms higher-spin algebras like w to $\mathcal{W}$, by introducing renormalizations —Moyal terms— (see next Section).

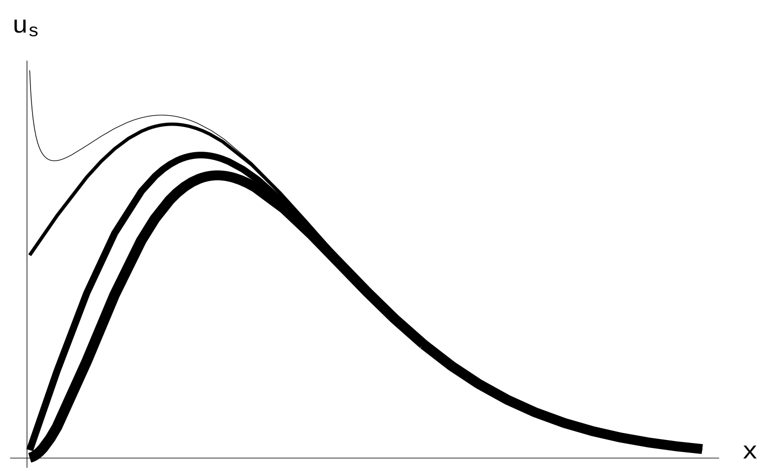

Figure 2: Higher-spin $s = \frac{1}{2}, 1$(critical),> 1(infrared divergence) deformations of the Planckian spectrum $s = 0$ (thickest line).

[§]Note that this limit can only be reached in going to the universal covering of $SU(1,1)$, which means to make time non-compact.

Unruh effect could be seen as a "first order effect" of gravity on quantum field theory, in the sense that uniform accelerations (special conformal transformations) are enough to account for this radiation phenomenon. Higher order (linear, quadratic, etc) accelerations can be consistently included in 1+1D by enlarging the $SO(2,2) \sim SU(1,1) \times SU(1,1)$ symmetry to two copies of the Virasoro algebra

$$\begin{array}{c} \leftarrow \dots L_{-3}, L_{-2}, \overbrace{L_{-1}, L_0, L_1}^{su(1,1)}, L_2, L_3 \dots \rightarrow \\ \leftarrow \dots \bar{L}_{-3}, \bar{L}_{-2}, \underbrace{\bar{L}_{-1}, \bar{L}_0, \bar{L}_1}_{su(1,1)}, \bar{L}_2, \bar{L}_3 \dots \rightarrow, \end{array} \quad (57)$$

for the generators $L_n, \bar{L}_n$ of holomorphic and antiholomorphic diffeomorphisms (left- and right-moving modes), respectively, with commutation relations:

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \\ [\bar{L}_m, \bar{L}_n] &= (m-n)\bar{L}_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \\ [L_m, \bar{L}_n] &= 0, \end{aligned} \quad (58)$$

where c denotes the typical central charge.

We have given a statistical mechanical description of the Unruh effect (vacuum radiation in uniformly accelerated frames) just by counting microscopic states of a conformally invariant quantum field theory —more precisely, the restriction to its left-(or right-)moving sector. There is strong evidence (see [40] and references therein) that conformal field theories also provide a universal description of low-energy black hole thermodynamics. Actually, Unruh's temperature (54) coincides with Hawking's temperature

$$T = \frac{\hbar c^3}{8\pi M k_B G} = \frac{2\pi G M \hbar}{\Sigma c k_B} \quad (59)$$

($\Sigma = 4\pi r_g^2 = 8\pi G^2 M^2/c^4$ stands for the surface of the event horizon) when the acceleration is that of a free falling observer on the surface Σ, i.e. $a = c^4/(4GM) = GM/r_g^2$. In fact, the Virasoro algebra (58) proves to be a physically important subalgebra of the gauge algebra of surface deformations that leave the horizon fixed for an arbitrary black hole. Thus, the fields on the surface must transform according to irreducible representations of the Virasoro algebra (58), which is the general symmetry principle that governs the density of microscopic states. Bekenstein-Hawking expression for the entropy can be then calculated from the Cardy formula [41] (see also [42] for logarithmic corrections).

The infinite-dimensional character of conformal symmetry seems to be exclusive patrimony of 1+1 dimensions. Indeed, conformal symmetry $SO(D,2)$ in D space-time dimensions is said to be finite-dimensional except for $D = 2$. However, one can consider "analytic continuations" (in a broad sense specified later) of the finite-dimensional $SO(4,2)$ symmetry, containing diffeomorphisms and higher-spin fields, which constitutes an extrapolation of the Virasoro and $\mathcal{W}$ symmetries to realistic (3+1) dimensions. Such infinite enlargement of $SO(4,2) \sim SU(2,2)$ has been given by the author [10, 11] in the (more general) context of a new class of infinite-dimensional tensor operator algebras of pseudo-unitary groups $U(N_+, N_-)$. Let us see how these abstract algebras admit a physical interpretation as Lie algebras of higher-spin fields and diffeomorphisms on homogeneous spaces like AdS_2 and AdS_5.

4 Higher-dimensional $\mathcal{W}$ algebras and quantum AdS$_5$ space.

The second quantization algebra (23,25,26) contains the strictly necessary space-time symmetries (25) from which to extract the space-time manifold. Nevertheless, in (1+1) dimensions, the $su(1,1)$ Lie algebra (35) can be enlarged to the Virasoro algebra (58) through an "analytic continuation" (57), that is, an extension of $su(1,1)=\{L_n, |n|\leq 1\}$ beyond the wedge $|n|\leq 1$ when we identify $\hat{L}_1\equiv \hat{X}_{\bar{z}}$, $\hat{L}_{-1}\equiv -\hat{X}_z, \hat{L}_0\equiv \frac{1}{2}\hat{X}_h$. This incorporation of new local transformations (diffeomorphisms) constitutes the main (gauge) guide principle to formulate two-dimensional gravity models, like the well known Polyakov's induced gravity [17]. There can be no doubt that a similar enlargement procedure promoting the finite-dimensional $SO(4,2)$ conformal symmetry to the infinite realm is a potentially valuable symmetry resource to formulate gravity models in realistic dimensions. Let us show how this infinite enlargement of $SO(4,2)\sim U(2,2)/U(1)$ can be given as a particular case of more general infinite extensions of pseudo-unitary symmetries $U(N_+,N_-)$, and how diffeomorphisms and higher-spin currents are embedded in this algebraic structure.

