ON A MORSE CONJECTURE FOR ANALYTIC FLOWS ON COMPACT SURFACES

HABIB MARZOUGUI AND GABRIEL SOLER LÓPEZ

ABSTRACT. The aim of this paper is to prove a Morse conjecture; in particular it is shown that a topologically transitive analytic flow on a compact surface is metrically transitive. We also build smooth topologically transitive flows on surfaces which are not metrically transitive.

1. INTRODUCTION

In what follows S will denote a *surface*, that is, a second countable Hausdorff topological space which is locally homeomorphic to the plane. For a set $U \subset S$ we will say that a continuous map $\Phi : A \subset \mathbb{R} \times U \to U$ is a *local flow* on U when the following properties hold:

(a): A is open in $\mathbb{R} \times U$ and, for any fixed $u \in U$, the set of numbers t for which $\Phi(t, u)$ is defined is an open interval $I_u \ni 0$;

(b): $\Phi(0, u) = u$ for any $u \in U$;

(c): If $\Phi(t, u) = v$ then $I_v = \{s - t : s \in I_u\}$; moreover, $\Phi(s, v) = \Phi(s, \Phi(t, u)) = \Phi(s + t, u)$ for any $s \in I_v$.

If $A = \mathbb{R} \times U$ then the local flow is said to be a *flow* on U. It is easy to check that any local flow on a compact connected surface is in fact a flow.

Recall that any surface S admits an analytic structure which is unique up to diffeomorphisms, see [12, 15], [16, Example 3.1.6] and [6, p. 16]. Then the notion of analytic local flow arises in a natural way and it is closely related to the one of analytic vector field. Namely, see S analytically embedded in $\mathbb{R}^m$ (with $m = 3$ or $m = 4$ depending on the surface). If $u \in S$ then its tangent plane TS_u can be seen as a subset of $\mathbb{R}^m$, and we can define an *analytic vector field* on an open set U as an analytic map $F : U \to \mathbb{R}^m$ such that $F(u) \in TS_u$

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for any $u \in U$. It turns out that if $\Phi : A \to U$ is an analytic local flow then there is an analytic vector field $F : U \to \mathbb{R}^m$ such that $\frac{\partial \Phi}{\partial t}(t,u) = F(\Phi(t,u))$ for any t and u and that, conversely, for any analytic vector field F on U there is an analytic local flow Φ on U such that $\frac{\partial \Phi}{\partial t}(t,u) = F(\Phi(t,u))$ for any t and u.

We remind some classical definitions related to the local flow notion. Let $\Phi : A \subset \mathbb{R} \times U \to U$ be a local flow, then the map $\Phi_u : I_u \to U$ is defined by $\Phi_u(t) = \Phi(t,u)$, the *orbit of* $u \in S$ is $\Phi_u(I_u) = \Phi(I_u \times \{u\})$ and we will generically refer to this set as a Φ-*orbit*. For any interval $I \subset I_u$, $\Phi_u(I)$ is called a *sub-orbit* of the orbit $\Phi_u(I_u)$. If the orbit of u just consists of u then it is called a *singular point* and the orbit $\{u\}$ is called a *singular orbit*, otherwise u is said to be a *regular point* and $\Phi_u(I_u)$ a *regular orbit*; when Φ_u is a periodic nonconstant map then the orbit of u is called *periodic*. The set of singular points from the flow Φ is denoted by $\mathrm{Sing}(\Phi)$.

