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Research article

Numerical computing approach for solving Hunter-Saxton equation arising in liquid crystal model through sinc collocation method

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ABSTRACT

In this study, numerical treatment of liquid crystal model described through Hunter-Saxton equation (HSE) has been presented by sinc collocation technique through theta weighted scheme due to its enormous applications including, defects, phase diagrams, self-assembly, rheology, phase transitions, interfaces, and integrated biological applications in mesophase materials and processes. Sinc functions provide the procedure for function approximation over all types of domains containing singularities, semi-infinite or infinite domains. Sinc functions have been used to reduce HSE into an algebraic system of equations that makes the solution quite superficial. These algebraic equations have been interpreted as matrices. This projected that sinc collocation technique is considerably efficacious on computational ground for higher accuracy and convergence of numerical solutions. Stability analysis of the proposed technique has ensured the accuracy and reliability of the method, moreover, as the stability parameter satisfied the condition the proposed solution of the problem converges. The solution of the HSE is presented through graphical figures and tables for different cases that are constructed on various values of θ and collocation points. The accuracy and efficiency of the proposed technique is analyzed on the basis of absolute errors.

1. Introduction

Liquid crystals (LCs) is a state between liquid and solid crystals, for instance, electronic displays, cell membranes, proteins, solutions of soap and detergents are well known examples of LCs. The history of LCs begins back in 1888, when an Austrian physiologist Friedrich Reinitzen started examining the properties of cholesterol. There were several speculations about the physico-chemical properties of cholesterol in the start but afterwards that class was called out the cholesteric liquid crystals [1, 2, 3]. However, neumatic is one of the naturally occurring phenomenon which allowed the mathematicians to convert it into a model for future predictions. Therefore, J. K. Hunter and Ralph Saxton derived an asymptotic partial differential equation which governs hyperbolic waves as a model of a weakly nonlinear orientational wave propagation in a massive nematic liquid crystal director field [4]. It is bi-variational and bi-Hamiltonian structure of integrable partial differential equations known as Hunter-Saxton equation (HSE). Some assumptions for HSE were made as follows;

- There will be no fluid flow,
- Orientation of molecules will be focused and shown by director field of neumatic LCs (i.e.) $v(x,t)$,
- No kinetic energy due to higher viscosity of LCs,
- Potential energy will consist upon bend, twist and elastic coefficient of splay,

The authors created a wave-marker situation using the initial boundary value problem (IBVP) of HSE purely for splay waves.

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The general HSE form, reported in [1, 5, 6], is defined as

$$\left(v_t + v v_x\right)_x = \frac{1}{2} v_x^2, \tag{1}$$

with conditions

$$v(x,0) = g(x),\ v(0,t) = 0,\ v(L,t) = 0,\ v_x(L,t) = 0.$$

In Hunter-Saxton equation, various terms presents as follows:

- x is scaled as position that moves along linearized wave velocity $v(x,t)$,
- t as time coordinate,
- When $x = 0$, the wave marker is mounted and at the other end of the channel the reflection of wave is ignored because it was far away from the wave marker.

As over a finite time scale this experiment takes place so at the other end of the channel no wave motion produced. The result of current work does not support the model if any reflection of waves produced backward or by twist. To take account of the twisted splay waves at the HSE approximation level, a director field with rotational inertia is needed as in [7].

$$\begin{cases} \left(u_t + w u_x\right)_x = 0, & -\infty < x < \infty, t = 0, \\ w_{xx} = u_x^2, & w_{xx} > 0, \end{cases} \tag{2}$$

where the second dependent variable $w(x,t)$ in Eq. (2) can be interpreted as the amplitude of splay wave or advection velocity of w, that is generated by twist waves $u(x,t)$.

Hunter and Saxton used variational principles to derive and analyze this equation. They studied the instability in the director field of a nematic liquid crystal. Smooth solutions break down in finite times of asymptotic equation [4]. Jonathan Lenells provided foundation for the geometric study of HS equation, this exhibits a geodesic flow. The system of nonlinear DEs like Hunter-Saxton, Camassa-Holm and Degasperis-Procesi was analyzed by variational principle to find the weak solutions in [8], Volterra-Fredholm integral equations [9], Boussinesq equations [10], Schrödinger equation [11, 12], telegraph PDEs [13, 14], Burgers equation [15]. There are several researches in the literature, that are examined by the different techniques [16, 17, 18, 19, 20]. A local solution of two-component HSE was provided using Kato's local existence theory in [21], general solution is provided through local discontinuous Galerkin (LDG) and new discontinuous Galerkin (DG) method in [22]. Furthermore, iterative methods like, variational iteration method (VIM), modified variational iteration method (VMIM), Adomian decomposition method (ADM), modified Adomian decomposition method (MADM) and Homotopy analysis method (HAM) are described [23]. The reciprocal transformations [24], Homotopy decomposition method [25], bivariate generalized fractional order of the Chebyshev functions (BGFCF) [26], cubic trigonometric B-Spline collocation method [27], collocation method [28], Harr wavelet quasilinearization approach [29], and Lipschitz metric [30], time marching scheme [31] are applied to study the diffusion of neumatic LCs. The generalized Hunter-Saxton equation is considered using integrability structures [32], Numerical solutions of HSE using Laguerre wavelet and by using efficient approach on time domains is presented in [33, 34, 35].

The proper treatments of BCs are one of the basic difficulties in evaluating any numerical scheme, study based on this concept is carried out in [36, 37, 38, 39, 40, 41, 42]. Among the computational methods, one of the important class of methods were spectral methods to solve linear and non-linear partial differential equations, somehow spectral methods have few drawbacks because these methods do not represent the physical process in spectral space. Secondly, these are restricted to those problems which has periodic boundary conditions [43]. Another technique of Lagrange interpolation polynomials (used as test functions) is not so much useful to determine the oscillating solution or problems with unbounded domains. A desirable use of sinc functions can deal with these adversities while keeping the experimental convergence rate. As the derivatives of sinc functions on boundaries or not defined, to overcome this difficulty sinc collocation methods along with finite difference method is used to calculate derivatives near boundaries for solving the Hunter-Saxton equation [44].

From the above reported literature it is noticed that the sinc collocation method with theta weighted scheme not yet used to solve Hunter-Saxton equation arising in liquid crystal model, this motivated the authors to study on the proposed model. In this paper, sinc collocation technique is used to evaluate the time-periodic behavior of the Hunter-Saxton equation, detailed stability analysis of proposed technique is presented after implementation to the problem. In section 1, an introduction is given, section 2 contains the preliminary information about how to apply a sinc function for partial differential equation and to approximate the solution and this method is applied on the proposed problem in section 3. In section 4, complete stability analysis is presented, then necessary and sufficient bounds for θ has been evaluated and confirmed numerically in next section 5 with numerical results and graphical representation of solution. In the last section 5, the conclusion is provided.