Let us denote by $X_{\alpha\beta}$, $\alpha,\beta=1,\ldots,N\equiv N_++N_-$, the generators (step operators) of the $u(N_+,N_-)$ Lie algebra (see Appendix A for another basis of generators) , which admit a matrix realization of the form:

$$X_{\alpha\beta}\equiv -X_\alpha{}^\gamma\eta_{\gamma\beta},\ \ \text{with}\ \ (X_\alpha{}^\gamma)_\sigma{}^\beta\equiv\delta_{\alpha\sigma}\delta^{\gamma\beta}, \tag{60}$$

where $\eta=\mathrm{diag}(1,\overset{N_+}{\ldots},1,-1,\overset{N_-}{\ldots},-1)$ is used to raise and lower indices. The commutation relations of these step operators are:

$$[X_{\alpha_1\beta_1},X_{\alpha_2\beta_2}]=(\eta_{\alpha_1\beta_2}X_{\alpha_2\beta_1}-\eta_{\alpha_2\beta_1}X_{\alpha_1\beta_2})\,. \tag{61}$$

The relation between the step operators $X_{\alpha\beta}$ and L_n for $u(1,1)$ is simply: $L_1=X_{12}, L_{-1}=X_{21}, L_0=\frac{1}{2}(X_{22}+X_{11}), \mathcal{N}=\frac{1}{2}(X_{22}-X_{11})$, where $\mathcal{N}$ is the generator associated with the trace $\eta_{\alpha\beta}X^{\alpha\beta}$ —the operator *number of particles* (49) or *electric charge* in second quantization. The key point is to note that the commutation relations (61) can also be written as:

$$\left[L^I_m,L^J_n\right]=\eta^{\alpha\beta}(J_\alpha m_\beta-I_\alpha n_\beta)L^{I+J-\delta_\alpha}_{m+n}\,, \tag{62}$$

where the lower index m of L now symbolizes an integral upper-triangular $N\times N$ matrix and the upper $U(N_+,N_-)$-spin index I represents a half-integral lattice vector; more schematic:

$$m=\begin{pmatrix} 0 & m_{12} & m_{13} & \ldots & m_{1N}\\ 0 & 0 & m_{23} & \ldots & m_{2N}\\ 0 & 0 & 0 & \ldots & m_{3N}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \vdots & \vdots & \vdots & \ddots & 0\end{pmatrix}_{N\times N},m_{\alpha\beta}\in\mathbb{Z};\quad I=(I_1,\ldots,I_N)\,,\ \ I_\alpha\in\mathbb{Z}/2\,. \tag{63}$$

We are also denoting $m_\alpha\equiv\sum_{\beta>\alpha}m_{\alpha\beta}-\sum_{\beta<\alpha}m_{\beta\alpha}$ and $\delta_\alpha\equiv(\delta_{\alpha,1},\ldots,\delta_{\alpha,N})$. Note that the generators L^I_m are labeled by $N+N(N-1)/2=N(N+1)/2$ indices, in the same way as wave functions ψ^I_m in the carrier space of irreps of $U(N)$ (see Appendix A).

There are many possible ways of embedding the $u(N_+,N_-)$ generators (61) inside (62), as there are many possible choices of $su(1,1)$ inside (58). However, we can establish a "canonical" choice, such as:

$$X_{\alpha\beta} \equiv L^{\delta_\alpha}_{x_{\alpha\beta}}\,, \quad x_{\alpha\beta} \equiv \mathrm{sign}(\beta-\alpha) \sum_{\sigma=\min(\alpha,\beta)}^{\min(\alpha,\beta)-1} x_{\sigma,\sigma+1}\,, \tag{64}$$

where $x_{\sigma,\sigma+1}$ denotes an upper-triangular matrix with zero entries, except for one at the $(\sigma,\sigma+1)$-position, i.e. $(x_{\sigma,\sigma+1})_{\mu\nu} = \delta_{\sigma,\mu}\delta_{\sigma+1,\nu}$. For example, the $u(1,1)$ generators correspond to:

$$X_{12} = L^{(1,0)}_{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \quad X_{21} = L^{(0,1)}_{\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}}, \quad X_{11} = L^{(1,0)}_{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}, \quad X_{22} = L^{(0,1)}_{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}. \tag{65}$$

If we allow the lower-index $m = x_{\alpha\beta}$ of $L^{\delta_\alpha}_m$ in Eq. (64) to run over arbitrary integral upper-triangular matrices m, then we arrive at the infinite-dimensional algebra:

$$\left[L^{\delta_\alpha}_m, L^{\delta_\beta}_n\right] = m^\beta L^{\delta_\alpha}_{m+n} - n^\alpha L^{\delta_\beta}_{m+n}\,, \tag{66}$$

which we shall denote by $\mathcal{L}^{(1)}_\infty(u(N_+,N_-))$. As a particular example, for the simplest case of $u(1,1)$, this "analytic continuation" leads to two Virasoro sectors: $L_{m_{12}} \equiv L^{(1,0)}_m$, $\bar{L}_{m_{12}} \equiv L^{(0,1)}_m$. Its (3+1)-dimensional counterpart $\mathcal{L}^{(1)}_\infty(u(2,2))$ contains four noncommuting Virasoro-like sectors $\mathcal{L}^{(1\alpha)}_\infty(u(2,2)) = \{L^{\delta_\alpha}_m\}$, $\alpha = 1,\ldots,4$, which, in their turn, hold three genuine Virasoro sectors for $m = ku_{\alpha\beta}$, $k \in \mathbb{Z}$, $\alpha,\beta = 1,\ldots,4$, where $(u_{\alpha\beta})_{\mu\nu} = \delta_{\alpha,\mu}\delta_{\beta,\nu}$ is an upper-triangular matrix. In general, $\mathcal{L}^{(1)}_\infty(u(N_+,N_-))$ contains $N(N-1)$ distinct and noncommuting Virasoro sectors, and holds $u(N_+,N_-)$ as the *maximal finite-dimensional subalgebra*.