Let us now consider that Φ is a flow, then a set $A \subset S$ is said to be *invariant* if $\Phi(\mathbb{R} \times A) = A$. The ω-*limit set of* u is defined by $\omega_\Phi(u) = \{v \in S : \exists (t_n)_{n=1}^\infty \to +\infty; (\Phi_u(t_n))_{n=1}^\infty \to v\}$ and the α-*limit set of* u is similarly defined by $\alpha_\Phi(u) = \{v \in S : \exists (t_n)_{n=1}^\infty \to -\infty; (\Phi_u(t_n))_{n=1}^\infty \to v\}$, both sets are closed and invariant and if S is compact they are also connected. The set $\omega_\Phi(u)$ (resp. $\alpha_\Phi(u)$) is also called the ω-*limit set of the orbit* $\Phi_u(\mathbb{R})$ (resp. α-*limit set of the orbit* $\Phi_u(\mathbb{R})$). A point $u \in S$ or its orbit $\Phi_u(\mathbb{R})$ is said to be *recurrent* if $u \in \omega_\Phi(u)$ or $u \in \alpha_\Phi(u)$. Singular points and those from periodic orbits are examples of recurrent points, so called *trivial*. All other recurrent points are *nontrivial*. Let $L = \Phi_u(\mathbb{R})$, the orbit L is called *proper* if $\overline{L} \setminus L$ is closed in S (for example periodic orbits are proper). If L is non proper, it is called *nontrivial recurrent*. In particular, any dense orbit L is nontrivial recurrent and $\overline{L} = \omega_\Phi(u)$ or $\overline{L} = \alpha_\Phi(u)$.

We call *class of the orbit* L to the union $cl(L)$ of Φ-orbits G such that $\overline{G} = \overline{L}$. We note that orbits which are in the same class are either all proper or recurrent. In particular, if L is proper, $cl(L) = L$.

The flow Φ is said to be *topologically transitive* if there is $u \in S$ so that $\overline{\Phi(u)} = S$. Denote by μ the Lebesgue measure of S. In particular, μ is positive on any open set of S. The flow Φ is called *metrically transitive* if any invariant closed set has either zero measure or full measure with respect to μ.

It is well known that a metrically transitive flow on a compact surface is always topologically transitive. The converse is not true in general. Morse conjectured that the converse is true for analytic flow. In [3], T. Ding constructed a topologically transitive C^∞-flow on any closed n-manifold ($n \geq 2$) which is not metrically transitive and proved that the Morse conjecture is true for analytic flow on the torus $\mathbb{T}^2$. The Morse conjecture is also true in the special cases like the stated in the following theorem.

Theorem 1 (Ding-Marzougui)**.** *Let S be a compact orientable surface and let $\Phi : \mathbb{R} \times S \to S$ be a topologically transitive C^1-flow. Then the following statements hold:*

- *If* $\mathrm{Sing}(\Phi)$ *is finite then Φ is metrically transitive (T. Ding,* [4]*).*
- *If* $\mathrm{Sing}(\Phi)$ *is countable then Φ is metrically transitive (H. Marzougui,* [10]*).*

For analytic flows, we mention the preprint [2] by Aranson and Zhuzhoma where they proved the Morse conjecture for analytic flows on compact orientable surfaces and on nonorientable ones if the flow is highly topologically transitive. Note that an analytic flow can have an uncountable singular points.

In this paper we prove the Morse conjecture for analytic flows by other methods different from those given in [2], moreover we show that we can dispose of the restriction on the cardinality of $\mathrm{Sing}(\Phi)$ and the orientability of the surface. Our main result is the following:

Main Theorem. *Let S be a compact surface (orientable or not) and let $\Phi : \mathbb{R} \times S \to S$ be a topologically transitive analytic flow. Then Φ is metrically transitive.*

We will denote by M_g (resp. N_g) the only —up to homeomorphisms— orientable (resp. nonorientable) surface of genus g. As a complement of the previous result we will prove:

Theorem A. *Let $S = M_g$ ($g \geq 1$) or $S = N_g$ ($g \geq 3$), then S admits a smooth topologically transitive flow which is not metrically transitive.*

The remainder of the present work is divided in three sections. The first one presents some theorems about singular points and ω-limit sets. In Sections 3 and 4 we prove, respectively, Main Theorem and Theorem A.

