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Solving a novel designed second order nonlinear Lane–Emden delay differential model using the heuristic techniques

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ABSTRACT

The aim of the present study is to present a new model based on the nonlinear singular second order delay differential equation of Lane–Emden type and numerically solved by using the heuristic technique. Four different examples are presented based on the designed model and numerically solved by using artificial neural networks optimized by the global search, local search methods and their hybrid combinations, respectively, named as genetic algorithm (GA), sequential quadratic programming (SQP) and GA-SQP. The numerical results of the designed model are compared for the proposed heuristic technique with the exact/explicit results that demonstrate the performance and correctness. Moreover, statistical investigations/assessments are presented for the accuracy and performance of the designed model implemented with heuristic methodology.

1. Introduction

Delay differential (DD) equation is historical and has seen in the work of Leibnitz and Newton at the end of the 16th century. DD equations have extensive applications in broad domains [1–4]. The DD equation have been widely studied by many researchers. Kondorse studied DD equations in 1777 but proper applications of DD equations have been used in the mid of the 19th century. To mention a few of them are Kuang [5] and Lasalle [6] proposed the theory, solution methods and applications of DD types of equations. Perko [7] discussed both types of equations based on linear/nonlinear differential models for dynamical structure. Forde [8] discussed DD equation type of models in the studies of mathematical biology. Beretta and Kuang [9] worked on the geometric stability of DD equations with the parameters of delay dependent and others Frazier [10], Rangkuti and Noorani [11] and Chapra [12]. The simple mathematical notation of DD equation is written as [13]:

$$\frac{du}{dx} = h\left(t, u(x-\tau)\right). \tag{1}$$

where τ represents the delayed term in Eq. (1).

The historical Lane–Emden (LE) equation designates a numerous phenomenon in gas clouds, polytropic stars, spherical cloud of gas, cooling of radiators and in the modeling of cluster galaxies. Due to the singularity of LE equation, it has achieved the attention of many researchers and has various applications in dusty fluid models [14], physical science models [15], catalytic diffusion reactions [16] and other include density profile of gaseous star, isothermal gas spheres, the theory of electromagnetic, quantum mechanics, astrophysics, oscillating magnetic field, stellar structure morphogenesis and isotropic continuous media [17–26]. It is always very difficult, tough and challengeable to tackle the singular models due to singularity. A limited number of existing methods are available to handle these types of singular models [27–32]. The mathematical formulation of the second order LE equation is written as [33–37]:

$$\begin{aligned} &\frac{d^2u}{dx^2} + \frac{\rho}{x}\frac{du}{dx} + g(u) = 0, \\ &u(0) = \alpha, \frac{du(0)}{dx} = 0. \end{aligned} \tag{2}$$

where $\rho \geq 1$ denotes the shape factor.

The objective/aim of the study is to present a novel model for nonlinear DD-LE equations and solved by stochastic numerical solver by using neural networks optimized through genetic algorithms and sequential quadratic programming.

The paper organized as: Section 2 for derivation of the proposed nonlinear DD equation of Lane–Emden type, Section 3

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for proposed methodology, Section 4 for outcomes of numerical experimentations with descriptive analysis and Conclusions is given in the last section.

2. Construction of second order nonlinear DD equation of Lane-Emden type

In this section of the study, the construction of second order nonlinear DD-LE equation will be discussed. By using the sense of standard LE and delay differential equations, the model along with its initial condition is achieved. The mathematical design of the nonlinear DD-LE model is written as:

$$x^{-\eta}\frac{d^n}{dx^n}\left(x^{\eta}\frac{d^m}{dt^m}u(x-\tau)\right)+g(u)=0, \tag{3}$$

where η represents the positive real number. To model the second order nonlinear DD-LE equation, fix the values of m and n as:

$$m+n=2, \qquad m,n\geq 1. \tag{4}$$

The Eq. (3) takes the form as:

$$\begin{aligned}&\frac{d^2}{dx^2}u(x-\tau)+\frac{\rho}{x}\frac{d}{dx}u(x-\tau)+g(u)=0,\\ &u(0)=\alpha,\ \frac{du(0)}{dx}=0.\end{aligned} \tag{5}$$

The singularity in the above equation appears single time at point $x=0$ with shape factor ρ. The above model clearly represents the DD equation of LE type.

3. Methodology and numerical experimentations

In this section, four different problems of the designed model have been and numerical solutions have been presented by applying the artificial neural networks (ANNs), optimized with genetic algorithm (GA), sequential quadratic programming (SQP) scheme and the hybrid of GA-SQP scheme. The details of this technique are extensively practice in the literature; see, for example, [38–55]. The feed-forward ANN models for proposed outcomes and their respective derivatives are mathematically given as:

$$\hat{u}=\sum_{i=1}^{n}\alpha_i f(w_i x+\beta_i), \tag{6}$$

$$\frac{d^2\hat{u}}{dx^2}=\sum_{i=1}^{n}a_i\frac{d^2}{dx^2}f(w_i x+\beta_i), \tag{7}$$

where a_i, w_i and β_i are the ith components of a, w and β vectors. The log-sigmoid function i.e., $f(x)=(1+\exp(-x))^{-1}$ and its second order derivative is used as an activation function. The simplified form of the Eqs. (6)–(7) is shown as:

$$\hat{u}=\sum_{i=1}^{n}\alpha_i\left(1+\exp\left(-(w_i x+\beta)\right)\right)^{-1}, \tag{8}$$

$$\frac{d^2\hat{u}}{dx^2}=\sum_{i=1}^{n}\alpha_i\frac{d^2}{dx^2}\left(1+\exp\left(-(w_i x+\beta)\right)\right)^{-1}. \tag{9}$$

The fitness formulation becomes as:

$$\varepsilon=\varepsilon_1+\varepsilon_2. \tag{10}$$

where ε_1 and ε_2 are the error based functions related to differential model and initial conditions.

$$\varepsilon=\frac{1}{N}\sum_{i=1}^{n}\left(\frac{d^2}{dx^2}\hat{u}(x_i-\tau)+\frac{\rho}{x_i}\frac{d}{dx}\hat{u}(x_i-\tau)+g_i\right)^2 + \frac{1}{2}\left(\left(\hat{u}_0-\alpha\right)^2+\left(\frac{d\hat{u}_0}{dx}\right)^2\right) \tag{11}$$

where $N=\frac{1}{h}$, $g_i=g(x_i)$, $\hat{u}=u(x_i)$, $x_m=mh$.

3.1. Optimization

In this subsection, the detail of GA and SQP is provided that is used to optimize the nonlinear DD-LE model.