2. Sinc bases functions

Generally, Sinc function [45] is defined by

$$sinc(z) = \begin{cases} 1 & z = 0, \\ \frac{\sin(\pi z)}{\pi z} & z \neq 0, \end{cases} \tag{3}$$

for all $z \in \mathbb{C}$.

Assuming the step size (evenly spaced nodes) $h > 0$, the translated sinc function defined and denoted as,

$$S(m,h)(x) = S_m(x) = Sinc\left(\frac{x - mh}{h}\right), \qquad m \in \mathbb{Z} \tag{4}$$

$$sinc(m) = \begin{cases} 1 & x = 0, \\ \frac{\sin\left(\frac{\pi}{h}(x-mh)\right)}{\frac{\pi}{h}(x-mh)} & x \neq mh. \end{cases} \tag{5}$$

If v is a function defined on the real line, then for step-size $h > 0$ the Whittaker cardinal expansion for v is defined by,

$$C(m,h)(x) = \sum_{m=-\infty}^{\infty} v(mh) S(m,h)(x), \tag{6}$$

whenever the series converges, v is approximated by using the finite number of terms in the above equation, where $x_m = mh$ and step size h is given by

$$h = \sqrt{\frac{\pi d}{\alpha N}}, \tag{7}$$

$$0 < \alpha \leq 1, \qquad d \leq \pi,$$

N is suitably chosen, α and d depend upon the asymptotic behavior of v. Sinc approximation can be constructed for semi-infinite, finite or infinite intervals.

Definition. Let us consider D_b be the infinite strip with width $2b$ about the real axis [46] and $b > 0$ is defined as

$$D_b = \{ z \in \mathbb{C} : |\Im(z)| < b \}.$$

Further, for ϵ belongs to unit interval, let $D_b(\epsilon)$ be the rectangle in the complex plane

$$D_b(\epsilon) = \left\{ z \in \mathbb{C} \mid |\Re(z)| < \frac{1}{\epsilon},\ |\Im(z)| < \frac{b}{(1-\epsilon)} \right\}. \tag{8}$$

Let $B(D_b)$ denote the class of functions p that are analytic in D_b, such that

$$\int_{-b}^{b} |p(x+iy)|dy \longrightarrow 0\ ,\ x \longrightarrow \pm\infty,$$

and

$$N\left(p, D_b\right) = \lim_{\epsilon\to 0}\left(\int_{\partial D_b(\epsilon)} |p(z)|^2|dz|\right)^{\frac{1}{2}} < \infty,$$

where ∂D_b describe boundary of D_b.

Lemma 1. *The Hilbert transform [43] of a function v can be approximated by*

$$\mathbb{H}[v](x) = \frac{1}{\pi}P\int_{-\infty}^{\infty} \frac{f(x-y)}{y}dy \approx \sum_{m=-N}^{N} v_m T_m(x), \tag{9}$$

where

$$T_m(x) = \frac{1-\cos\left[\frac{\pi}{h}(x-mh)\right]}{\frac{\pi}{h}(x-mh)}.$$

The entries of Hilbert transforms matrix at collocation points $x_m = mh$ are

$$\mathbb{H} = T_m\left(x_i\right) = \begin{cases} 0 & m=i, \\ \frac{1-(-1)^{i-m}}{\pi(i-m)} & m\neq i. \end{cases} \tag{10}$$

The general formula for nth derivative of function v at collocation points $x_m = mh$ can be approximated by

$$v^n\left(x_m\right) = \sum_{m=-N}^{N} v(mh)\frac{d^n}{dx^n}\left[S_m(x)\right], \tag{11}$$

where S represents the sinc derivative and h represents the step size. Some general notations are,

$$S^{(n)} = S^{(n)}_{mi},\ S_m(x) = [S(m,h)(x)]|_{x=x_i}. \tag{12}$$

For calculation of derivatives, a general formula has been written below to calculate even and odd derivatives [43].

For even,

$$S^{(2s^*)} = \frac{1}{h^{2s^*}}\frac{d^{2s^*}}{dx^{2s^*}}[S(m,h)(x)]|_{x=x_i} = \frac{1}{h^{2s^*}}\begin{cases} (\pi)^{2s^*}\frac{(-1)^{s^*}}{(2s^*+1)}, & m=i, \\ \frac{(-1)^{(i-m)}}{(i-m)^{2s^*}}.\sum_{l=0}^{(s^*-1)}(-1)^{(l+1)}\frac{2s^*!}{(2l+1)!}\pi^{2l}(i-m)^{2l}, & m\neq i. \end{cases} \tag{13}$$

For odd,

$$S^{(2s^*+1)} = \frac{1}{h^{(2s^*+1)}}\frac{d^{(2s^*+1)}}{dx^{(2s^*+1)}}[S(m,h)(x)]|_{x=x_i} = \frac{1}{h^{(2s^*+1)}}\begin{cases} 0, & m=i, \\ \frac{(-1)^{(i-m)}}{(i-m)^{(2s^*+1)}}.\sum_{l=0}^{s^*}(-1)^l\frac{(2s^*+1)!}{(2l+1)!}\pi^{2l}(i-m)^{2l}, & m\neq i. \end{cases} \tag{14}$$

All those derivatives which has been used in this work are written below.

In particular put $s=0$, in Eq. (13) we have 0th Sinc derivative

$$S^{(0)} = [S(m,h)(x)]|_{x=x_i} = \begin{cases} 1, & m=i, \\ 0, & m\neq i. \end{cases} \tag{15}$$

For 1st Sinc derivative, put $s=0$, in Eq. (14) we have.

$$S^{(1)} = \frac{1}{h}\frac{\mathrm{d}}{\mathrm{d}x}[S(m,h)(x)]|_{x=x_i} = \frac{1}{h}\begin{cases} 0, & m=i, \\ \frac{(-1)^{(i-m)}}{(i-m)}, & m\neq i. \end{cases} \tag{16}$$

For 2nd Sinc derivative, put $s=1$, in Eq. (13) we have.

$$S^{(2)} = \frac{1}{h^2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}[S(m,h)(x)]|_{x=x_i} = \frac{1}{h^2}\begin{cases} \frac{-\pi^2}{3}, & m=i, \\ \frac{-2(-1)^{(i-m)}}{(i-m)^2}, & m\neq i. \end{cases} \tag{17}$$

Properties of Sinc collocation method

- The Sinc Collocation method is applicable over all types of domains where other technique fail to handle singularity, this method is applicable on even semi infinite or infinite domain, because it did not provide the infinite domain as infinite.
- The SCM can be successively applied on the higher order partial derivative due to its property of converting higher order derivatives into algebraic equations that are easy to tackle.
- The SCM can be imposed on that problems that have singularities with higher order partial derivative or on boundary, mostly the singularity is removed when this is applied on the problem if not a suitable polynomial is applied to remove the singularity.
- Moreover, The SCM is used extensively not because of expedient reaction against the problem having singularity but also due to interesting convergence.