2. Structure of singular points from an analytical flow

We begin by introducing some common notions. A *curve* B in S is the image $B = \varphi(I)$ of a continuous one-to-one map $\varphi : I \to S$, with I being an interval or $I = \mathbb{S}^1$. We call any such map φ a *parametrization* of B. If I is a compact interval or $I = \mathbb{S}^1$ then we call B an *arc* or a *circle*, respectively. If A is an arc, $\varphi : [a, b] \to A$ is a parametrization of A, $\varphi(a) = u$ and $\varphi(b) = v$ we will use the notation $A = [u; v]$. If A is additionally contained in a curve B (resp. in an orbit of Φ) we will also write $A = [u; v]_B$ (resp. $A = [u; v]_\Phi$).

As usually a *disk* is any set homeomorphic to $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and for any $U \subset S$, $\operatorname{Bd} U$ and $\operatorname{Int} U$ respectively denote the topological boundary and the interior of U. By an *r-star* we mean a topological space R homeomorphic to $\{z \in \mathbb{C} : z^r \in [0, 1]\}$, the homeomorphism maps 0 to a point p which is called the *vertex* of the star and maps the r-roots of the unity to the *endpoints* of the star. When r is 1 or 2 then R is just an arc. Any single point will be said to be a 0-star.

Theorem 2 ([7], Th. 4.3)**.** *Let $\Phi : \mathbb{R} \times S \to S$ be an analytic flow and let $u \in \operatorname{Sing}(\Phi)$. Then $\operatorname{Sing}(\Phi)$ is locally a $2n$-star having u as its vertex for some nonnegative integer n.*

Neighbourhoods of isolated singular points from a topologically transitive analytic flow have a simple structure. In order to describe it we need the classical notion of hyperbolic sector. Let $h(x, y) = (x, -y)$, $S_h = \{(x, y) \in \mathbb{R}^2 : 0 \le x < 1, 0 \le y < 1, xy < \frac{1}{2}\}$ and Φ_h the local flow defined by $z' = h(z)$ on S_h. Let $\Phi : \mathbb{R} \times S \to S$ be a flow and $u \in \operatorname{Sing}(\Phi)$, a set $N \ni u$ is said to be an *hyperbolic sector of u* if the induced local flow by Φ on N is *topologically equivalent* to the induced local flow by Φ_h on S_h; that is, there exists an homeomorphism $g : N \to S_h$ which maps Φ-orbits onto Φ_h-orbits and moreover g preserves the orientation of the orbits.

Theorem 3. *Let $\Phi : \mathbb{R} \times S \to S$ be a topologically transitive analytic flow and let u be an isolated singular point, then it has a neighbourhood which is a union of an even number of hyperbolic sectors. Moreover, the orbits separating the sectors approach u from definite directions.*

Proof. From [7, Th. 4.4] and the topological transitivity of Φ u has a neighbourhood which is a union of $n_h \in \mathbb{N}$ hyperbolic sectors. Moreover by [1, p.36, Th. 4.1] $\frac{2-n_h}{2} \in \mathbb{N}$ (this number is the *index of u*), then n_h is even. □

As it is remarked in [7] next theorem is an old folklore result, however it is proved in the cited paper since it is not easy to provide a reference, see [14].

Theorem 4 ([7], Th. 4.5)**.** *Let $g = (g_1, g_2) : O \subset \mathbb{R}^2 \to \mathbb{R}^2$ be an analytic map, $(0, 0) \in O$, and assume that $g(0, 0) = (0, 0)$. Then there are an open neighbourhood U of the origin and* *analytic functions* $C, f_1, f_2 : U \to \mathbb{R}$ *such that* $g_1 = Cf_1$ *and* $g_2 = Cf_2$ *in* U*, and* $z' = f(z)$*,* $f = (f_1, f_2)$*, has no singular points in* $U \backslash \{(0,0)\}$*.*

Last result shows that, in a sense, it is possible to apply Theorem 3 to non isolated singular points. In particular combining theorems 2, 3 and 4 we obtain:

Theorem 5. *Let* $\Phi : \mathbb{R} \times S \to S$ *be a topologically transitive analytic flow and let* u *be a non isolated singular point. Then there is a disk* U *neighbouring* u *and a finite set of arcs,* $\{A_i = [a_i; u]\}_{i=1}^{2k}$*, so that:*

- $R = \bigcup_{i=1}^{2k} A_i$ *is a* $2k$*-star whose vertex is* u*.*
- R *intersects* $\operatorname{Bd} U$ *exactly at its endpoints.*
- $A_i \subset \operatorname{Sing}(\Phi)$ *or* A_i *is a* Φ*-sub-orbit so that if* $v \in A_i$ *then* $\alpha_\Phi(v) = \{u\}$ *or* $\omega_\Phi(v) = \{u\}$*.*
- *If* V *is a component from* $U \backslash R$ *then the induced local flow by* Φ *on* V *is topologically equivalent to the local flow induced by* Φ_h *on* $\operatorname{Int} S_h$*.*

Proof. By Theorem 2 there are an open disk U containing u and a $2l$-star (l is a nonnegative integer), T, so that: $\operatorname{Sing}(\Phi) \cap \overline{U} = T$, u is the vertex of T and the endpoints of T are in $\operatorname{Bd} U$. It is not restrictive to assume that we can apply Theorem 4 to U and define a local flow on U, $\Psi : A \subset \mathbb{R} \times U \to U$, satisfying: (i) $\operatorname{Sing}(\Psi) = \{u\}$; (ii) any nonsingular sub-orbit of Φ in U is a Ψ-orbit; (iii) any branch of T is a Ψ-orbit separating hyperbolic sectors from Ψ.

Apply Theorem 3 to define the $2k$-star (k is a nonnegative integer) R whose vertex is u and whose branches are the Ψ-orbits separating hyperbolic sectors. Now it is clear that T is (possibly strictly) contained in R and then the theorem follows by the relation between Ψ and Φ described in the previous paragraph.

□

The open set V in the previous theorem is said to be a *quasi-hyperbolic sector* of u.

Lemma 6. *Let* Φ *be a topologically transitive analytic flow on a compact surface* S*. Then the set of singular points having more than two quasi-hyperbolic sectors is finite.*

Proof. Let us denote by n_u the number of quasi-hyperbolic sectors from $u \in \operatorname{Sing}(\Phi)$. Assume that the result does not hold, then there is a set of singular points $\{w_m\}_{m \in \mathbb{N}}$ converging to

w such that $n_{w_m} \geq 4$. Since $w \in \mathrm{Sing}(\Phi)$ there is a disk $W \ni w$ and a finite set of arcs, $\{A_i^w = [a_i^w; w]\}_{i=1}^{n_w}$, with the properties stated in Theorem 5.

Recall that any component from $W \setminus \bigcup_{i=1}^{n_w} A_i^w$ is a quasi-hyperbolic sector and thus it does not contain any singular point, then there exists $l \in \mathbb{N}$ so that $w_m \in \bigcup_{i=1}^{n_w} A_i$ if $m \geq l$. For any $m \geq l$ there is a set of arcs $\{A_i^{w_m} = [a_i^{w_m}; w_m]\}_{i=1}^{n_{w_m}}$ as in Theorem 5, two of them are contained in the arc A_j^w to which w_m belongs. Since $n_{w_m} \geq 4$ there exists some $A_p^{w_m}$ included in a quasi hyperbolic sector of w, but either $A_p^{w_m} \subset \mathrm{Sing}(\Phi)$ or for any point $v \in A_p^{w_m}$ we have $\omega_\Phi(v) = \{w_m\}$ or $\alpha_\Phi(v) = \{w_m\}$. A contradiction.