Genetic Algorithm: It is a universal search scheme presented by Holland and widely used by numerous researchers to produce consistent, feasible, well-organized and accurate, consistent and well-organized outcomes of optimization models [56–58]. GA belongs to a computational scheme based on evolutionary intelligence and works as a natural genetic process. In the process of adaptation, the objective function is applied through its dynamic operators named as crossover; mutation; and selection operators. Some recent applications are a simulation of protein-folding [59], engineering electromagnetics [60], optimization of the design of green building [61], prediction of stock price [62], metrics and chemistry [63], selection of feature subset [64], modeling of bankruptcy prediction [65], planning of berth allocation [66] and schedule of multiprocessors [67].

Sequential Quadratic Programming: It is a local search scheme and applied to various optimization sub-problems. Some recent applications of SQP are produced of multiproduct [68], economically load dispatch model [69], dynamical walking robot [70], deformation in the blood vessels [71], a temporary hydrothermal organization [72], thermal cycling blow mold [73], easy LNG procedure [74], structures of wind turbine [75], vector recovery of flight for large aircraft transport [76], power flow of optimal problems [77] and applied for solving the convex quadratic bi-level programming models [78].

3.2. Performance indices

The statistical performance is based on three operators named as MAD, ENSE and TIC and their performance statistical based operators G.MAD, G.ENSE and G.TIC are used to check the accuracy and performance of the designed scheme. The workflow diagram of the designed methodology is plotted in Fig. 1 and the mathematical form of these operators is described as:

$$\text{MAD}=\frac{1}{n}\sum_{i=1}^{n}\left|u_i-\hat{u}_i\right|, \tag{12}$$

$$\text{TIC}=\frac{\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(u_i-\hat{u}_i\right)^2}}{\left(\sqrt{\frac{1}{n}\sum_{m=1}^{n}u_i^2}+\sqrt{\frac{1}{n}\sum_{m=1}^{n}\hat{u}_i^2}\right)} \tag{13}$$

$$\text{NSE}=\left\{1-\frac{\sum_{i=1}^{n}\left(u_i-\hat{u}_i\right)^2}{\sum_{i=1}^{n}\left(u_i-\bar{u}_i\right)^2},\ \bar{u}_i=\frac{1}{n}\sum_{i=1}^{n}u_i\right. \tag{14}$$

$$\text{ENSE}=1-\text{NSE} \tag{15}$$

4. Simulations and results

The numerical surveys for solving the DD-LE model using the ANN optimized with GA-SQP are provided in this section. Four different examples are discussed that represents the designed model as well as the numerical results.

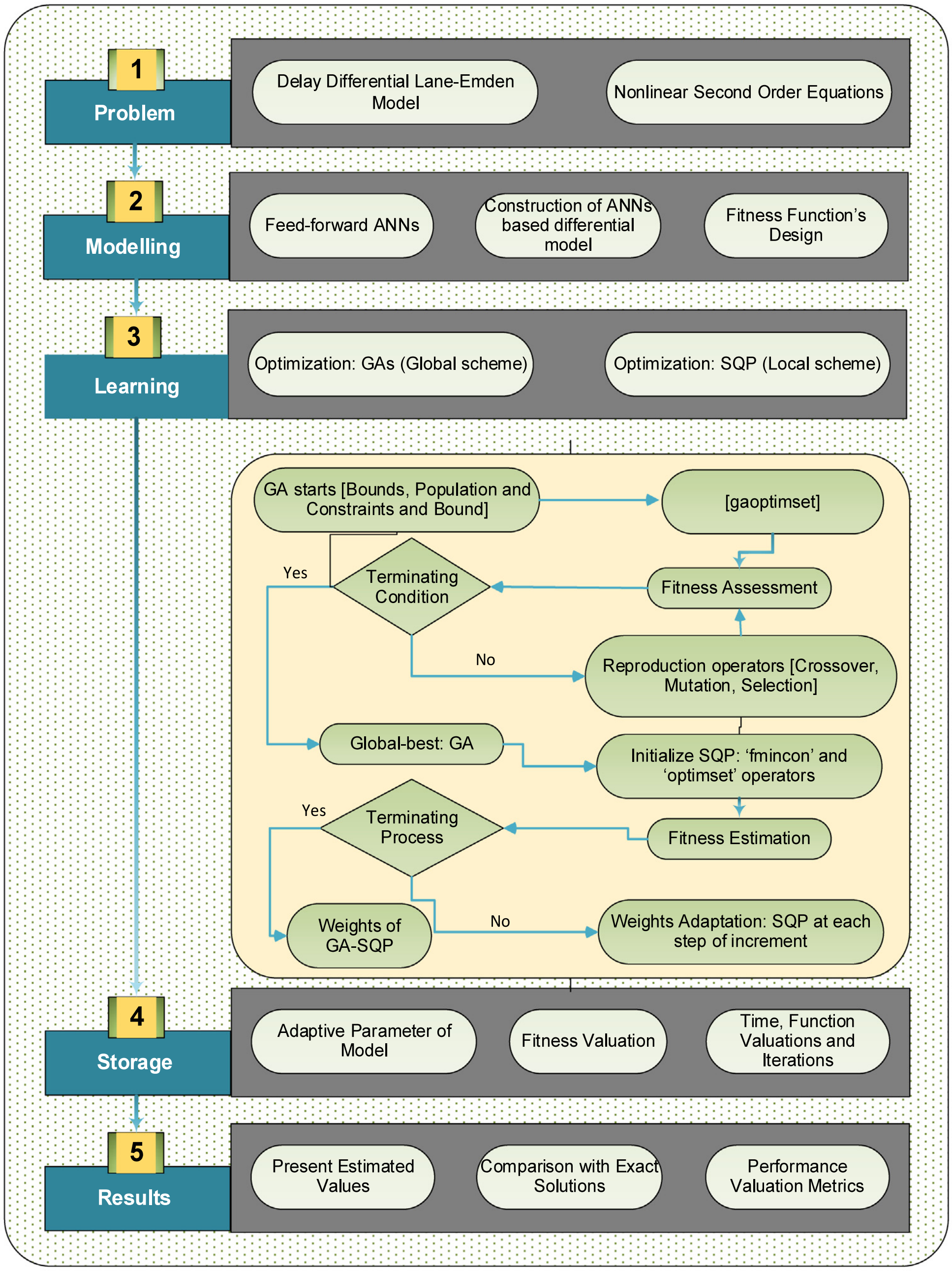


Fig. 1. Workflow diagram of the designed methodology.