The graphical abstract of Sinc Collocation methodology and implementation of SCM on Hunter Saxton equation is illustrated in Fig. 1.

3. Solving Hunter-Saxton equation

Consider the Hunter Saxton equation.

$$\left(v_t + vv_x\right)_x = \frac{1}{2}v_x^2, \tag{18}$$

$$v_{xt} + vv_{xx} + v_x^2 - \frac{1}{2}v_x^2 = 0,$$

$$v_{xt} + vv_{xx} + \frac{1}{2}v_x^2 = 0, \tag{19}$$

with initial condition,

$$v(x,0) = 1. \tag{20}$$

Now applying θ-weighted scheme on Eq. (19), where δt represents the time step

$$\begin{aligned} &\left(\frac{v_x^{k+1} - v_x^k}{\delta t}\right) + \theta\left[\left(vv_{xx}\right)^{k+1} + \frac{1}{2}\left(v_x^2\right)^{k+1}\right] \\ &\quad + (1-\theta)\left[\left(vv_{xx}\right)^k + \frac{1}{2}\left(v_x^2\right)^k\right] = 0, \\ &v_x^{k+1} - v_x^k + \theta\delta t\left[\left(vv_{xx}\right)^{k+1} + \frac{1}{2}\left(v_x^2\right)^{k+1}\right] \\ &\quad + (1-\theta)\delta t\left[\left(vv_{xx}\right)^k + \frac{1}{2}\left(v_x^2\right)^k\right] = 0. \end{aligned} \tag{21}$$

Now linearizing some nonlinear terms for this, we write the Taylor expansion of $\left(v_x^{k+1}\right)^2$

$$\begin{aligned} &v_x^{k+1} = v_x\left(t_k + \delta t\right) \\ &v_x^{k+1} = v_x\left(t_k\right) + \delta t v_{xt}\left(t_k\right) + O(\delta t)^2, \\ &v_x^{k+1}v_x^{k+1} = \left(v_x\left(t_k\right) + \delta t v_{xt}\left(t_k\right) + O(\delta t)^2\right) \\ &\qquad \left(v_x\left(t_k\right) + \delta t v_{xt}\left(t_k\right) + O(\delta t)^2\right) \\ &= \left(v_x^k + \delta t\left(\frac{v_x^{k+1} - v_x^k}{\delta t}\right) + O(\delta t)^2\right) \\ &\quad \left(v_x^k + \delta t\left(\frac{v_x^{k+1} - v_x^k}{\delta t}\right) + O(\delta t)^2\right), \end{aligned} \tag{22}$$

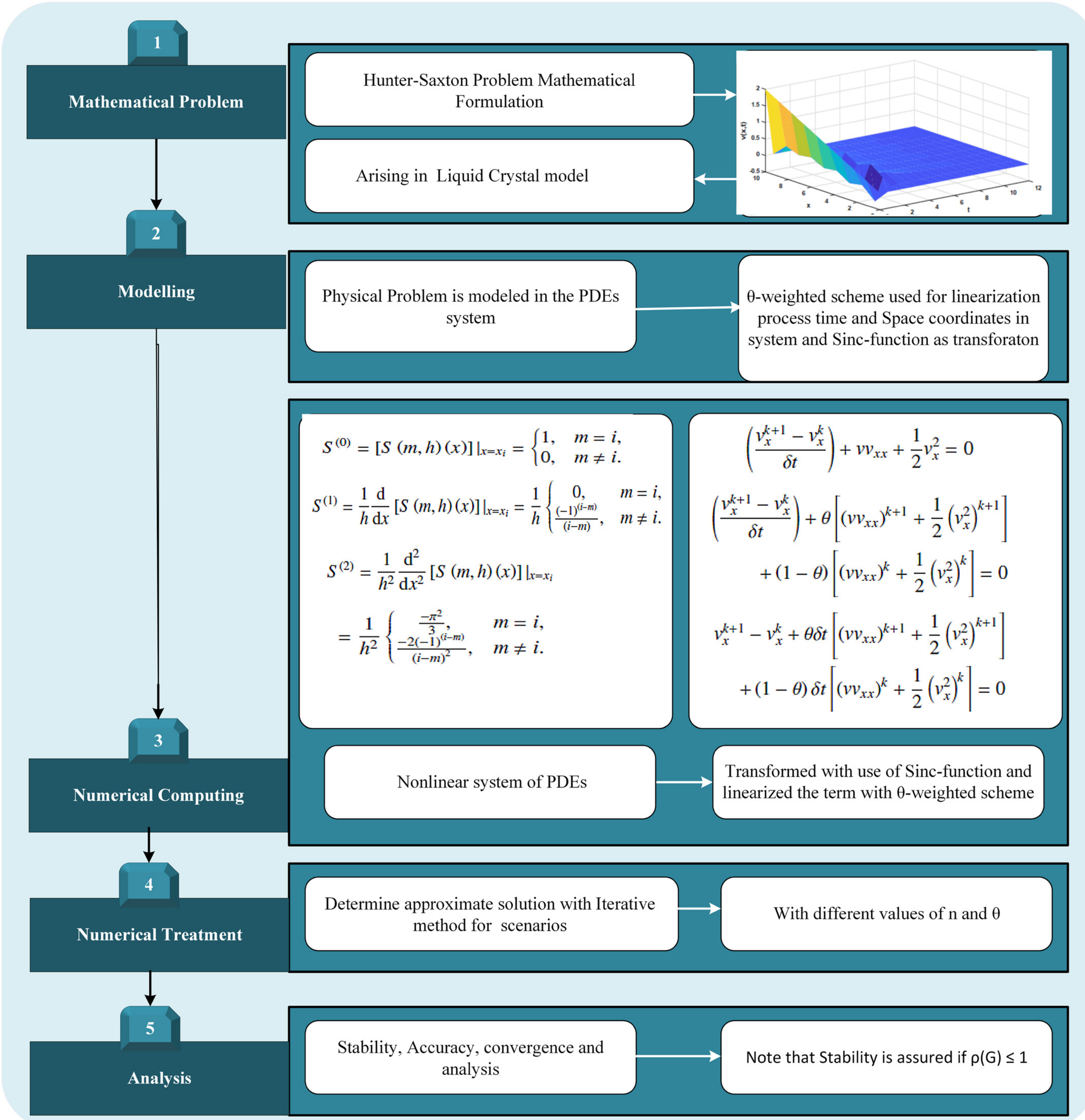

Fig. 1. Graphical Abstract of Sinc collocation method for HSE.