□

Proposition 7. *Let S be a compact surface and Φ a topologically transitive analytic flow on S. Then the set* $\mathrm{Sing}(\Phi)$ *has zero Lebesgue measure.*

Proof. Use that S is compact, Theorem 5 and Lemma 6 to decompose $\mathrm{Sing}(\Phi)$ as the following disjoint union set

$$\{u_i\}_{i=1}^r \cup \{v_i\}_{i=1}^s \cup \bigcup_{i=1}^t A_i,$$

where: (i) for any $1 \leq i \leq r$, u_i is an isolated point of $\mathrm{Sing}(\Phi)$; (ii) for any $1 \leq i \leq s$, the set $\mathrm{Sing}(\Phi)$ is locally an $2n_i$-star in a neighbourhood of v_i, for some integer $n_i \geq 2$; (iii) for any $1 \leq i \leq t$, A_i is either a circle or an arc whose endpoints are in $\{v_i\}_{i=1}^s$. Thus $\mathrm{Sing}(\Phi)$ has zero Lebesgue measure. □

We now introduce a notion related to the flow box one. Let $h_s : (-1,1) \times [0,1) \to S$ (resp. $h : (-1,1) \times (-1,1) \to S$) be an embedding satisfying that $h_s((-1,1) \times \{s\})$ (resp. $h((-1,1) \times \{s\}))$ is a sub-orbit of Φ for every $s \in (0,1)$ (resp. $s \in (-1,1)$), then we call $h_s((-1,1) \times [0,1))$ a *semi-open flow box* (resp. $h((-1,1) \times (-1,1))$ a *flow box*). The set $h_s((-1,1) \times \{0\})$ is the *border* of the semi-open flow box and for any interval $I \subset (0,1)$ (also $J \subset (-1,1)$) the set $h_s(\{0\} \times I)$ (also $h(\{0\} \times J)$) is said to be a *transversal curve* to Φ. A *closed tranversal* to Φ is a smoothly imbedded circle in S and nowhere tangent to Φ.

Lemma 8 (Local structure of ω-limit sets)**.** *Let Φ be an analytic flow on S and let $u \in S$ so that $\omega_\Phi(u)$ does not consist of a single singular point. Then, for every $v \in \omega_\Phi(u)$ there are a disk U neighbouring v and an n-star $R \subset \omega_\Phi(u) \cap U$, $n \geq 2$, with the following properties:*

(1) *v is the vertex of R;*

(2) *R intersects* $\mathrm{Bd}\, U$ *exactly at its endpoints;*

(3) *if O is any of the components of* $\mathrm{Int}(U \backslash R)$*, then either $O \cap \Phi_u(\mathbb{R}) = \emptyset$, or $O \cup B$ is a semi-open flow box with border B the boundary of O in* $\mathrm{Int}\, U$*.*

Proof. See [7, Proof of Lemma 4.6]. □

Proposition 9. *Let Φ be a continuous flow on a compact and connected surface S and let $u \in S$. If $\omega_\Phi(u)$ or $\alpha_\Phi(u)$ contains a periodic orbit then it reduces to this periodic orbit.*

In particular, If Φ is topologically transitive then it does not admit periodic orbits.

Proof. This follows from [5, p. 67, Proposition 7.11]. □

Proposition 10. *Let Φ be a continuous flow on a compact and connected surface S and let $u \in S$ so that $L = \Phi_u(\mathbb{R})$ is nontrivial recurrent. Then every orbit contained in $\overline{L} \backslash cl(L)$ is closed in $S \backslash \mathrm{Sing}(\Phi)$.*

Proof. This result corresponds with [11, Proposition 2.1] for orientable surfaces. It remains valid for non orientable ones by pulling-back the flow to the orientable 2-cover. □

Proposition 11. *Let Φ an analytic flow on a compact and connected surface S, let $u \in S$ so that $L = \Phi_u(\mathbb{R})$ is non trivial recurrent and let $v \in \overline{L} \backslash cl(L)$. Then $\omega_\Phi(v)$ (resp. $\alpha_\Phi(v)$) is a singular point.*