Example 1. Consider the following singular, highly nonlinear delay differential model of Lane–Emden type (5) involving trigonometric functions with parameters $\tau = 1$, $\rho = 3$ and the function $g(u) = u^{-3} + \cos(1-x) - \sec^3 x - 3x^{-1}\sin(1-x)$, as:

$$\begin{cases} \dfrac{d^2u(x-1)}{dx^2} + \dfrac{3}{x}\dfrac{du(x-1)}{dx} + u^{-3} \\ \quad = -\cos(1-x) + \sec^3 x + \frac{3}{x}\sin(1-x), \\ u(0) = 1,\ \dfrac{du(0)}{dx} = 0. \end{cases} \tag{16}$$

The exact solution $u(x)$ of Eq. (16) is $u(x) = \cos x$. The inputs x are taken between 0 and 1 for numerical solution of said example.

The error based fitness function using Eqs. (6)–(11) is given as follows:

$$\varepsilon = \frac{1}{N}\sum_{i=1}^{N}\left(\begin{array}{l} x_i \dfrac{d^2\hat{u}(x_i-1)}{dx^2} + 3\dfrac{d\hat{u}(x_i-1)}{dx} + x_i\hat{u}^{-3} \\ + x_i\cos(1-x_i) - x_i\sec^3 x_i - \dfrac{3}{x_i}\sin(1-x_i) \end{array} \right)^2 + \frac{1}{2}\left((\hat{u}_0 - 1)^2 + \left(\frac{d\hat{u}_0}{dx}\right)^2 \right). \tag{17}$$

where the $\hat{u}(x)$ is approximate numerical solution of exact solution $u(x)$ of Eq. (16).

Example 2. Consider the nonlinear delay differential model of LE type as represented in (5) having a trigonometric functions with parameters $\tau = 1$, $\rho = 3$ and the function $g(u) = u^2 - \sec^3(1 -$

$x) - \sec^2 x + x^{-1} 3\sec(1-x)\tan(1-x) - \sec x(1-x)\tan^2(1-x)$, as:

$$\begin{cases} \dfrac{d^2u(x-1)}{dx^2} + \dfrac{3}{x}\dfrac{du(x-1)}{dx} + u^2 = \sec^3(1-x) + \sec^2 x \\ -\dfrac{3\sec(1-x)\tan(1-x)}{x} + \sec x(1-x)\tan^2(1-x), \\ u(0) = 1,\ \dfrac{du(0)}{dx} = 0. \end{cases} \tag{18}$$

The exact solution $u(x)$ of Eq. (18) is $u(x) = \sec x$, and The inputs x are taken between 0 and 1 for numerical solution of said example, while the error based fitness function using Eqs. (6)–(11) is formulated as given below:

$$\varepsilon = \frac{1}{N}\sum_{i=1}^{N}\begin{pmatrix} x_i\dfrac{d^2\hat{u}(x_i-1)}{dx^2} + 3\dfrac{d\hat{u}(x_i-1)}{dx} + x_i\hat{u}^2 \\ -x_i\sec^3(1-x_i) - x_i\sec^2(x_i) \\ +3\sec(1-x_i)\tan(1-x_i) \\ -x_i\sec(1-x_i)\tan^2(1-x_i) \end{pmatrix}^2 + \frac{1}{2}\left((\hat{u}_0-1)^2 + \left(\frac{d\hat{u}_0}{dx}\right)^2\right). \tag{19}$$

where the $\hat{u}(x)$ is approximate numerical solution of exact solution $u(x)$ of Eq. (18).

Example 3. Consider the nonlinear differential model of LE type as represented in (5) having exponential functions with parameters $\tau = 1$, $\rho = 3$ and the function $g(u) = e^u - 88 - e^{1+x^2-x^5} + 21x^{-1} + 150x - 120x^2 + 35x^3$, as:

$$\begin{cases} \dfrac{d^2u(x-1)}{dx^2} + \dfrac{3}{x}\dfrac{du(x-1)}{dx} + e^u \\ = 88 + e^{1+x^2-x^5} - \dfrac{21}{x} - 150x + 120x^2 - 35x^3 \\ u(0) = 1,\ \dfrac{du(0)}{dx} = 0. \end{cases} \tag{20}$$

The exact solution of Eq. (20) is $u(x) = 1 + x^2 - x^5$ and the inputs x are taken between 0 and 1 for numerical solution of said example, while the error based fitness function is given as:

$$\varepsilon = \frac{1}{N}\sum_{i=1}^{N}\begin{pmatrix} x_i\dfrac{d^2\hat{u}(x_i-1)}{dx^2} + 3\dfrac{d\hat{u}(x_i-1)}{dx} + x_ie^{\hat{u}} - 88x_i \\ -x_ie^{1+x_i^2-x_i^5} + 21 + 150x_i^2 - 120x_i^3 + 35x_i^4 \end{pmatrix}^2 + \frac{1}{2}\left((\hat{u}_0-1)^2 + \left(\frac{d\hat{u}_0}{dx}\right)^2\right). \tag{21}$$

where the $\hat{u}(x)$ is approximate numerical solution of exact solution $u(x)$ of Eq. (20).

Example 4. Consider the nonlinear DD model of LE type (5) having strong forcing function with parameters $\tau = 1$, $\rho = 3$ and the function $g(u) = u^4 - \frac{9}{x} + 23 - 15x - 4x^3 - 6x^6 - 4x^9 - x^{12}$, as

$$\begin{cases} \dfrac{d^2u(x-1)}{dx^2} + \dfrac{3}{x}\dfrac{du(x-1)}{dx} + u^4 \\ = \dfrac{9}{x} - 23 + 15x + 4x^3 + 6x^6 + 4x^9 + x^{12} \\ u(0) = 1,\ \dfrac{du(0)}{dx} = 0. \end{cases} \tag{22}$$

The exact solution of Eq. (22) is $u(x) = 1 + x^2 - x^5$ and the inputs x are taken between 0 and 1 for numerical solution of said example. The error based fitness function using (6)–(11) is given as follows:

$$\varepsilon = \frac{1}{N}\sum_{i=1}^{N}\begin{pmatrix} x_i\dfrac{d^2\hat{u}(x_i-1)}{dx^2} + 3\dfrac{d\hat{u}(x_i-1)}{dx} + x_ie^{\hat{u}} - 9 \\ +23x_i - 15x_i^2 - 4x^4 - 6x^7 - 4x^{10} - x^{13} \end{pmatrix}^2 + \frac{1}{2}\left((\hat{u}_0-1)^2 + \left(\frac{d\hat{u}_0}{dx}\right)^2\right), \tag{23}$$

where the $\hat{u}(x)$ is approximate numerical solution of exact solution $u(x)$ of Eq. (22).