$$v_x^{k+1} v_x^{k+1} \approx 2 v_x^k v_x^{k+1} - v_x^k v_x^k. \tag{23}$$

Also, we need to linearize $(v v_{xx})^{k+1}$ so we write its Taylor expansion.

$$\begin{aligned} v^{k+1} &= v(t_k + \delta t) \\ &= v(t_k) + \delta t v_t(t_k) + O(\delta t)^2, \end{aligned} \tag{24}$$

$$\begin{aligned} v_{xx}^{k+1} &= v_{xx}(t_k + \delta t) \\ &= v_{xx}(t_k) + \delta t v_{xxt}(t_k) + O(\delta t)^2, \end{aligned} \tag{25}$$

$$\begin{aligned} v^{k+1} v_{xx}^{k+1} &= \left(v(t_k) + \delta t v_t(t_k) + O(\delta t)^2\right) \left(v_{xx}(t_k) + \delta t v_{xxt}(t_k) + O(\delta t)^2\right) \\ &= \left(v^k + \delta t\left(\frac{v^{k+1} - v^k}{\delta t}\right) + O(\delta t)^2\right) \left(v_{xx}^k + \delta t\left(\frac{v_{xx}^{k+1} - v_{xx}^k}{\delta t}\right) + O(\delta t)^2\right), \\ v^{k+1} v_{xx}^{k+1} &\approx v^{k+1} v_{xx}^k + v^k v_{xx}^{k+1} - v^k v_{xx}^k. \end{aligned} \tag{26}$$

Substituting values from Eqs. (23), (26) into Eq. (21) we get,

$$\begin{aligned} &v_x^{k+1} - v_x^k + \theta \delta t\left[\left(v^{k+1} v_{xx}^k + v^k v_{xx}^{k+1} - v^k v_{xx}^k\right)\right. \\ &\left.+ \frac{1}{2}\left(2 v_x^k v_x^{k+1} - v_x^k v_x^k\right)\right](1-\theta)\, \delta t\left[v^k v_{xx}^k + \frac{1}{2} v_x^k v_x^k\right] = 0. \end{aligned} \tag{27}$$

After simplification

$$
\begin{aligned}
&v_x^{k+1}-v_x^k+\theta\delta t\left[v^{k+1}v_{xx}^k+v^k v_x^{k+1}+v_x^k v_x^{k+1}\right]\\
&\qquad +(1-2\theta)\,\delta t\left[v^k v_{xx}^k+\frac{1}{2}v_x^k v_x^k\right]=0,\\
&v_x^{k+1}+\theta\delta t\left[v^{k+1}v_{xx}^k+v^k v_x^{k+1}+v_x^k v_x^{k+1}\right]\\
&\quad =v_x^k-(1-2\theta)\,\delta t\left[v^k v_{xx}^k+\frac{1}{2}v_x^k v_x^k\right].
\end{aligned}
\tag{28}
$$

Assume, the solution can be interpolated as

$$
v\left(x_i,t_k\right)\equiv v^k\left(x_i\right)\approx\sum_{m=1}^{N}v_m^k S_m\left(x_i\right).
$$

Now we plug in the assumed solution into above Eq. (28)

$$
\begin{aligned}
&\sum_{m=1}^{N}v_m^{k+1}S_m^{(1)}(x_i)+\theta\delta t\Big[\sum_{m=1}^{N}v_m^{k+1}S_m^{(0)}(x_i)\sum_{n=1}^{N}v_n^k S_m^{(2)}(x_i)\\
&\quad+\sum_{m=1}^{N}v_m^k S_m^{(0)}(x_i)\sum_{n=1}^{N}v_n^{k+1}S_m^{(1)}(x_i)\\
&\quad+\sum_{m=1}^{N}v_m^k S_m^{(1)}(x_i)\sum_{n=1}^{N}v_n^{k+1}S_m^{(1)}(x_i)\Big]\\
&\quad=\sum_{m=1}^{N}v_m^k S_m^1(x_i)-(1-2\theta)\delta t\Big[\sum_{m=1}^{N}v_m^k S_m^{(0)}(x_i)\\
&\quad\sum_{n=1}^{N}v_n^k S_m^{(2)}(x_i)+\frac{1}{2}\sum_{m=1}^{N}v_m^k S_m^{(1)}(x_i)\sum_{n=1}^{N}v_n^k S_m^{(1)}(x_i)\Big].
\end{aligned}
\tag{29}
$$

We write non-linear terms of Eq. (29) as,

$$
\begin{aligned}
N_1&=S^{(2)}v^k * S^{(0)},\\
N_2&=S^{(0)}v^k * S^{(1)},\\
N_3&=S^{(1)}v^k * S^{(1)},
\end{aligned}
\tag{30}
$$

where '$*$' stands for component by component multiplication

$$
\begin{aligned}
&\left[S^{(0)}+\theta\delta t(N_1+N_2+N_3)\right]v^{k+1}=\\
&\left[S^{(1)}-\delta t N_1-\frac{\delta t}{2}N_3+2\theta\delta t N_1+\theta\delta t N_3\right]v^k.
\end{aligned}
\tag{31}
$$

Eq. (29) now becomes the following matrix equation by using Eq. (30) into it

$$
Gv^{k+1}=Hv^k,
\tag{32}
$$

where

$$
G=S^{(0)}+\theta\delta t(N_1+N_2+N_3),
\tag{33}
$$

$$
H=S^{(1)}-\delta t N_1-\frac{\delta t}{2}N_3+2\theta\delta t N_1+\theta\delta t N_3,
\tag{34}
$$

$$
v^{k+1}=I^0 * \left(G^{-1}H\right)v^k.
\tag{35}
$$

4. Stability analysis

In this section stability analysis of Sinc collocation method for solving Hunter Saxton equation (31) has been presented. The error term can be written as

$$
\begin{aligned}
&\left[S^{(0)}+\theta\delta t(N_1+N_2+N_3)\right]e^{k+1}\\
&=\left[S^{(1)}-\delta t N_1-\frac{\delta t}{2}N_3+2\theta\delta t N_1+\theta\delta t N_3\right]e^k,
\end{aligned}
\tag{36}
$$

where $S^{(0)}, S^{(1)}, N_1, N_2$, and N_3 are defined by, error is defined as:

$$
e^k=|v_{exact}^k-v_{approximate}^k|,
\tag{37}
$$

where v_{exact}^k and $v_{approximate}^k$ are exact and approximated solution at time t_k.