The algebra $\mathcal{L}^{(1)}_\infty(u(N_+,N_-))$ can be seen as the *minimal* infinite continuation of $u(N_+,N_-)$ representing the diffeomorphism algebra diff(N) of the corresponding N-dimensional manifold (locally the Minkowski space-time for $u(2,2)$). Indeed, the algebra (66) formally coincides with the algebra of vector fields $L^\mu_{f(y)} = f(y)\frac{\partial}{\partial y_\mu}$, where $y = (y_1,\ldots,y_N)$ denotes a local system of coordinates and $f(y)$ can be expanded in a plane wave basis, such that $L^\mu_{\vec{m}} = e^{im^\alpha y_\alpha}\frac{\partial}{\partial y_\mu}$ constitutes a basis of vector fields for the so called generalized Witt algebra [43], of which there are studies about its representations (see e.g. [44, 45, 46]). Note that, for us, the N-dimensional lattice vector $\vec{m} = (m_1,\ldots,m_N)$ is constrained by $\sum_{\alpha=1}^N m_\alpha = 0$ (see the definition of m_α in paragraph after Eq. 62), which introduces some novelties as regards the Witt algebra. In fact, the algebra (66) can be split into one "temporal" piece, constituted by an Abelian ideal generated by $\tilde{L}^N_m \equiv \eta_{\alpha\alpha}L^{\delta_\alpha}_m$, and a "residual" symmetry generated by the spatial diffeomorphisms

$$\tilde{L}^j_m \equiv \eta_{jj}L^{\delta_j}_m - \eta_{j+1,j+1}L^{\delta_{j+1}}_m,\ j = 1,\ldots,N-1 \text{ (no sum on } j)\,, \tag{67}$$

which act semi-directly on the temporal part. More precisely, the commutation relations (66) in this new basis adopt the following form:

$$\begin{aligned} \left[\tilde{L}^j_m, \tilde{L}^k_n\right] &= \tilde{m}^k \tilde{L}^j_{m+n} - \tilde{n}^j \tilde{L}^k_{m+n}\,, \\ \left[\tilde{L}^j_m, \tilde{L}^N_n\right] &= -\tilde{n}^j \tilde{L}^N_{m+n}\,, \\ \left[\tilde{L}^N_m, \tilde{L}^N_n\right] &= 0\,, \end{aligned} \tag{68}$$

where $\tilde{m}_k \equiv m_k - m_{k+1}$. Only for $N=2$, the last commutator admits a central extension of the form $\sim n_{12}\delta_{m+n,0}$ compatible with the rest of commutation relations (68). This result amounts to the fact that the (unconstrained) diffeomorphism algebra diff(N) does not admit any non-trivial central extension except when $N=1$ [47].

Additionally, after the restriction $I=\delta_\alpha$ in (64) is also relaxed to arbitrary half-integral lattice vectors I, the commutation relations (62) define a *higher-$u(N_+,N_-)$-spin algebra* $\mathcal{L}_\infty(u(N_+,N_-))$ (in a sense similar to that of Ref. [8]), which contains $\mathcal{L}^{(1)}_\infty(u(N_+,N_-))$ as a subalgebra as well as all *currents* L^I_m with all $U(N_+,N_-)$-spins I.

For example, for the conformal-spin s quantum fields $\hat\phi^{(s)}$ in (46) on AdS$_2=SO(1,2)/SO(1,1)$, one can check that the identification:

$$L^{(-s,0)}_{|m|} = \sqrt{N^{(s)}_{|m|-s}}\,\hat{a}^{(s)\dagger}_{|m|-s}\,, \quad L^{(-s,0)}_{-|m|} = \sqrt{N^{(s)}_{|m|-s}}\,\hat{a}^{(s)}_{|m|-s}\,,$$
$$L^{(1,0)}_{\begin{pmatrix}0&1\\0&0\end{pmatrix}} = \hat{X}_z\,,\quad L^{(1,0)}_{\begin{pmatrix}0&-1\\0&0\end{pmatrix}} = -\hat{X}_{\bar z}\,,\quad L^{(1,0)}_{\begin{pmatrix}0&0\\0&0\end{pmatrix}} = -\frac{1}{2}\hat{X}_h \tag{69}$$

between the $\mathcal{L}_\infty(u(1,1))$ generators $L^{(I_1,I_2)}_m$, the space-time $SU(1,1)$ generators (35) and the Fourier coefficients $\hat{a}^{(s)}_m,\hat{a}^{(s)\dagger}_m$ of the field $\hat\phi^{(s)}$ —where $|m|$ denotes the absolute value of all the entries of the upper-triangular matrix $m=\begin{pmatrix}0&m\\0&0\end{pmatrix}$—, give us an embedding of the second quantization algebra (35,47) in (62), except for central terms. Thus, the algebra $\mathcal{L}_\infty(u(1,1))$ contains not just the basic space-time symmetry $u(1,1)$ of AdS$_2$, but also its prolongation to the whole algebra $\mathcal{L}^{(1)}_\infty(u(1,1))\sim$ diff(2) of diffeomorphisms of AdS$_2$, together with all fields $\hat\phi^{(s)}$ with arbitrary conformal spin. This embedding of $SU(1,1)$-spin s quantum fields $\hat\phi^{(s)}$ on AdS$_2$ in $\mathcal{L}_\infty(u(1,1))$ is straightforwardly generalized to $SU(2,2)$-spin S quantum fields $\hat\phi^{(S)}$ on AdS$_5$ (see Appendix A for a explicit expression of coordinates, wave functions and propagators in AdS$_5$) by considering the (4+1)-dimensional counterpart $\mathcal{L}_\infty(u(2,2))$ of the (1+1)-dimensional higher-spin algebra $\mathcal{L}_\infty(u(1,1))$. Note that the generators L^I_m of $\mathcal{L}_\infty(u(N_+,N_-))$ carry an extra spin label (the trace $S_N=I_1+\ldots+I_N$) with respect to the Fourier coefficients $a^{(S)}_m$, $S=(S_1,\ldots,S_{N-1})$, of the $SU(N_+,N_-)$ quantum fields $\hat\phi^{(S)}$ [see paragraph before Eq. (88)]; the relevance of this extra phase invariance and its possible connection with charge conjugation remains to be studied.

As we have already said, the algebra (62) reproduces the commutation relations of higher-$U(N_+,N_-)$-spin quantum fields except for central terms (propagators). In fact, besides such central extensions, the quantization process entails unavoidable renormalizations (mainly due to ordering problems) of the form:

$$\left[\hat{L}^I_m,\hat{L}^J_n\right] = \hbar\eta^{\alpha\beta}(J_\alpha m_\beta - I_\alpha n_\beta)\hat{L}^{I+J-\delta_\alpha}_{m+n} + O(\hbar^3) + \hbar^{(\sum_{\alpha=1}^N I_\alpha+J_\alpha)}c^{(I,J)}(m)\delta_{m+n,0}\hat{1}\,, \tag{70}$$

where $\hat{1}\sim\hat{L}^0_0$ denotes a central generator and $c^{(I,J)}(m)$ are central charges. The higher order terms $O(\hbar^3)$ can be captured in a classical construction by extending the classical (Poisson-Lie) bracket (62) to the Moyal bracket (see [10] for more information on Moyal deformation).