Example 5. Consider the nonlinear differential equation model of LE type (5) having hyperbolic function with parameters $\tau = 1$, $\rho = 3$ and the function $g(u) = u^2 - \cosh(x-1) - 3x^{-1}\sinh(x-1) - \cosh^2 x$ as:

$$\begin{cases} \dfrac{d^2u(x-1)}{dx^2} + \dfrac{3}{x}\dfrac{du(x-1)}{dx} + u^2 \\ = \cosh(x-1) + \dfrac{3}{x}\sinh(x-1) + \cosh^2 x \\ u(0) = 1,\ \dfrac{du(0)}{dx} = 0. \end{cases} \tag{24}$$

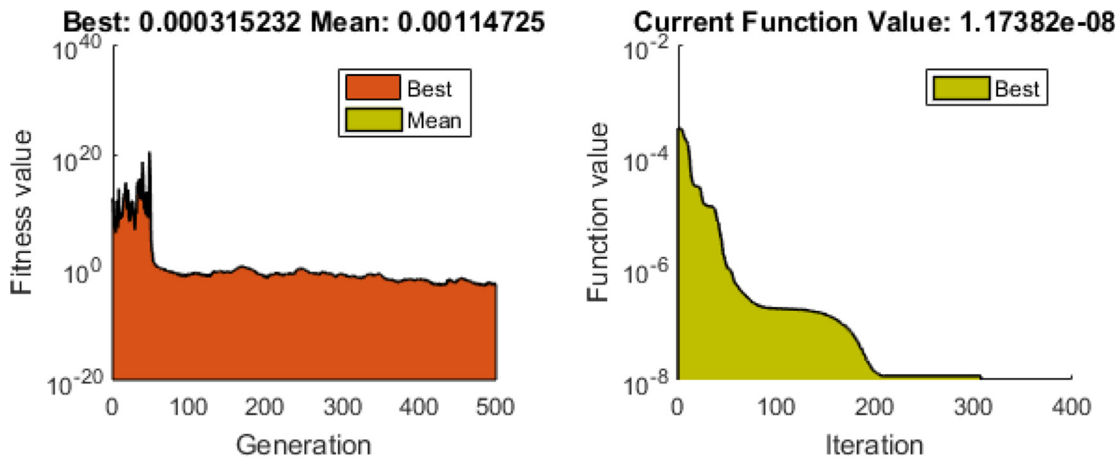
The exact/explicit solution of Eq. (24) is unknown and the inputs x are taken between 0 and 1 for numerical solution of said example, while the error based fitness function is using (6)–(11) for (24) is given as:

$$\varepsilon = \frac{1}{N}\sum_{i=1}^{N}\begin{pmatrix} x_i\dfrac{d^2\hat{u}(x_i-1)}{dx^2} + 3\dfrac{d\hat{u}(x_i-1)}{dx} + x_i\hat{u}^2 - x_i\cosh^2(x_i) \\ -x_i\cosh(x_i-1) - 3\sin\mathrm{h}(x_i-1) \end{pmatrix}^2 + \frac{1}{2}\left((\hat{u}_0-1)^2 + \left(\frac{d\hat{u}_0}{dx}\right)^2\right). \tag{25}$$

where the $\hat{u}(x)$ is approximate numerical solution of exact solution $u(x)$ of Eq. (24).

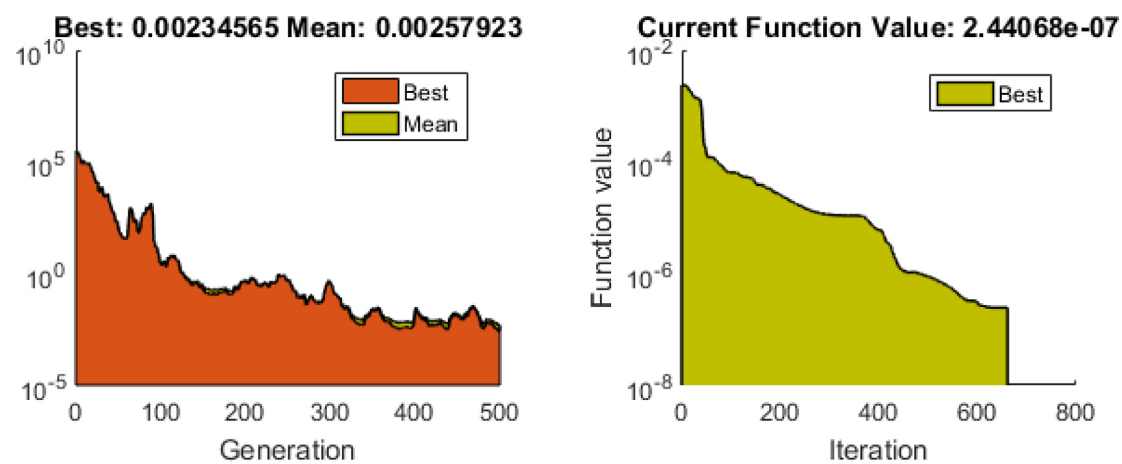
The fitness function in Eqs. (17), (19), (21), (23) and (25) are optimization with hybrid computing of GA-SQP for Examples 1, 2, 3, 4 and 5, respectively, while the convergence or learning curves, iterative update of the objective functions, are plotted in Figs. 2–6 for Examples 1–5, respectively. One may observed that the initially GAs performed speedy optimization of fitness function but after consuming iteration its ability of convergence decrease which is further enhance by the process of hybridization with SQP. So the GA-SQP methodology provided consistent convergent solution for all five case studies. Additionally, one may observed the perform of the stochastic methodology is consistent for LE equation with known and unknown exact solutions.