Eq. (36) can be written as

$$
e^{k+1}=De^k,
\tag{38}
$$

where

$$
\begin{aligned}
&D\equiv G^{-1}H=[S^{(0)}+\theta\delta t(N_1+N_2+N_3)]^{-1}\\
&\times[S^{(1)}-\delta t N_1-\frac{\delta t}{2}N_3+2\theta\delta t N_1+\theta\delta t N_3].
\end{aligned}
\tag{39}
$$

Sinc collocation method is considered numerically stable if $\rho(D)\le 1$, where $\rho(\cdot)$ denotes the spectral radius. It will be stable if

$$
\left|\frac{\left(S^{(1)}-\delta t N_1-\frac{\delta t}{2}N_3+2\theta\delta t N_1+\theta\delta t N_3\right)}{S^{(0)}+\theta\delta t\left(N_1+N_2+N_3\right)}\right|\le 1.
\tag{40}
$$

Now we have eigenvalues of $S^{(0)}, S^{(1)}, N_1, N_2, N_3$ as $\lambda_0, \lambda_1, \lambda_{N_1}, \lambda_{N_2}$ and λ_{N_3} respectively. From section 2 $S^{(0)}$ is just an identity matrix so $\lambda_0=1$. Furthermore

$$
\left|\frac{\lambda_1-\delta t(\lambda_{N_1}+\frac{1}{2}\lambda_{N_3})+2\theta\delta t(\lambda_{N_1}+\frac{1}{2}\lambda_{N_3})}{1+\theta\delta t(\lambda_{N_1}+\lambda_{N_2}+\lambda_{N_3})}\right|\le 1,
\tag{41}
$$

$$
\left\{S^{(1)}\right\}_{m,i}=\frac{(-1)^{m-i}}{h(m-i)}=-\frac{(-1)^{i-m}}{h(i-m)}=-\left\{S^{(1)}\right\}_{i,m},
\tag{42}
$$

where $\left\{S^{(1)}\right\}_{m,m}=0$. This shows that $S^{(0)}$ is skew-symmetric and have purely imaginary values as $\lambda_1=i\mid\lambda_1\mid$

$$
\begin{aligned}
&\left|\frac{(2\theta-1)\lambda_{N_1}^R+(\theta-0.5)\lambda_{N_3}^R}{1+\theta\left(\lambda_{N_1}^R+\lambda_{N_2}^R+\lambda_{N_3}^R\right)+i\theta\left(\lambda_{N_1}^I+\lambda_{N_2}^I+\lambda_{N_3}^I\right)}\right|+\\
&i\left|\frac{\left(\lambda_1+(2\theta-1)\lambda_{N_1}^I+(\theta-0.5)\lambda_{N_3}^I\right)}{1+\theta\left(\lambda_{N_1}^R+\lambda_{N_2}^R+\lambda_{N_3}^R\right)+i\theta\left(\lambda_{N_1}^I+\lambda_{N_2}^I+\lambda_{N_3}^I\right)}\right|\le 1,
\end{aligned}
\tag{43}
$$

which implies that

$$
\begin{aligned}
&\left|(2\theta-1)\lambda_{N_1}^R+\left(\theta-\frac{1}{2}\right)\lambda_{N_3}^R\right|+\\
&\qquad i\left|\left(\lambda+(2\theta-1)\lambda_{N_1}^I+\left(\theta-\frac{1}{2}\right)\lambda_{N_3}^I\right)\right|\\
&\quad\le\left|1+\theta\left(\lambda_{N_1}^R+\lambda_{N_2}^R+\lambda_{N_3}^R\right)+i\theta\left(\lambda_{N_1}^I+\lambda_{N_2}^I+\lambda_{N_3}^I\right)\right|,
\end{aligned}
\tag{44}
$$

i.e.,

$$
\begin{aligned}
&\left[(2\theta-1)\lambda_{N_1}^R+\left(\theta-\frac{1}{2}\right)\lambda_{N_3}^R\right]^2+\\
&\qquad\left[\lambda+(2\theta-1)\lambda_{N_1}^I+\left(\theta-\frac{1}{2}\right)\lambda_{N_3}^I\right]^2\\
&\quad\le\left[1+\theta\left(\lambda_{N_1}^R+\lambda_{N_2}^R+\lambda_{N_3}^R\right)\right]^2+\left[\theta\left(\lambda_{N_1}^I+\lambda_{N_2}^I+\lambda_{N_3}^I\right)\right]^2,
\end{aligned}
\tag{45}
$$

after algebraic treatment, the above condition becomes

$$
\begin{aligned}
&2(2\theta-1)\left(\theta-\frac{1}{2}\right)\lambda_{N_1}^R\lambda_{N_3}^R+2(2\theta-1)\lambda_1\lambda_{N_1}^I+\\
&\qquad(2\theta-1)^2\lambda_{N_1}^I\lambda_{N_2}^I+(2\theta-1)\lambda_1\lambda_{N_3}^I\\
&\qquad-\theta^2\left(2\lambda_{N_1}^R\lambda_{N_2}^R+2\lambda_{N_2}^R\lambda_{N_3}^R+2\lambda_{N_3}^R\lambda_{N_1}^R\right)-\\
&\qquad 2\theta\left(\lambda_{N_1}^R+\lambda_{N_2}^R+\lambda_{N_3}^R\right)-\\
&\qquad\theta^2\left(2\lambda_{N_1}^I\lambda_{N_2}^I+2\lambda_{N_2}^I\lambda_{N_3}^I+2\lambda_{N_3}^I\lambda_{N_1}^I\right)+|\lambda_1|^2\\
&\quad\le 1+(3\theta+1)(1-\theta)\left|\lambda_{N_1}\right|^2+2\theta^2\left|\lambda_{N_2}\right|^2+\left(\theta-\frac{1}{8}\right)\left|\lambda_{N_3}\right|^2.
\end{aligned}
\tag{46}
$$

Note that stability is assured if $\rho(G)\le 1$. The spectral radius is computed by iteration matrix, for values of theta between 0 and 1. The stability of the method is assured as Fig. 2 shows the condition of theta $0\le\theta\le 1$ is nearly sufficient. This condition must hold for all eigen values of the corresponding matrices for the method to be stable. Notice that if $\frac{2}{5}\le\theta\le 1$, the left hand side may be negative, depending upon the choice of eigen-values, whereas the right hand side of Eq. (46) is non-negative. Which conclude that the condition $\frac{2}{5}\le\theta\le 1$ is necessary but

Table 1. Numerical Solution of HSE by Sinc Collocation method for case 2.