Central extensions provide the essential ingredient required to construct invariant geometric action functionals on coadjoint orbits of the corresponding groups. When applied to the infinite continuation (70) of $u(2,2)$, this would lead to Wess-Zumino-Witten-like models for *induced conformal gravities in* $3+1$ *dimensions*, as happens for the Virasoro and $\mathcal{W}$ algebras in relation with (1+1)-dimensional gravity (see e.g. Ref. [24]). This is the main claim of this article: *the potential (gauge) guiding principle of the higher-$u(2,2)$-spin algebra (70) towards the formulation of consistent gravity models in realistic dimensions.*

Moreover, as we have already commented, the statistical mechanical explanation of radiation phenomena like black hole evaporation proves to rely heavily on the count of microscopic states of a given gauge (diffeomorphism) invariant field theory. Previous to this, an exhaustive study of central charges and representation theory of the corresponding symmetry algebras —like (70)— would be necessary. The general study of infinite-dimensional algebras and groups has not progressed very far, except for some important achievements in one and two dimensions. Thus, we do not expect the representation theory of these algebras to be an easy task. Perhaps, some results imported from the representation theory of toroidal Lie algebras (see e.g. [48]) could be of help. "Two-toroidal Lie algebras" are higher-dimensional generalizations of the affine Kac-Moody algebra in the sense that they replace the loop with a two-torus, that is, the integer lower-index m (loop winding number) of the Kac-Moody algebra generator T^a_m is replaced by an integral four-vertor $\vec{m} = (m_1, \ldots, m_4)$. These algebras appear as current symmetry algebras of the four-dimensional Kähler WZW model, also denoted by WZW_4 (see e.g. [49]). WZW_4 model also arises as the induced surface theory of a 4+1 dimensional AdS Chern-Simons gravity theory with gauge group $SO(4,2) \times U(1) \sim U(2,2)$ —except for discrete symmetries— (see [50, 51]), in analogy with what happens in 2+1 dimensions [52, 53].

4.1 Quantum AdS space

Before passing on to the Conclusions, let us comment on the potential relevance of the C^*-algebras (70) on tractable non-commutative versions of curved spaces like AdS_5. It is a remarkable fact that the algebra (70) is actually a member of a N-parameter family $\tilde{\mathcal{L}}_{\vec{\rho}}(u(N_+, N_-))$, $\vec{\rho} \equiv (\rho_1, \ldots, \rho_N)$ of non-isomorphic algebras of $U(N_+, N_-)$ tensor operators (see [10]), the classical limit $\hbar \to 0, \rho_\alpha \to \infty$ corresponding to the classical (Poisson-Lie) algebra $\mathcal{L}_\infty(u(N_+, N_-))$ with commutation relations (62). A very interesting feature of $\tilde{\mathcal{L}}_{\vec{\rho}}(u(N_+, N_-))$ is that it *collapses* to $\mathrm{Mat}_d(\mathbb{C})$ (the full matrix algebra of $d \times d$ complex matrices) whenever the (complex) parameters ρ_α coincide with the eigenvalues q_α of the Casimir operators C_α of $u(N_+, N_-)$ in a d-dimensional irrep $D_{\vec{q}}$ of $u(N_+, N_-)$. This fact can provide discrete, 'fuzzy' or 'cellular' descriptions of the non-commutative counterpart of AdS_5 when applying the ideas of *non-commutative geometry* (see e.g. [32]) to $\tilde{\mathcal{L}}_{\vec{\rho}}(u(2,2))$. The appealing feature of a non-commutative space M is that a G-invariant 'lattice structure' can be constructed in a natural way, a desirable property as regards finite models of quantum gravity (see e.g. [14] and Refs. therein).

It is also a very important feature of $\mathcal{L}_{\vec{\rho}}(u(N_+, N_-))$ that the quantization deformation scheme (10) does not affect the maximal finite-dimensional subalgebra $su(N_+, N_-)$ ('good observables' or preferred coordinates [54]) of non-commuting 'position operators'

$$y_{\alpha\beta} = \tfrac{\lambda\!\!\!^-}{2}(X_{\alpha\beta} + X^\dagger_{\alpha\beta}), \quad y_{\beta\alpha} = \tfrac{i\lambda\!\!\!^-}{2}(X_{\alpha\beta} - X^\dagger_{\beta\alpha}), \quad \alpha < \beta, \\ y_\alpha = \lambda\!\!\!^-(\eta_{\alpha\alpha} X_{\alpha\alpha} - \eta_{\alpha+1,\alpha+1} X_{\alpha+1,\alpha+1}), \tag{71}$$

on the algebraic manifold $F_{N_+N_-}$, where $\lambda\!\!\!^-$ gives y dimensions of length. The 'volume' v_j of the $N-1$ submanifolds F_j of the *flag manifold* $F_{N_+N_-} = F_N \supset \ldots \supset F_2$ (see e.g. [12, 55] for a definition of flag manifolds) is proportional to the eigenvalue ρ_j of the $su(N_+, N_-)$ Casimir operator $\hat{C}_j$ (84) in those coordinates: $v_j = \lambda\!\!\!^{-j} \rho_j$. Large volumes (flat-like spaces) correspond to a high density of quantum points (large ρ). In the classical limit $\lambda \to 0$, $\rho \to \infty$, the coordinates y commute.

The notion of "quantum space" itself could be the collection of all of them, enclosed in a more general algebraic structure. A suitable candidate for this purpose, where different quantum spaces can coexist with different multiplicities, is the *group algebra* $C^*(G)$ of complex functions on G. Indeed, the group algebra decomposes as (see e.g. [38, 56]):

$$C^*(G) \simeq \bigoplus_{\vec{\rho}\in\hat{G}} \mathcal{L}_{\vec{\rho}}(\mathcal{G}), \tag{72}$$

where $\hat{G}$ denotes the set of unirreps of G. From this point of view, classical, flat-like spaces (large ρ) would be "more likely" to be found inside the whole ensemble $C^*(G)$, as it should be!.