The graphical representations of the results for Examples 1–4 are plotted in Figs. 7 to 9 using 30 variables based on the hybrid combination of GA-SQP optimization methodology. Fig. 7 represents the set of best weights using 10 neurons and comparison of the results of the exact and proposed results. The first four subfigures a to d show the set of best weights which represent the proposed solution using 30 variables in the form of a_i, w_i and β_i. The rest of the subfigures e to h represent the comparison of exact solutions and proposed best solution for all examples. The overlapping of both plotted results indicate the exactness and correctness of the designed model as well as proposed scheme. Fig. 8 represent the absolute error (AE) and performance indices of the hybrid approach of GA-SQP for Examples 1–4. The best, worst and mean solution have been plotted in the subfigures for AE and performance indices. It is clear that the best values of AE lie in the range of 10^{-06} to 10^{-07}, 10^{-05} to 10^{-06}, 10^{-03} to 10^{-05} and 10^{-04} to 10^{-06} for Examples 1, 2, 3 and 4, respectively, while



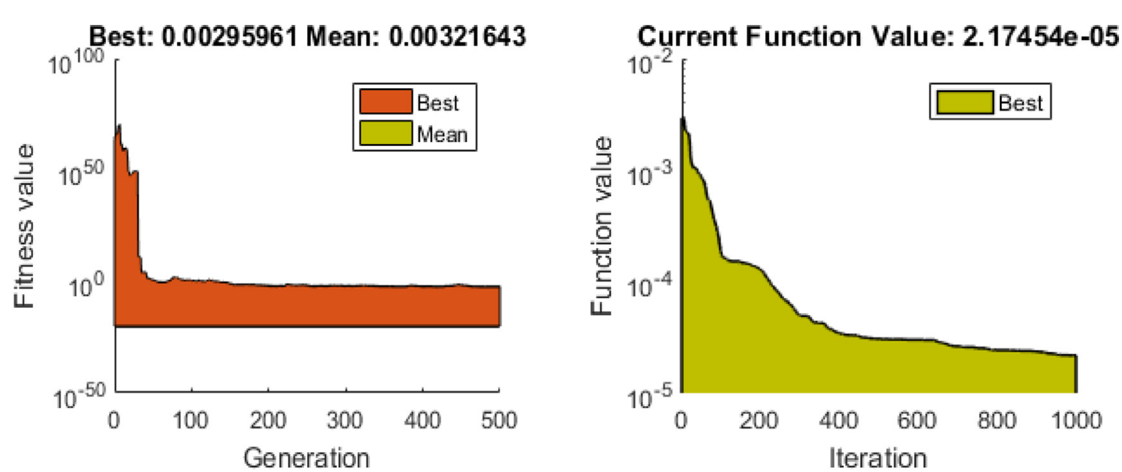
(a) Convergence or learning curve of GAs (b) Convergence or learning curve of GA-SQP

Fig. 2. Convergence of optimization mechanism for Example 1.



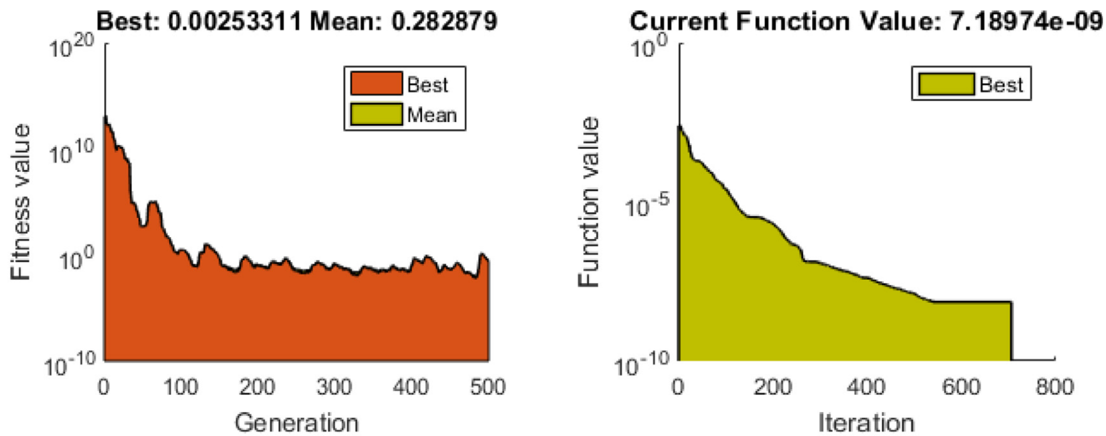
(a) Convergence or learning curve of GAs (b) Convergence or learning curve of GA-SQP

Fig. 3. Convergence of optimization mechanism for Example 2.



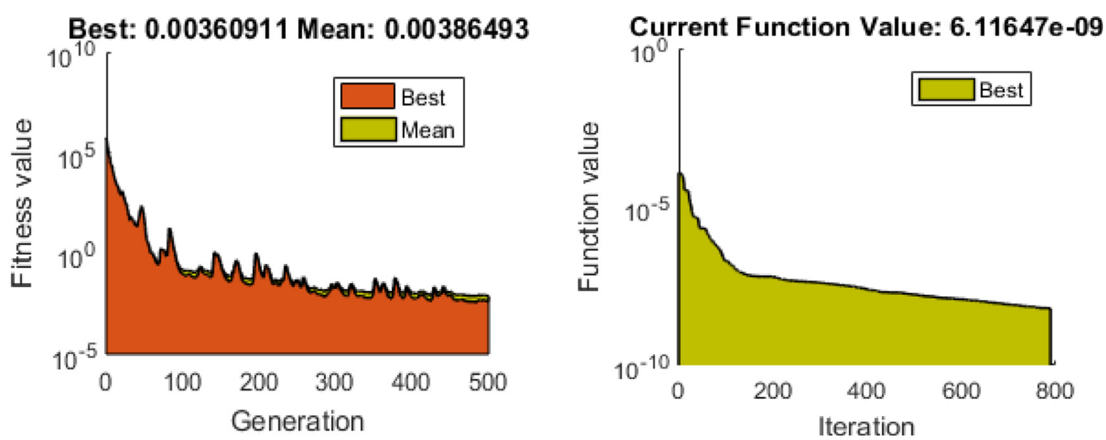
(a) Convergence or learning curve of GAs (b) Convergence or learning curve of GA-SQP

Fig. 4. Convergence of optimization mechanism for Example 3.



(a) Convergence or learning curve of GAs (b) Convergence or learning curve of GA-SQP

Fig. 5. Convergence of optimization mechanism for Example 4.