x	t=0	t=2	t=4	t=6	t=8	t=10	t=12
0.000000	0.000000	0.000000	0.000000	4.78E-18	-1.02E-18	2.26E-19	-3.96E-21
1.111111	0.222222	-0.03945	0.012182	0.004022	0.000678	0.000172	4.35E-05
2.222222	0.444444	0.088753	0.036620	0.009826	0.001588	0.000401	0.000101
3.333333	0.666667	0.096572	0.045222	0.012246	0.001982	0.000501	0.000126
4.444444	0.888889	0.197649	0.063955	0.016628	0.002665	0.000672	0.000169
5.555556	1.111111	0.172924	0.063622	0.016836	0.002711	0.000684	0.000172
6.666667	1.333333	0.278434	0.083603	0.021510	0.003439	0.000867	0.000218
7.777778	1.555556	0.208779	0.072398	0.019117	0.003077	0.000777	0.000196
8.888889	1.777778	0.355958	0.102456	0.026248	0.004192	0.001057	0.000266
10.00000	2.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000

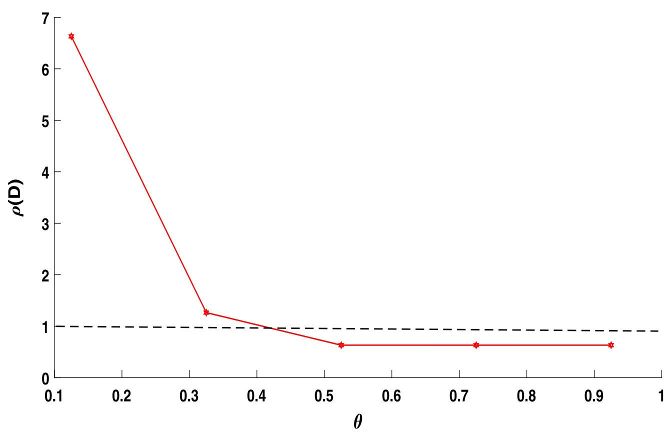

Fig. 2. Illustration of convergence dependence on θ for HSE.

not sufficient, for stability of the Sinc collocation method we insert $\theta = 1$ in Eq. (43) it gives

$$\left| \frac{\lambda^R_{N_1} + \left(\frac{1}{2}\right)\lambda^R_{N_3} + i\left(\lambda_1 + \lambda^I_{N_1} + \left(\frac{1}{2}\right)\lambda^I_{N_3}\right)}{1 + \left(\lambda^R_{N_1} + \lambda^R_{N_2} + \lambda^R_{N_3}\right) + i\left(\lambda^I_{N_1} + \lambda^I_{N_2} + \lambda^I_{N_3}\right)} \right| \le 1 , \tag{47}$$

$$\left|\lambda^R_{N_1} + \frac{1}{2}\lambda^R_{N_3} + i\left(\lambda_1 + \lambda^I_{N_1} + \frac{1}{2}\lambda^I_{N_3}\right)\right| \le \left|1 + \left(\lambda^R_{N_1} + \lambda^R_{N_2} + \lambda^R_{N_3}\right) + i\left(\lambda^I_{N_1} + \lambda^I_{N_2} + \lambda^I_{N_3}\right)\right|, \tag{48}$$

using the triangle inequality, the above equation becomes

$$\left|\lambda^R_{N_1} + \frac{1}{2}\lambda^R_{N_3} + i\left(\lambda_1 + \lambda^I_{N_1} + \frac{1}{2}\lambda^I_{N_3}\right)\right| \le \left|1 + \lambda^R_{N_1} + i\lambda^I_{N_1}\right| + \left|\lambda^R_{N_2} + i\lambda^I_{N_2}\right| + \left|\lambda^R_{N_3} + i\lambda^I_{N_3}\right|,$$

which implies that

$$0 \le \left|1 + \lambda^R_{N_1} + i\lambda^I_{N_1}\right| + \left|\lambda^R_{N_2} + i\lambda^I_{N_2}\right| + \left|\lambda^R_{N_3} + i\lambda^I_{N_3}\right|.$$

Fig. 2 expresses the sufficient condition for $0 \le \theta \le 1$. Eq. (43) is true for all choice of eigenvalues. Thus we obtain $\theta = 1$, is the sufficient condition for stability of the technique. When $\theta = 0$ the technique is restrictively stable with the limit on time step:

$$\delta t \le \frac{1 - \mathrm{Re}^2\left(\lambda_1\right) - \mathrm{Im}^2\left(\lambda_1\right)}{\frac{1}{4}\left(\mathrm{Re}^2\left(\lambda_{N_3}\right) + \mathrm{Im}^2\left(\lambda_{N_3}\right)\right) + \mathrm{Re}\left(\lambda_{N_3}\lambda_{N_1}\right) + \mathrm{Im}\left(\lambda_{N_1}\lambda_1\right)}. \tag{49}$$

5. Solutions for Hunter-Saxton equation

The solution of the HSE is obtained by the SCM with considering different values of θ along with collocation points and number of time steps, where Matlab is used to reduce the lap time. The general steps to obtain the solution of the HSE for all the proposed cases are illustrated in Algorithm 1.

Algorithm 1 The step by step description of the solution obtaining for HSE through Sinc collocation method.

```
procedure 1(Inputs)
    Input1: Enter the collocation points = {x_m}_{m=0}
    Input2: Enter the number of time steps = n_t
    Calculate: δx and δt
procedure 2(Derivatives of Sinc function)
    for i = 1,...,x_m do
        for j = 1,...,x_m do
            if i == j then
                S^(0)(i, j) = 1,
                S^(1)(i, j) = 0,
                S^(2)(i, j) = −π^{2/3}.
            else
                S^(0)(i, j) = 0,
                S^(1)(i, j) = (−1)^{j−i} / (h(j−i)),
                S^(2)(i, j) = −2(−1)^{j−i} / (h^2(j−i)^2).
            end
        end
    end
procedure 3(Iteration)
    for k = 2,...,n_t do
        for i = 1,...,x_m do
            v^{k+1}(i, k) = I^0 * (G^{−1} H) v^k
        end
    end
    Output: Numerical solution of the HSE
```

5.1. Case 1

The numerical solution of Hunter Saxton equation is presented below, using the graphical figures. The graphical figures represent the wave propagation in a massive nematic liquid crystal director field. In Case 1, the solution of HSE is presented. The number of collocation points and iteration numbers are taken 10 and 20, respectively. Fig. 3 represents the numerical solution of HSE for various values of θ, it represents the wave propagation in a massive nematic liquid crystal director field, for four values of θ, taken to be, $\theta = 0.2, 0.4, 0.6$ and $\theta = 0.8$, respectively. Fig. 3 represents the solution of HSE in 3-dimension, elaborating the wave propagation, furthermore, the solution contains the singularities at various points of x, t. The Fig. 3a-b are graphical representation of the $\theta = 0.2, 0.4$ and these are not converges because the stability occurs for $\theta > 0.4$ where other two subfigures illustrates the convergent solution of HSE. Approximation has done by setting $\delta x = L/(nx - 1)$ and $\delta t = Tmax/(nt - 1)$ and $\theta = 1$. Fig. 4 expresses the wave propagation in a massive nematic liquid crystal director field at $\theta = 1$ corresponding to different values of x and t. Fig. 5 represents the solution of HSE at $v(t = 0)$. Fig. 6 expresses the solution of HSE for four different values of x, at $x = 0$, $x = L/4$, $x = L/2$ and $x = L$, respectively. At $x = 0$, the solution of HSE is $v(x, t) = 0$. At $x = L/4$, $x = L/2$, $x = L$, the solution is increasing near the values $t = 0.65$.