This breaking down of $C^*(G)$ into factor algebras $\mathcal{L}_{\vec{\rho}}(\mathcal{G})$ (matrix algebras for $\vec{\rho} \in \hat{G}$) is the quantum analogue of the standard foliation $\mathcal{G}^* \simeq \bigcup_j F_j$ of the coalgebra $\mathcal{G}^*$ into coadjoint orbits F_j (symplectic leaves) [56].

There is a natural invariant associative and noncommutative $*$-product between functions $\psi^I_m, \psi^J_n \in C^*(G)$ (the *convolution product*):

$$(\psi^I_m * \psi^J_n)(g') \equiv \int_G d^L g\, \psi^I_m(g)\psi^J_n(g^{-1} \bullet g'), \tag{73}$$

and an *involution* $\psi^*(g) \equiv \bar{\psi}(g^{-1})$, which define $C^*(G)$ as a noncommutative C^*-algebra. We also define the "negative modes" ψ^I_{-m} of basic anti-holomorphic functions (85,86) as the corresponding holomorphic ones, i.e. $\psi^I_{-m} \equiv \bar{\psi}^I_m$. The classical limit of the convolution commutator

$$\{\psi, \psi'\} = \lim_{\lambda\to 0} \frac{i}{\lambda^2}(\psi * \psi' - \psi' * \psi) \tag{74}$$

corresponds to the Poisson-Lie bracket between functions on $C^\infty(\mathcal{G}^*)$. The explicit form (in coordinates) of the Poisson-Lie bracket for $G = U(N_+, N_-)$ is:

$$\{\psi, \psi'\} = (\eta_{\alpha_1\beta_2} x_{\alpha_2\beta_1} - \eta_{\alpha_2\beta_1} x_{\alpha_1\beta_2}) \frac{\partial\psi}{\partial x_{\alpha_1\beta_1}} \frac{\partial\psi'}{\partial x_{\alpha_2\beta_2}}, \tag{75}$$

where $x_{\alpha\beta}, \alpha, \beta = 1, \ldots, N$ denote the coordinates in the coalgebra $\mathcal{G}^*$ (seen as a N^2-dimensional vector space) of $G = U(N_+, N_-)$. The constraints $C_\alpha(x) = \lambda^\alpha \rho_\alpha \equiv v_\alpha$ induced by the Casimir operators (84) foliate $C^\infty(\mathcal{G}^*)$ into Poisson algebras $C^\infty(F_\alpha)$. The leaves F (coadjoint orbits, flag manifolds) of $\mathcal{G}^*$ admit a symplectic structure (F, Ω), where Ω denotes a closed two-form —Kähler form (90)— which can be obtained from a Kähler potential (91) as described at the end of Appendix A. After this foliation, the Poisson bracket (75) becomes

$$\{\psi_a, \psi_b\} = \sum_{\alpha_j>\beta_j} \Omega^{\alpha_1\beta_1;\alpha_2\beta_2} \frac{\partial\psi_a}{\partial z^{\alpha_1\beta_1}} \frac{\partial\psi_b}{\partial \bar{z}^{\alpha_2\beta_2}} = \sum_c f^c_{ab}\psi_c, \tag{76}$$

where $\psi_{a,b,c}$ belong to a given irrep $\mathcal{H}_{\vec{v}}(G)$. The structure constants for (76) can be obtained through $f^c_{ab} = \langle \psi_c \mid \{\psi_a, \psi_b\}\rangle$ when the set $\{\psi_a\}$ is chosen to be orthonormal. To each function $\psi \in C^\infty(F)$, one can assign its *Hamiltonian vector field* $H_\psi \equiv \{\psi, \cdot\}$, which is obviously divergence-free and preserves the natural volume form $\Omega^{N(N-1)/2}$. Thus, the space $\mathrm{Ham}(F)$

of Hamiltonian vector fields is, in general, a subalgebra of the algebra sdiff(F) of symplectic (volume-preserving) diffeomorphisms of F, as stated in (1). They only (essentially) coincide in two dimensions, where $F_2 = S^2$ and $F_{1,1} = S^{1,1}$ (the sphere and the hyperboloid) [57].

The author is aware that some of the previous physical ideas are still rather conjectural and not quite developed or founded. Nevertheless, the mathematical structure, as such, is somewhat promising and deserves further study. A more explicit interconnection between tensor operator, group, Poisson and symplectic diffeomorphism algebras, and non-commutative geometry on $U(N_+, N_-)$ is in progress [58].

5 Conclusions and outlook

The long sought-for unification of all interactions and exact solvability of (quantum) field theory and statistics parallels the quest for new symmetry principles. Symmetry is an essential resource when facing those two fundamental problems, either as a gauge guide principle or as a valuable classification tool. The representation theory of infinite-dimensional groups and algebras has not progressed very far, except for some important achievements in one and two dimensions (mainly Virasoro and Kac-Moody symmetries), and necessary breakthroughs in the subject remain to be carried out. This article intends to fill just part of this gap by providing tractable higher-dimensional analogies of the infinite two-dimensional conformal symmetry and its generalizations in the context of higher-conformal-spin fields on anti-de Sitter spaces. We have discussed the potential role of these new symmetry algebras in the understanding of important aspects of (the still unknown) quantum gravity like: radiation phenomena and quantum (non-commutative) structure of the space at Planck scale. The next step will consist in an exhaustive study of the representations of these tensor operator (and group) algebras and the construction of invariant geometric action functionals. Other facets in which these symmetries could be of use are: integrable (classical and quantum) nonlinear field models and phase transitions in higher dimensions.

A Appendix: Higher-$U(2,2)$-spin fields on AdS$_5$

In order to put coordinates on AdS$_5$, the ideal choice is the Bruhat cell decomposition [12] of $SU(2,2)$ for the flag manifold $F_{2,2} = SU(2,2)/U(1)^3$ (e.g. the maximal coadjoint orbit of $SU(2,2)$). According to this Bruhat decomposition, the flag manifold $F_{2,2}$ can be covered with several coordinate patches. We shall restrict ourselves to the largest cell which provides a complex coordinatization $\{z_{\alpha\beta}, \alpha > \beta = 1,2,3\}$ of nearly all $F_{2,2}$, missing only lower-dimensional subspaces. A triangularization process

$$\begin{array}{c} \begin{matrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{u}_4 \end{matrix} \\ \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ u_{21} & u_{22} & u_{23} & u_{24} \\ u_{31} & u_{32} & u_{33} & u_{34} \\ u_{41} & u_{42} & u_{43} & u_{44} \end{pmatrix} \end{array} \longrightarrow \begin{array}{c} \begin{matrix} \vec{z}_1 & \vec{z}_2 & \vec{z}_3 & \vec{z}_4 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ z_{21} & 1 & 0 & 0 \\ z_{31} & z_{32} & 1 & 0 \\ z_{41} & z_{42} & z_{43} & 1 \end{pmatrix} \end{array} \tag{77}$$