(a) Convergence or learning curve of GAs (b) Convergence or learning curve of GA-SQP

Fig. 6. Convergence of optimization mechanism for Example 5.

the residual error, i.e., ability of proposed solutions to how closely satisfy the differential Eq. (22), is calculated for Example 5 and it values varies between 10^{-04} to 10^{-06}. The mean values of AE lie around 10^{-03} to 10^{-04}, 10^{-02} to 10^{-03}, 10^{-01} to 10^{-03} and 10^{-03} to 10^{-04} for Examples 1 to 4. However, the worst values show also very good result and found in very good agreements. The rest of the subfigures show the performance analysis of statistical operators MAD, TIC and ENSE. The best values of MAD lie around 10^{-06} to 10^{-08}, 10^{-04} to 10^{-06}, 10^{-02} to 10^{-04} and 10^{-04} to 10^{-06} for Examples 1, 2, 3 and 4, respectively. The mean values of MAD for all examples lie around 10^{-02} to 10^{-04}. The best values of TIC for Examples 1, 2 and 4 lie around 10^{-08} to 10^{-10}, while for Example 3 these values exist in the range of 10^{-06} to 10^{-08}. The mean and worst values of TIC for all examples lie around 10^{-06} to 10^{-08} and 10^{-04} to 10^{-06} for all examples. The best ENSE value in performance indices for Example 1 lie 10^{-10} to 10^{-12}. However, the mean and worst values for ENSE lie also in good measures. The experiential optimal results establish the correctness, the worth and efficiency of the designed model and proposed scheme. The plots of Fig. 9 are based on the convergence analysis of FIT, MAD, ENSE and TIC values using 30 variables based on the hybrid of GA-SQP. It is easy to understand that the values of the statistical operators are found in very good measures and huge part of the independent executions achieved best values of the statistical based operators for Examples 1–4. Consequently, the designed model's performance and the proposed technique is accurate in terms of the statistical operators TIC, MAD, FIT and ENSE.

For more satisfaction of the designed model and numerical scheme, precision analysis based statistics is implemented in terms of minimum (Min) values, Maximum (Max) values, median (Med) values and semi interquartile range (S.I.R) values for solving the singular DD-LE model. The numerical experimentations are drawn for 40 executions to check the accuracy and precision of the present scheme. The mathematical formulations of S.I.R is $-0.5\,(Q_1 - Q_3)$, where Q_1 and Q_3 denote the first and third quartile values. The statistics operators in sense of Min, Max, Med and S.I.R are provided in Tables 1 and 2 for solving the singular DD-LE model. The best and worst runs are denoted as the Min and Max errors for the independent runs of the algorithm. The numerical outcomes for Min, Max, Med and S.I.R indicate precise values for all examples of DD-LE model. The global performance for all examples based on mean and Med values are indicated in Table 3 and satisfactory numerical results are obtained.

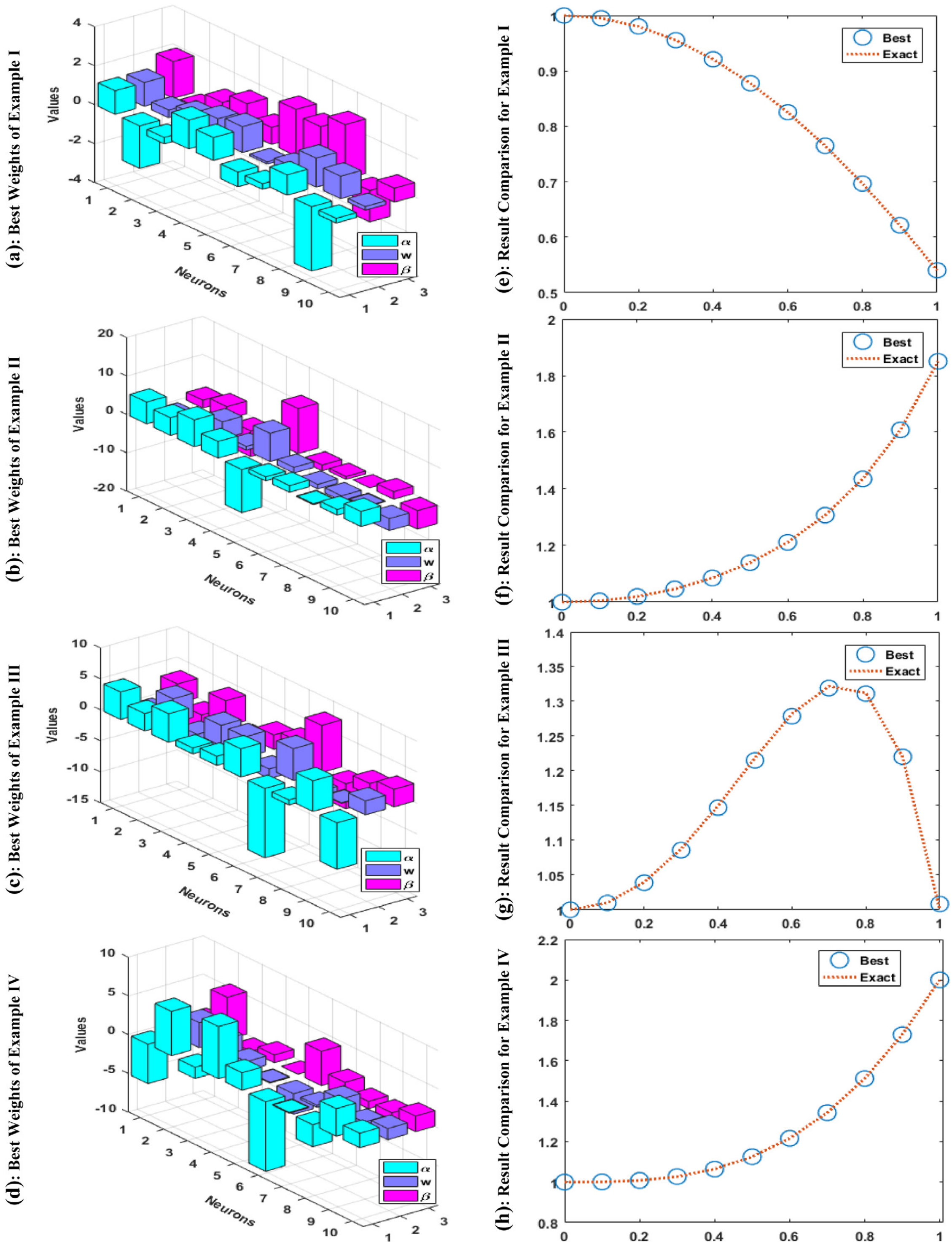


Fig. 7. Set of best weights and results comparison for Examples 1–4 using 10 neurons.

5. Conclusion

The goal of the present study is to model the nonlinear second order DD-LE equation and produce the numerical outcomes using neural networks models enhanced by the integration of genetic algorithms and sequential quadratic programming. The designed model and scheme is confirmed preciously to compare each example with the exact solutions. Some concluded remarks are as follows

- The nonlinear DD model of LE type is designed and numerical investigated successfully.
- Exploitation of computational heuristics is applied to achieved the brilliance outcomes for nonlinear DD model of LE type with superior accuracy and reliability.
- The presented computing heuristics worth is verified through statistical assessments based outcomes for large number of autonomous runs.