5.2. Case 2

In Case 2, the graphical solution of HSE is presented for $\theta = 0.6$, number of collocation points and iteration numbers are taken to be 10 and 20, respectively. The HSE presents the wave propagation in a

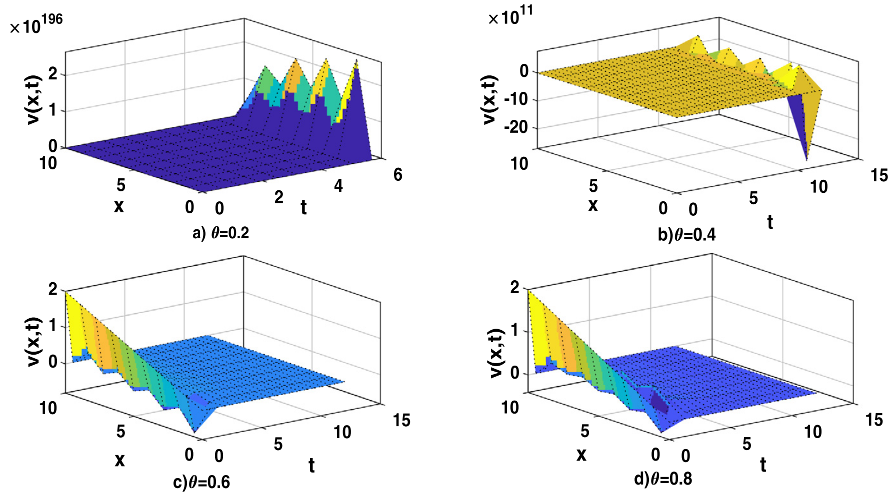

Fig. 3. Case 1 wave propagation of HSE for various values of θ.

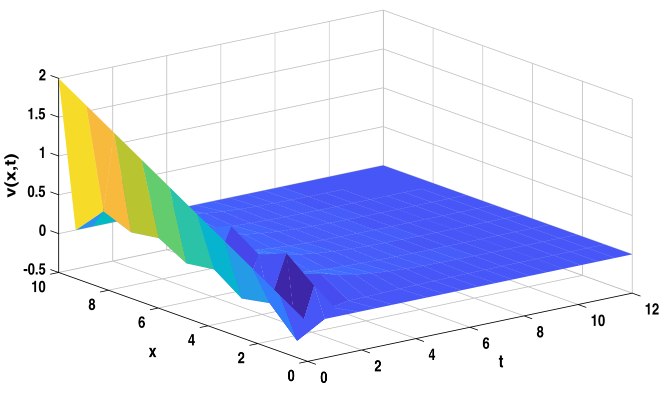

Fig. 4. Wave propagation of HSE for case 1 corresponding to various values of x and t.

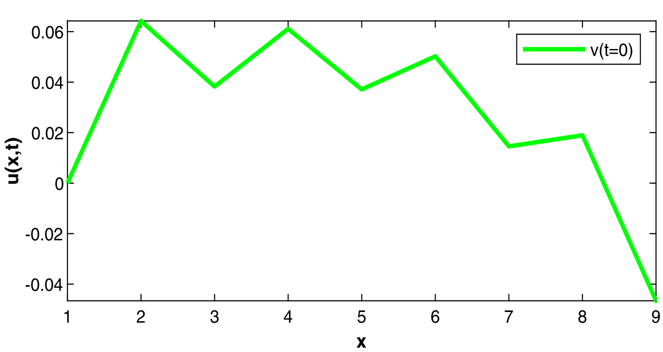

Fig. 5. Wave propagation of HSE for $v(t = 0)$ of case 1.

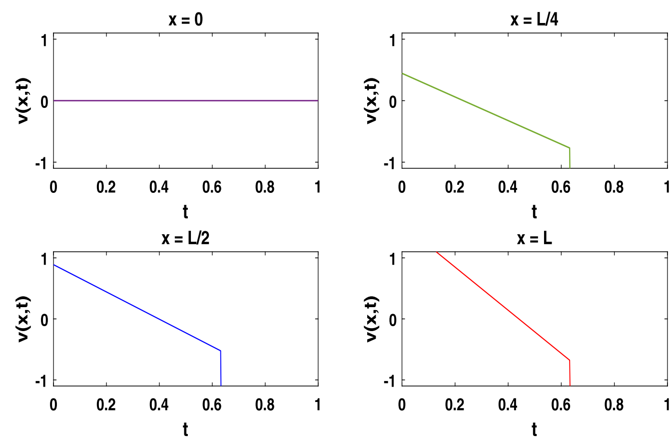

Fig. 6. Solution of HSE for different values of x for case 1.

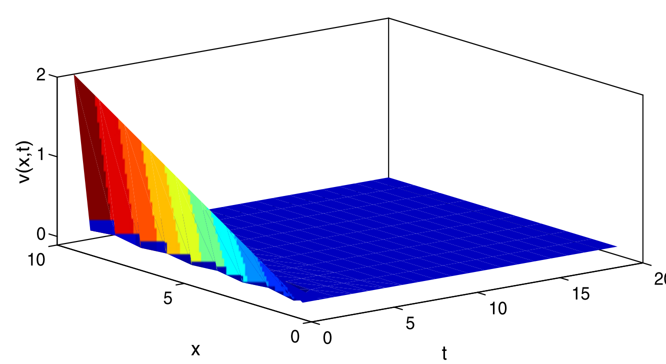

Fig. 7. Wave propagation of HSE for case 2 corresponding to various values of x and t.

massive nematic liquid crystal director field. Fig. 7 expresses the wave propagation in a massive nematic liquid crystal director field at $\theta = 0.6$. Fig. 8 represents the solution of HSE at $v(x = L/2)$. Fig. 9 expresses the solution of HSE for four different values of x, at $x = 0, x = L/4, x = L/2$ and $x = L$, respectively. At $x = 0$, the solution of HSE is $v(x,t) = 0$. At $x = L/4, x = L/2, x = L$, the behavior of the solution of HSE is decreasing. The numerical solution of the HSE for this case is tabulated in Table 1 at different times ranges from 0 to 12 corresponding to varying x values from 0 to 10.