Proof. If $v \in \mathrm{Sing}(\Phi)$ then the result follows trivially. Assume that $v \in S \backslash \mathrm{Sing}(\Phi)$ and write $G = \Phi_v(\mathbb{R})$. Since L is nontrivial recurrent then $\overline{L} = \omega_\Phi(u)$ or $\overline{L} = \alpha_\Phi(u)$, one can suppose for example that $\overline{L} = \omega_\Phi(u)$. So, there exists a closed transversal $\tau_a \subset S \backslash \overline{G}$ meeting L infinitely many times, take $y_n = \Phi(s_n, a) \in \tau_a \cap L$ where $\{s_n\}_{n \in \mathbb{N}} \subset [0, +\infty[$ and $\lim_{n \to +\infty} s_n = +\infty$. Let $x_0 \in G$ and τ_{x_0} be a transversal curve passing by x_0. Since $G \subset \omega_\Phi(u)$, there exist $x_n = \Phi(t_n, a) \in \tau_{x_0} \cap L$ such that $\lim_{n \to +\infty} x_n = x_0$, $\{t_n\}_{n \in \mathbb{N}} \subset [0, +\infty[$ and $\lim_{n \to +\infty} t_n = +\infty$. One can then choose for every $n \in \mathbb{N}$, $s_n < t_n < s_{n+1} < t_{n+1}$. Define $L_n = [y_n; y_{n+1}]_\Phi \subset L$, hence G is adherent to $\bigcup_{n \in \mathbb{N}} L_n$. For every $n \in \mathbb{N}$ we let $\theta_n = L_n \cup [y_n; y_{n+1}]_{\tau_a}$. Then, each θ_n defines a class $[\theta_n]$ in $H_1(S; \mathbb{Z})$. Since the subgroup $\mathcal{H}$ of $H_1(S; \mathbb{Z})$ generated by $\{[\theta_n]\}_{n \in \mathbb{N}}$ is of finite type, there exists $k \in \mathbb{N}$ such that $\{[\theta_1], [\theta_2], \dots, [\theta_k]\}$ generates $\mathcal{H}$.

Now suppose that $\omega_\Phi(v)$ is not reduced to a singular point. Since $\omega_\Phi(v) \subset \mathrm{Sing}(\Phi)$ (by propositions 9 and 10), take $p \in \omega_\Phi(v)$ and apply Lemma 8 to obtain a semi-open flow box

$h : (-1, 1) \times [0, 1) \to S$ where $h(0, 0) = p$, and $h(\{0\} \times (0, 1))$ is a transversal curve meeting G infinitely many times. Hence, there exists a closed transversal τ_1 meeting G and disjoint from $L_1 \cup L_2 \cup \cdots \cup L_k$. Therefore, τ_1 has no intersection with each generator of $\mathcal{H}$; this contradicts the fact that G is adherent to the union of L_n, $n \in \mathbb{N}$. □

3. Main Theorem

We assume that we are in the conditions of Main Theorem.

From Proposition 11, we have:

Corollary 12. *Let $L = \Phi_u(\mathbb{R})$ be a dense orbit, then $S \backslash cl(L)$ is a union of singularities and non periodic orbits whose ω-limit set (resp. α-limit set) is a singular point.*

Proposition 13. *The number of regular Φ-orbits whose ω-limit set (resp. α-limit set) has exactly one point is finite.*

Proof. We must show that the number of orbits whose ω-limit set consists of one point from $\text{Sing}(\Phi)$ is finite. Apply Theorem 5 to decompose $\text{Sing}(\Phi)$ as the disjoint union $A \cup B \cup C$, where (i) A is the set of isolated singular points; (ii) any point from B is not isolated in $\text{Sing}(\Phi)$ and has more than two quasi-hyperbolic sectors; (iii) any point from C is not isolated in $\text{Sing}(\Phi)$ and has exactly two quasi-hyperbolic sectors.