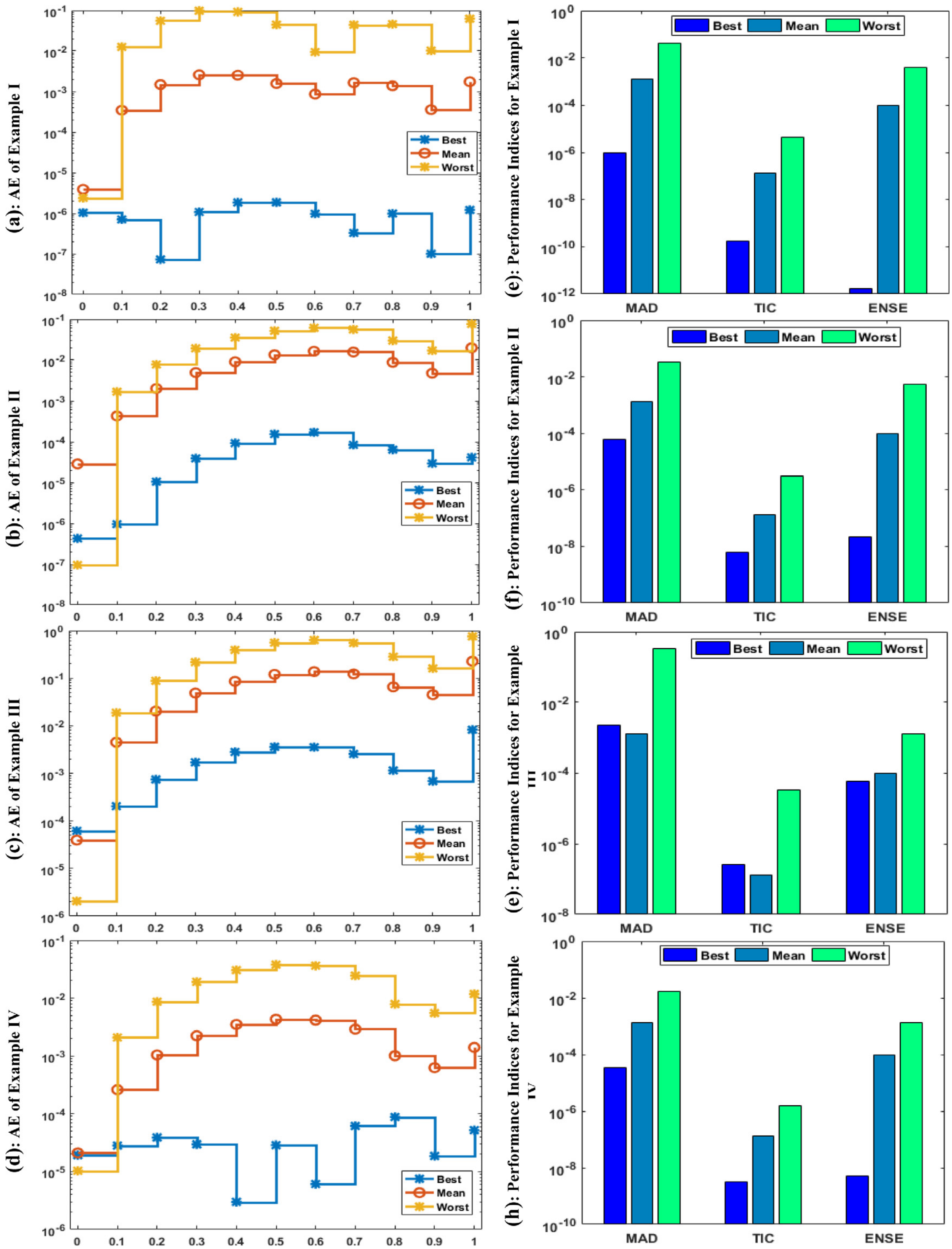


Fig. 8. AE and performance indices for Examples 1–4 using 10 neurons.

- The correctness of the model as well as proposed method is analyzed through the overlapping of the exact and present results.
- Nonlinear DD-LE model are not easy to handle because of its stiffer nature. Both DD equations and LE equations are always very challengeable to solve for researchers. However, ANN has ability to handle these system with ease.

One may explore/exploit/implement the theoretical analysis for the convergence of proposed integrated heuristics with detailed mathematical development in future for further development in related domain. Additionally, the proposed heuristic computing paradigm looks promising to be implemented as alternate solver for fractional order systems [79,80], bioinformatics model [81, 82], circuit theory [83,84], plasma physics [85,86], mathematical model in financial dynamics [87,88]. computer virus models [89,90], system identification [91,92], control [93,94] and fluid mechanics [95,96].