The absolute errors (AEs) of the obtained results with reference solution where, reference solution is obtained through Adomian's numerical method for the accuracy and efficiency analysis of the proposed technique with $\theta = 0.9$. The graphical illustration of AEs is described in the Fig. 10.

Table 2 presents the absolute error of HSE with Adomian's method. A comparison of proposed technique is presented in Table 3 with the existing numerical techniques, Haar wavelet quasi-linearization method [29], a collocation method [52] and Chebyshev functions (B-GFCF) collocation method [53]. It is concluded that our proposed technique provides the better results on the basis of absolute errors.

Table 2. Absolute Error with Adomian's method.

	t				
	0.1	0.2	0.3	0.4	0.5
0.1	3.92E-01	4.18E-01	6.49E-01	7.11E-01	9.27E-01
0.2	1.80E-01	3.05E-01	4.87E-01	6.22E-01	8.01E-01
0.3	9.75E-02	2.42E-01	4.00E-01	5.47E-01	7.03E-01
0.4	4.47E-02	1.86E-01	3.29E-01	4.71E-01	6.14E-01
0.5	8.75E-04	1.32E-01	2.66E-01	3.99E-01	5.32E-01
0.6	4.34E-02	8.15E-02	2.07E-01	3.31E-01	4.57E-01
0.7	8.33E-02	3.43E-02	1.52E-01	2.70E-01	3.87E-01
0.8	1.20E-01	9.31E-03	1.02E-01	2.13E-01	3.24E-01
0.9	1.55E-01	4.93E-02	5.60E-02	1.61E-01	2.67E-01
1	1.86E-01	8.57E-02	1.43E-02	1.14E-01	2.14E-01

Table 3. Comparative study on the basis of max AE.

t	Ref. [29] for N = 256	Ref. [52] for N = 128	Ref. [53] for N = 225	SCM for N = 10	
0.1	4.61 E-06	5.45E-09	7.75E-16	5.86E-18	nt = 11
0.01	7.36 E-09	1.30E-07	3.83E-17	1.76E-19	nt = 101
0.001	1.00E-11		6.05E-19	1.85E-19	nt = 1001

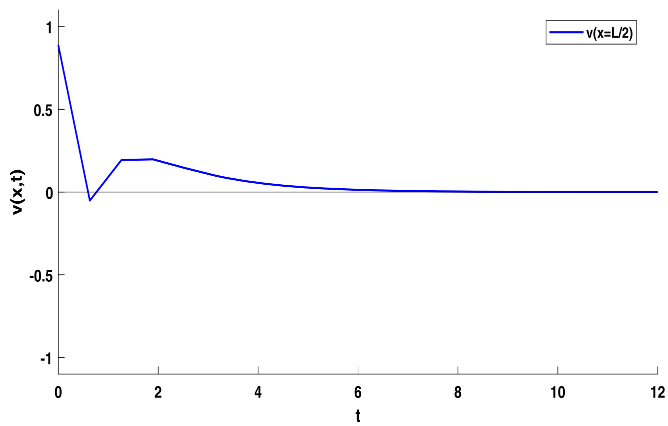

Fig. 8. Case 2 solution of HSE for $x = L/2$.

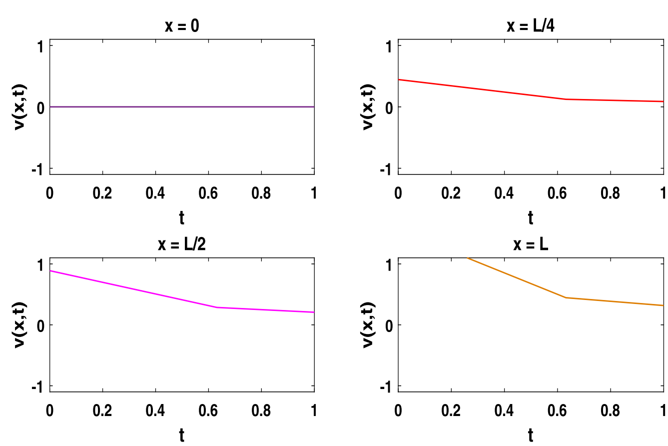

Fig. 9. Solutions of HSE at different values of x for case 2.

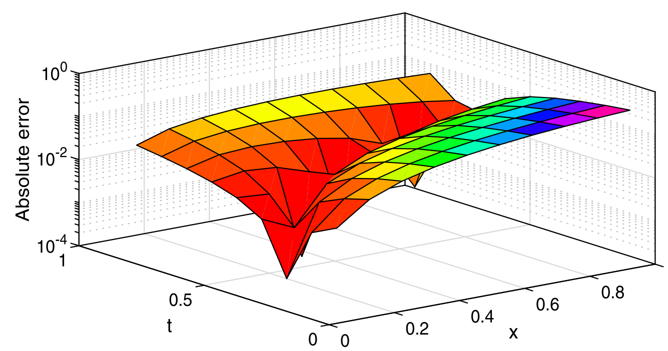

Fig. 10. Absolute errors of the HSE with $\theta = 0.9$.

6. Conclusion

It is concluded, on the basis of the results obtained in the previous section that the proposed technique has been proved to be more reliable, accurate, stable, and attractive. Initially, cardinal expansion of Sinc base functions has been used to approximate the space dimension, thus it allows the avoidance of finite difference grid for 2nd and 3rd order nonlinear partial differential equations. θ-weighted finite difference scheme has been applied to approximate time derivative, which reduces the complexity of equations. The results showed that SCM successfully tackled the proposed problem, in the presence of the singularities. Sinc bases functions have been used to minimize the mathematical calculations. One of the sound advantages has gained by Sinc collocation method is ability to compose quite accurate results with Matlab coding. The convergence of the Sinc collocation method is examined by running the algorithm many times with same collocation points through Mesh points the behavior is determined, most of time the different mash point is provided where in other technique we cannot find the exact behavior between the integer domain but from Sinc collocation method we calculate. The stability of the proposed equation is examined and graphically illustrated in Fig. 2 that the solution is stable for $\theta > 0.4$. The wave propagation varies as the values of θ corresponding to the x and t values. Moreover, the variation of x values also effects the wave propagation values as the x increases the slop of flow increases i-e., they have direct relation for both cases. Similarly, at specific x values with variation in time values the solution of the Hunter Saxton equation through the proposed method decreases.

In Future, such equations which may have infinite domains or difficult to solve by other numerical techniques can be solved by Sinc collocation method, quite successfully, moreover, this technique provides more accurate and efficient solutions for the higher order equations [47, 48], also can applied on the system of equations [49, 50, 51].

Declarations

Author contribution statement

I. Ahmad, H. Ilyas, K. Kutlu, V. Anam, S. I. Hussain, J. L. C. Guirao: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

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Data availability statement

No data was used for the research described in the article.

Declaration of interests statement

The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

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