gives us the coordinates

$$z_{21} = \frac{u_{21}}{u_{11}},\ z_{31} = \frac{u_{31}}{u_{11}},\ z_{41} = \frac{u_{41}}{u_{11}},$$

$$z_{32} = \frac{u_{11}u_{32} - u_{12}u_{31}}{u_{11}u_{22} - u_{12}u_{21}},\ z_{42} = \frac{u_{11}u_{42} - u_{12}u_{41}}{u_{11}u_{22} - u_{12}u_{21}}, \tag{78}$$
$$z_{43} = \frac{u_{13}(u_{21}u_{42} - u_{22}u_{41}) - u_{23}(u_{11}u_{42} - u_{12}u_{41}) + u_{43}(u_{11}u_{22} - u_{12}u_{21})}{u_{13}(u_{21}u_{32} - u_{22}u_{31}) - u_{23}(u_{11}u_{32} - u_{12}u_{31}) + u_{33}(u_{11}u_{22} - u_{12}u_{21})},$$

on the 6-dimensional complex (non-compact) manifold $F_{2,2}$. These coordinates are defined where the denominators are non-zero. Other patches basically correspond to different locations of 0's and 1's in the "triangularization" process (77). The Cartan subgroup $U(1)^3$ corresponds to diagonal matrices $h = \mathrm{diag}(h_1, h_2/h_1, h_3/h_2, 1/h_3)$ with coordinates

$$h_1 = \left(\frac{u_{11}}{\bar u_{11}}\right)^{1/2},\ h_2 = \left(\frac{u_{11}u_{22} - u_{12}u_{21}}{\bar u_{11}\bar u_{22} - \bar u_{12}\bar u_{21}}\right)^{1/2}, \tag{79}$$
$$h_3 = \left(\frac{u_{13}(u_{21}u_{32} - u_{22}u_{31}) - u_{23}(u_{11}u_{32} - u_{12}u_{31}) + u_{33}(u_{11}u_{22} - u_{12}u_{21})}{\bar u_{13}(\bar u_{21}\bar u_{32} - \bar u_{22}\bar u_{31}) - \bar u_{23}(\bar u_{11}\bar u_{32} - \bar u_{12}\bar u_{31}) + \bar u_{33}(\bar u_{11}\bar u_{22} - \bar u_{12}\bar u_{21})}\right)^{1/2}.$$

Let us regard any pseudo-unitary matrix $v \in SU(2,2)$ as a juxtaposition of four column vectors $v = (\vec v_1, \vec v_2, \vec v_3, \vec v_4)$. The expression of v in minimal coordinates $z_{\alpha\beta}$ can be obtained (up to torus elements h) by means of an "adapted" Gramm-Schmidt orthonormalization process of the set $\{\vec z_\alpha\}$ in (77) as follows:

$$\vec v'_\alpha = \left(\vec z_\alpha - \frac{(\vec z_\alpha, \vec v_{\alpha-1})}{(\vec v_{\alpha-1}, \vec v_{\alpha-1})}\vec v_{\alpha-1} - \ldots - \frac{(\vec z_\alpha, \vec v_1)}{(\vec v_1, \vec v_1)}\vec v_1\right),\ \ \vec v_\alpha = \frac{\vec v'_\alpha}{(\eta^{\alpha\alpha}(\vec v'_\alpha, \vec v'_\alpha))^{1/2}}, \tag{80}$$

where $(\vec u_\alpha, \vec v_\beta) \equiv \bar u_{\alpha\mu}\eta^{\mu\nu}v_{\beta\nu}$ denotes a scalar product with indefinite metric $\eta = \mathrm{diag}(1,1,-1,-1)$. The explicit expression of (80) proves to be:

$$\begin{aligned}
\vec v_1 &= \frac{1}{\Delta_1}\begin{pmatrix} 1 \\ z_{21} \\ z_{31} \\ z_{41}\end{pmatrix},\ \vec v_2 = \frac{1}{\Delta_1\Delta_2}\begin{pmatrix} -\bar z_{21} + z_{32}\bar z_{31} + z_{42}\bar z_{41} \\ 1 + z_{32}z_{21}\bar z_{31} - z_{31}\bar z_{31} + z_{42}z_{21}\bar z_{41} - z_{41}\bar z_{41} \\ z_{32} + z_{32}z_{21}\bar z_{21} - \bar z_{21}z_{31} + z_{42}z_{31}\bar z_{41} - z_{32}z_{41}\bar z_{41} \\ z_{42} + z_{42}z_{21}\bar z_{21} - z_{42}z_{31}\bar z_{31} - \bar z_{21}z_{41} + z_{32}\bar z_{31}z_{41}\end{pmatrix} \\
\vec v_3 &= \frac{1}{\Delta_2\Delta_3}\begin{pmatrix} [-\bar z_{32}\bar z_{21} - \bar z_{42}z_{43}\bar z_{21} + \bar z_{31} - z_{42}\bar z_{42}\bar z_{31} \\ +z_{32}\bar z_{42}z_{43}\bar z_{31} + \bar z_{32}z_{42}\bar z_{41} + z_{43}\bar z_{41} - z_{32}\bar z_{32}z_{43}\bar z_{41}] \\ [\bar z_{32} + \bar z_{42}z_{43} - z_{42}\bar z_{42}z_{21}\bar z_{31} + z_{32}\bar z_{42}z_{43}z_{21}\bar z_{31} - \bar z_{42}z_{43}z_{31}\bar z_{31} + \bar z_{42}\bar z_{31}z_{41} \\ +\bar z_{32}z_{42}z_{21}\bar z_{41} - z_{32}\bar z_{32}z_{43}z_{21}\bar z_{41} + \bar z_{32}z_{43}z_{31}\bar z_{41} - \bar z_{32}z_{41}\bar z_{41}] \\ [1 - z_{42}\bar z_{42} + z_{32}\bar z_{42}z_{43} - z_{42}\bar z_{42}z_{21}\bar z_{21} + z_{32}\bar z_{42}z_{43}z_{21}\bar z_{21} - \bar z_{42}z_{43}\bar z_{21}z_{31} \\ +\bar z_{42}\bar z_{21}z_{41} + z_{42}z_{21}\bar z_{41} - z_{32}z_{43}z_{21}\bar z_{41} + z_{43}z_{31}\bar z_{41} - z_{41}\bar z_{41}] \\ [\bar z_{32}z_{42} + z_{43} - z_{32}\bar z_{32}z_{43} + \bar z_{32}z_{42}z_{21}\bar z_{21} - z_{32}\bar z_{32}z_{43}z_{21}\bar z_{21} + \bar z_{32}z_{43}\bar z_{21}z_{31} \\ -z_{42}z_{21}\bar z_{31} + z_{32}z_{43}z_{21}\bar z_{31} - z_{43}z_{31}\bar z_{31} - \bar z_{32}\bar z_{21}z_{41} + \bar z_{31}z_{41}]\end{pmatrix} \\
\vec v_4 &= \frac{1}{\Delta_3}\begin{pmatrix} -\bar z_{42}\bar z_{21} + \bar z_{32}\bar z_{43}\bar z_{21} - \bar z_{43}\bar z_{31} + \bar z_{41} \\ \bar z_{42} - \bar z_{32}\bar z_{43} \\ -\bar z_{43} \\ 1\end{pmatrix}
\end{aligned} \tag{81}$$