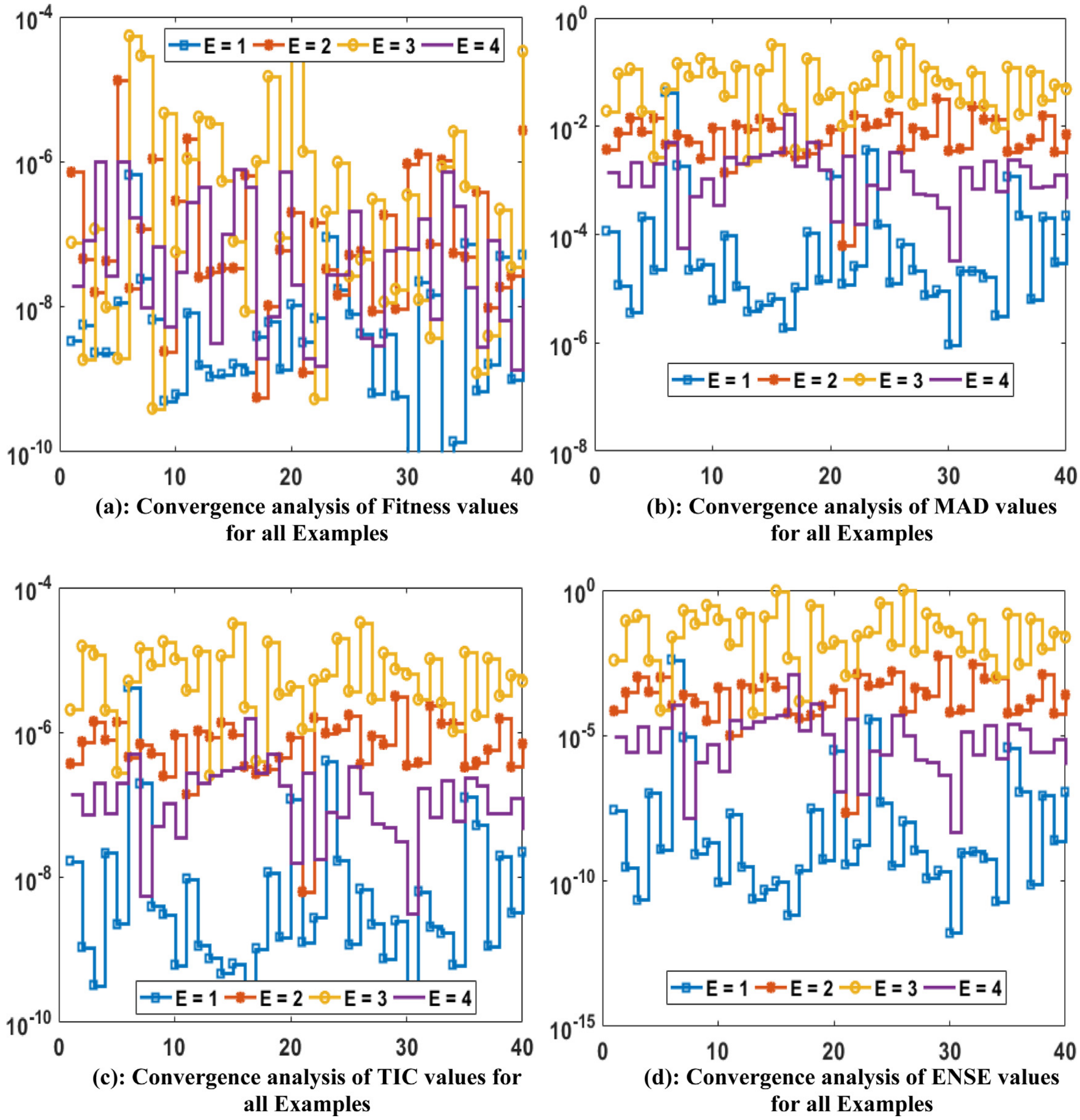


Fig. 9. Convergence of Fitness, MAD, TIC and ENSE values for Examples 1–4.

Table 1
Statistics based values for Examples 1 and 2.

x	Example 1				Example 2			
	Min	Max	Med	S.I.R	Min	Max	Med	S.I.R
0	8.039E−08	2.074E−05	2.327E−06	2.768E−06	9.485E−08	2.561E−04	3.567E−06	2.091E−05
0.1	2.866E−07	1.254E−02	5.354E−06	6.803E−06	9.572E−07	1.681E−03	3.807E−04	2.032E−04
0.2	6.949E−08	5.568E−02	5.074E−06	2.856E−05	1.049E−05	7.811E−03	1.709E−03	9.862E−04
0.3	9.206E−07	9.761E−02	1.001E−05	7.306E−05	3.903E−05	1.936E−02	4.160E−03	2.467E−03
0.4	8.644E−07	9.141E−02	2.146E−05	1.201E−04	9.223E−05	3.540E−02	7.572E−03	4.427E−03
0.5	4.298E−07	4.430E−02	3.668E−05	1.183E−04	1.529E−04	5.189E−02	1.125E−02	6.549E−03
0.6	9.691E−07	1.062E−02	4.420E−05	1.252E−04	1.695E−04	6.166E−02	1.409E−02	8.020E−03
0.7	3.291E−07	4.394E−02	4.326E−05	1.064E−04	8.381E−05	5.632E−02	1.381E−02	7.364E−03
0.8	1.464E−07	4.577E−02	1.638E−05	5.686E−05	6.357E−05	2.998E−02	7.903E−03	3.991E−03
0.9	1.000E−07	1.019E−02	1.174E−05	2.292E−05	2.931E−05	1.712E−02	4.217E−03	2.067E−03
1	2.235E−07	6.221E−02	7.250E−06	3.695E−05	4.124E−05	7.665E−02	1.704E−02	1.006E−02

Table 2
Statistics based values for Examples 3 and 4.

x	Example 3				Example 4			
	Min	Max	Med	S.I.R	Min	Max	Med	S.I.R
0	1.945E−08	9.120E−04	3.900E−06	1.015E−05	2.798E−07	1.512E−04	7.507E−06	9.958E−06
0.1	1.266E−04	1.870E−02	3.251E−03	2.672E−03	1.068E−05	2.078E−03	1.631E−04	1.407E−04
0.2	5.453E−04	8.760E−02	1.346E−02	1.168E−02	1.169E−05	8.656E−03	7.055E−04	4.697E−04
0.3	1.305E−03	2.172E−01	3.190E−02	2.641E−02	2.948E−05	1.905E−02	1.552E−03	9.880E−04
0.4	2.416E−03	3.915E−01	5.546E−02	4.408E−02	2.977E−06	3.042E−02	2.451E−03	1.537E−03
0.5	3.594E−03	5.564E−01	7.741E−02	6.133E−02	2.865E−05	3.786E−02	3.019E−03	1.890E−03
0.6	3.533E−03	6.319E−01	8.862E−02	7.192E−02	6.077E−06	3.628E−02	2.899E−03	1.837E−03
0.7	2.550E−03	5.496E−01	8.052E−02	6.802E−02	6.131E−05	2.447E−02	2.002E−03	1.304E−03
0.8	1.151E−03	2.819E−01	4.354E−02	4.072E−02	3.155E−05	7.858E−03	6.749E−04	4.746E−04
0.9	6.858E−04	1.601E−01	3.460E−02	2.345E−02	7.314E−06	5.514E−03	4.157E−04	2.951E−04
1	8.268E−03	7.540E−01	1.725E−01	1.335E−01	2.955E−05	1.180E−02	9.581E−04	6.665E−04

Table 3
Global measure values for Examples 1 to 4.

Index	Example	GFIT		GMAD		GTIC		GENSE	
		Mean	Med	Mean	Med	Mean	Med	Mean	Med
$\hat{u}(x)$	1	2.8E−08	3.7E−09	1.3E−03	2.2E−05	1.3E−07	2.4E−09	1.0E−04	1.0E−09
	2	6.5E−07	5.0E−08	8.7E−03	7.4E−03	8.6E−07	7.2E−07	5.9E−04	2.8E−04
	3	4.9E−06	1.6E−07	8.0E−02	5.4E−02	8.3E−06	5.7E−06	1.2E−01	3.1E−02
	4	1.8E−07	2.9E−08	1.9E−03	1.3E−03	1.9E−07	1.3E−07	5.0E−05	8.3E−06

CRediT authorship contribution statement

Zulqurnain Sabir: Writing, Editing, Supervision, Validation. **Juan L.G. Guirao:** Conceptualization, Methodology, Software. **Tareq Saeed:** Conceptualization, Methodology, Software.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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