Since S is compact A is finite and by Lemma 6, B is also finite. Then by theorems 3 and 5 only finitely many orbits have exactly one point from $A \cup B$ as ω-limit set.

Let now $u \in C$, let $\{[a_1^u; u], [a_2^u; u]\}$ be the set of arcs from Theorem 5 and let U be the open disk neighbouring u from the same theorem. Then any orbit from $S \backslash ([a_1^u; u] \cup [a_2^u; u])$ does not have $\{u\}$ as ω-limit set. Moreover by theorems 2 and 5, $[a_1^u; u] \cup [a_2^u; u] \subset \text{Sing}(\Phi)$ and then we conclude that $\{u\}$ is not the ω-limit set of any regular Φ-orbit.

Therefore the number of regular Φ-orbits which have one point as ω-limit set is finite. Reasoning analogously we obtain that that the number of regular Φ-orbits having one point as α-limit set is finite and the proposition follows. □

3.1. Proof of Main Theorem. Now we are in position to prove our main result. Let M be a closed invariant set so that $S \neq M$ and let L be a dense orbit of Φ. Then $M \subset S \backslash cl(L)$. Let S_1 be the union set of orbits, $\Phi_u(\mathbb{R})$, so that the cardinal of $\omega_\Phi(u)$ and $\alpha_\Phi(u)$ is equal to 1. By Corollary 12, M is contained in $\text{Sing}(\Phi) \cup S_1$ and then $\mu(M) \leq \mu(\text{Sing}(\Phi) \cup S_1)$.

Finally, by Propositions 7 and 13, $\mu(\mathrm{Sing}(\Phi) \cup S_1) = 0$. Therefore, $\mu(M) = 0$. This means that Φ is a metrically transitive flow.

4. Theorem A

In order to prove Theorem A we introduce a definition and a theorem which characterizes surfaces admitting topologically transitive flows.

Two orientable circles on S are said to be a pair of *crossing circles* if they intersect transversally at exactly one point.

Theorem 14 ([9], Th. A)**.** *Let S be a connected surface. Then the following statements are equivalent:*

- *(i) S admits smooth topologically transitive flows;*
- *(ii) S admits topologically transitive flows;*
- *(iii) S is not homeomorphic to* $\mathbb{S}^2$ (the sphere), $\mathbb{P}^2$ (the projective plane)*, nor to any surface in* $\mathbb{B}^2$ (the Klein bottle)*;*
- *(iv) S contains a pair of crossing circles.*

Proof of Theorem A. Let A and B a pair of crossing circles on S, let D be a disk on $S \backslash (A \cup B)$ and take a compact set $\mathcal{K} \subset D$ homeomorphic to $\mathcal{C} \times [0,1]$ where $\mathcal{C}$ is a Cantor set and $\mu(\mathcal{K}) > 0$.

The surface $T = S \backslash \mathcal{K}$ is connected and contains $A \cup B$, a pair of crossing circles. Then, by the previous theorem, T admits a smooth topologically transitive flow, $\Psi : \mathbb{R} \times T \to T$. Finally we apply [8, Lemma 2.1] to obtain a smooth topologically transitive flow, $\Phi : \mathbb{R} \times S \to S$, so that $\mathcal{K} \subset \mathrm{Sing}(\Phi)$ and the orbits from Ψ coincide with those of Φ contained in $S \backslash \mathcal{K}$. Now it is clear that Φ is topologically transitive but it is not metrically transitive since $\mathcal{K}$ is invariant and its measure is not 0 neither the full measure of S.

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H. Marzougui address: Departament of Mathematics, Faculty of Sciences of Bizerte, Zarzouna 7021, Tunisia

E-mail address: habib.marzouki@fsb.rnu.tn

G. Soler López address: Universidad Politécnica de Cartagena, Departamento de Matemática Aplicada, Paseo Alfonso XIII nº 52, 30203 Cartagena, Spain

E-mail address: gabriel.soler@upct.es