where

$$\begin{aligned}
\Delta_1(z,\bar z) &= \sqrt{1 + |z_{21}|^2 - |z_{31}|^2 - |z_{41}|^2} \\
\Delta_2(z,\bar z) &= \sqrt{1 + |z_{32}z_{41} - z_{42}z_{31}|^2 - |z_{32}|^2 - |z_{42}|^2 - |z_{32}z_{21} - z_{31}|^2 - |z_{42}z_{21} - z_{41}|^2} \\
\Delta_3(z,\bar z) &= \sqrt{1 + |z_{43}|^2 - |z_{42} - z_{43}z_{32}|^2 - |z_{41} + z_{43}z_{32}z_{21} - z_{42}z_{21} - z_{43}z_{31}|^2}
\end{aligned} \tag{82}$$

are three characteristic lengths that will play a central role in what follows.

Using a relativistic notation, we may say that the vectors $\vec{v}_1$ and $\vec{v}_2$ are "spacelike" (that is, $(\vec{v}_{1,2}, \vec{v}_{1,2}) = 1$) whereas $\vec{v}_3$ and $\vec{v}_4$ are "timelike" (i.e, $(\vec{v}_{3,4}, \vec{v}_{3,4}) = -1$); this ensures that $v\eta v^\dagger = \eta$. Any $u \in SU(2,2)$ matrix in the present patch (which contains the identity element $z = 0 = \bar{z}, h = 1$) can be written in minimal coordinates $z_{\alpha\beta}, \bar{z}_{\alpha\beta}, h_\beta$, $\alpha > \beta = 1,2,3$ as the product $u = vh$ of an element v on the base $F_{2,2}$ times an element h on the fibre $U(1)^3$.

Once we have the expression of a general $SU(2,2)$ group element $g = u$ in terms of the minimal coordinates $u = u(z_{\alpha\beta}, \bar{z}_{\alpha\beta}, h_\beta)$, we can easily write the group law $u'' = u' \bullet u$ and compute the left- and right-invariant vector fields (21,24), as for the $SU(1,1)$ case (33,34). We shall not give the (rather cumbersome) explicit expression of the, let us say, right-invariant vector fields, but the relation between them and the step operators (60) with commutation relations (61). This correspondence is:

$$X^R_{z_{\alpha\beta}} \to X_\alpha{}^\beta, \;\; X^R_{\bar{z}_{\alpha\beta}} \to -X^\beta{}_\alpha, \;\; X^R_{h_\beta} \to X_\beta{}^\beta - X_{\beta+1}{}^{\beta+1}, \quad \alpha > \beta = 1,2,3. \tag{83}$$

The Casimir operators for $SU(2,2)$ are easily written in the basis of step operators as follows:

$$C_2 = X_\alpha{}^\beta X_\beta{}^\alpha, \;\; C_3 = X_\alpha{}^\beta X_\beta{}^\gamma X_\gamma{}^\alpha, \;\; C_4 = X_\alpha{}^\beta X_\beta{}^\gamma X_\gamma{}^\sigma X_\sigma{}^\alpha. \tag{84}$$

where the trace $C_1 = X_\alpha{}^\alpha$ is zero for special groups. Associated with the previous three Casimir operators, there are three possible choices of time-generators $X^L_{t_\beta} \equiv \frac{\omega_\beta}{2} X^L_{h_\beta}$ to construct the Hilbert space $\mathcal{H}_S(SU(2,2))$ of holomorphic (i.e. $X^L_z \psi = 0$) wave functions ψ on AdS$_5$. The general solution to the equations of motion proves to be:

$$\left.\begin{array}{l} X^L_{t_\beta}\psi = -\omega_\beta S_\beta \psi \\ X^L_{z_{\alpha\beta}}\psi = 0 \end{array}\right\} \Rightarrow \psi^{(S)}(u) = W^{(S)}(h, z, \bar{z})\varphi(\bar{z}), \tag{85}$$

where

$$W^{(S)}(h, z, \bar{z}) \equiv \prod_{\beta=1}^{3} \left(h_\beta \Delta_\beta(z, \bar{z})\right)^{-2S_\beta}, \quad \varphi(\bar{z}) \equiv \sum_m a^{(S)}_m \prod_{\alpha>\beta} (\bar{z}_{\alpha\beta})^{m_{\beta\alpha}} \tag{86}$$

represents the vacuum wave function $W^{(S)}$, which is written in terms of the toral (angular) coordinates h_β and the typical lengths Δ_β in (82), and φ is an analytic power series, with complex coefficients $a^{(S)}_m$, on its arguments $\bar{z}_{\alpha\beta}$, respectively. The index m denotes an integral upper-triangular 4×4 matrix (see Eq. (63)) and $S = (S_1, S_2, S_3)$ is the $SU(2,2)$-spin three-dimensional vector which lies on a half-integer lattice. Let us discuss the range of variation of the S_β and $m_{\alpha\beta}$ indices.

The sign of the $SU(2,2)$-spin indices S_β depends on the (non-)compact character of the corresponding simple roots (the ones whose generators $X_{\alpha\beta}$ are labeled by $\alpha\beta = 12, 23, 34$). In this notation, the roots $\alpha\beta = 12, 34$ are of compact type, which implies $S_1, S_3 \in \mathbb{Z}^+/2$, and the root $\alpha\beta = 23$ is of non-compact type, which leads to $S_2 \in \mathbb{Z}^-/2$. This also guarantees the finiteness of the scalar product (20) with left-invariant integration measure

$$d^L u = \frac{i}{(2\pi)^6} \frac{1}{\prod_{\beta=1}^3 (\Delta_\beta)^4} \bigwedge_{\beta=1}^{3} h_\beta^{-1} dh_\beta \bigwedge_{\alpha>\beta} d\Re(z_{\alpha\beta}) \wedge d\Im(z_{\alpha\beta}), \tag{87}$$

and the unitarity of the representation $\rho(u')\psi^{(S)}(u) = \psi^{(S)}(u'^{-1} \bullet u)$ of $SU(2,2)$. We can still keep track of the extra $U(1)$ quantum number S_4 that diferentiates $U(2,2) \simeq (SU(2,2) \times U(1))/\mathbb{Z}_4$

from $SU(2,2)$ representations. The $U(2,2)$ wave functions $\tilde{\psi}^{(I)}$ depend on an extra $U(1)$-factor $(h_4)^{-2S_4}, h_4 \in U(1)$ in the vacuum wave function $W^{(S)}$ in (86), where the relation between the $U(2,2)$-spin labels $I = (I_1, I_2, I_3, I_4)$ of Sec. (4) and the $SU(2,2) \times U(1)$-spin labels $S = (S_1, S_2, S_3, S_4)$ is $S_\beta = I_\beta - I_{\beta+1}, \beta = 1,2,3$ and $S_4 = I_1 + I_2 + I_3 + I_4$ (the Casimir C_1 —trace— eigenvalue). For completeness, we shall give the action of the Cartan subalgebra (Hamiltonian operators) on basic wave functions $\psi_m^{(S)} \equiv W^{(S)} \prod_{\alpha>\beta} (\bar{z}_{\alpha\beta})^{m_{\beta\alpha}}$ of $\mathcal{H}_S(SU(2,2))$. It can be easily calculated from the explicit expression of the right-invariant differential operators $X^R_{h_\beta}$ and is proven to be:

$$X^R_{h_\beta} \psi_m^{(S)} = (2S_\beta + m_\beta - m_{\beta+1}) \psi_m^{(S)}, \tag{88}$$

where m_β is defined after Eq. (63); note that the eigenvalue $(2S_\beta + m_\beta - m_{\beta+1})$ of $X^R_{h_\beta}$ can also be written as $2(\Gamma^+_\beta - \Gamma^+_{\beta+1})$, where $\Gamma^\pm_\beta \equiv I_\beta \pm m_\beta/2$ is a characteristic quantity that plays a central role (for example, the structure constants of the algebra (62) and its higher order quantum corrections (70) could also be written as powers of Γ's). It is also worth noticing that, from (88), the quantities $E^{(0)}_\beta \equiv \hbar\omega_\beta S_\beta$ represent the vacuum expectation values (zero-point energies) of the Hamiltonian operators $X^R_{t_\beta}$ defined above. The action of the remainder operators $X^R_{z,\bar{z}}$ on basic functions can also be calculated in the same way, although we shall not give the explicit expression here.

In order to find the domain of the matrix indices $m_{\alpha\beta}$ for each set of spin labels $\{S_\beta\}$, an "orbit-through-the-vacuum" procedure could be used, which consists in an iterative application of raising (creation) operators $X^R_{z_{\alpha\beta}}$ on the vacuum $\psi_0^{(S)}$. Another possibility is to look at the propagator $\Delta^{(S)}(u,u')$, whose particular form can be inferred from the expression (48) for the AdS_2 case; indeed, it can be written in terms of the lengths Δ_β as follows:

$$\Delta^{(S)}(u,u') = \sum_m \frac{1}{N_m^{(S)}} \psi_m^{(S)}(u) \bar{\psi}_m^{(S)}(u') = \kappa_S \prod_{\beta=1}^{3} \frac{\Delta_\beta(z',\bar{z})^{4S_\beta}}{(h_\beta \Delta_\beta(z,\bar{z}))^{2S_\beta} (\bar{h}'_\beta \Delta_\beta(z',\bar{z}'))^{2S_\beta}}, \tag{89}$$

where $N_m^{(S)}$ are the normalization (squared) constants of the basic functions $\psi_m^{(S)}$ and κ_S is a global constant depending on S. The expansion of the factor $\Delta_\beta(z',\bar{z})^{4S_\beta}$ in (89) in powers of $z'_{\alpha\beta}\bar{z}_{\alpha\beta}$ tells us which $m_{\beta\alpha}$ are present in the summation (89) —and (86)— for a given $SU(2,2)$-spin S. Taking into account that $S_3 \leq 0$, it is clear that some of the $m_{\alpha\beta}$ indices have no upper-bound, as corresponds in general to unitary irreducible representations of non-compact groups.

Before concluding this Appendix, let us briefly comment on Kähler geometric structures on the flag manifold $F_{2,2}$ (it also applies to general F_{N_+,N_-}). Since the flag manifold $F_{2,2}$ is a Kähler manifold [59], it possesses complex local coordinates $z_{\alpha\beta}$ (78), an Hermitian Riemannian metric g and a corresponding closed two-form (Kähler form) Ω

$$ds^2 = g^{\alpha\beta,\mu\nu} dz_{\alpha\beta} d\bar{z}_{\mu\nu}, \quad \Omega = i g^{\alpha\beta,\mu\nu} dz_{\alpha\beta} \wedge d\bar{z}_{\mu\nu}, \tag{90}$$

which can be obtained from the Kähler potential

$$K^{(S)}(z,\bar{z}) = \sum_{\beta=1}^{3} 2S_\beta \ln \Delta_\beta(z,\bar{z}) \tag{91}$$

through the formula $g^{\alpha\beta,\mu\nu} = \frac{\partial}{\partial z_{\alpha\beta}} \frac{\partial}{\partial \bar{z}_{\mu\nu}} K^{(S)}$, for a given coadjoint orbit of $SU(2,2)$ characterized by the indices S_β. Keep in mind that, in complex diferential calculus, $\partial^2 = 0 = \bar{\partial}^2$ and $\partial\bar{\partial} + \bar{\partial}\partial = 0$. Note also that, the Kähler potential $K^{(S)}(z,\bar{z})$ essentially corresponds to the natural logarithm of the vacuum $W^{(S)}(h,z,\bar{z})$ in (86), up to toral coordinates h.

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