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Technical communique

Imaginary-axis crossing points in root loci that depend polynomially on the gain ☆

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ABSTRACT

This technical note discusses the crossing with the stability boundary for the root loci that polynomially depend on the gain. A simple calculation method based on reflected polynomials is provided. The trend of the roots is also determined, which allows studying the gain margin as well as the minimum gain so that all the roots are within the stable region. Finally, these concepts are illustrated with two case studies.

1. Introduction

Affine Root Loci (ARLs) are not high-gain stabilizing for a plant with a relative degree greater than two. This is quite evident since in such a case there are asymptotes that cross the imaginary axis. For certain design specifications, the system is unable to move the closed-loop poles within the feasible design region. One possibility to solve this problem is to use lead–lag networks, or similar compensators that modify the ARL, (Franklin, Powell, & Emami-Naeini, 2018). Another possibility is to use general controllers of the form $G_c(s) = N_c(s;K)/D_c(s;K)$ where $N_c, D_c \in \mathbb{R}[s,K]$ (ring of bivariate polynomials on the real ones in the unknowns s and K), and have a polynomial dependence with respect to K with degree greater than one (Hoagg & Bernstein, 2004; Hoagg & Bernstein, 2007; Mareels, 1984; Miller & Davison, 1991). For example a plant given by a quadruple integrator $1/s^4$ can be stabilized by an appropriate controller with $N_c(s;K) = K^3 p_3(s)$ and $D_c(s;K) = K^2 p_2(s) + K p_1(s) + p_0(s)$, where $p_i \in \mathbb{R}[s]$ for $i = 0, \ldots, 3$. In general, in a closed loop with unity feedback we have a polynomial characteristic equation in K of the type $p_m(s) = K^m + \cdots + p_1(s)K + p_0(s) = 0$. The locus of the closed-loop poles in this equation is called the polynomial root locus (PRL). Throughout this work, only PRLs with positive gains will be considered, excluding complementary PRLs.

The plotting rules for the quadratic root locus have been analyzed in Wellman and Hoagg (2014), and for the cubic in Wellman (2013). Recently in Mulero-Martínez (2016), the general problem of determining the angles of the asymptotes for arbitrary PRLs was addressed. However, none of these studies analyze the crossover frequencies at the stability boundary. Motivated by this lack in the literature, this technical note addresses this problem in a simple way. Thus, the objective of this work is the analysis of the crossing points of the PRL with the imaginary axis and of the critical gains at these points. The crossing can occur from the stable to the unstable region and vice versa, so it is important to study the trend of the roots at the intersection with the imaginary axis. From these trends, we study the gain margin as well as the minimum gain so that all poles are within the stable region.

1.1. Notation

The ring of polynomials on the reals, in the unknowns $s \in \mathbb{C}$ and $K \in \mathbb{R}$, is represented by $\mathbb{R}[s,K]$. Similarly, $\mathbb{R}[s]$ is the ring of polynomials over the reals in the unknown s. Given a polynomial p in a ring of polynomials, the set of its roots will be denoted by $\mathcal{Z}(p)$. The set $\mathbb{R}^+$ represents the positive reals. $\mathbb{C}^-$ and $\mathbb{C}^+$ denote the left half-plane and right half-plane of the complex plane respectively. The branch i of the root locus, $r_i(s;K)$, is a curve in the complex plane parameterized by the gain K.

2. Imaginary axis crossing in PRLs

Let the bivariate polynomial $p \in \mathbb{R}[s,K]$ be given by

$$p(s;K) = \sum_{i=0}^{m} p_i(s) K^i, \quad (1)$$

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where $p_i \in \mathbb{R}[s]$. We define the reflected polynomial of $p(s;K)$ as the polynomial resulting from reversing the order of the polynomials in (1).

Definition 1. The reflected polynomial of $p(s;K)$ is defined as

$$p^*(s;K) = K^m \sum_{i=0}^{m} \overline{p_{m-i}(s)} K^{-i}.$$

From these definitions, it is immediate to see that if $j\omega$ is a root of $p(s;K)$, also $-j\omega$ is a root of $p^*\left(-s;\frac{1}{K}\right)$.

Lemma 2. *For a given K, if $j\omega \in \mathcal{Z}(p(\cdot;K))$ then $-j\omega \in \mathcal{Z}\left(p\left(-\cdot;K^{-1}\right)\right)$.*

Proof. This is straightforward from the definition of the reflected polynomial p^*:

$$p^*\left(-j\omega;K^{-1}\right) = K^{-m} \sum_{i=0}^{m} \overline{p_{m-i}(-j\omega)} K^{i}.$$

Since p has its coefficients on the field $\mathbb{R}$, it follows that $\overline{p_{m-i}(-j\omega)} = p_{m-i}(j\omega)$. Making the change of variable $l = m - i$ we get that

$$K^{-m} \sum_{i=0}^{m} p_{m-i}(j\omega) K^{i} = K^{-m} \sum_{l=0}^{m} p_{l}(j\omega) K^{m-l}.$$

And therefore,

$$p^*\left(-j\omega;K^{-1}\right) = p(j\omega;K) = 0. \quad \blacksquare$$

We will successively multiply $p(j\omega;K)$ by $1, \frac{1}{K}, \ldots, \frac{1}{K^{m-1}}$, which leads to a system of polynomial equations:

$$\begin{cases} \sum_{i=0}^{m} p_i(j\omega) K^i = 0, \\ \sum_{i=-1}^{m-1} p_{i+1}(j\omega) K^i = 0, \\ \vdots \\ \sum_{i=-(m-1)}^{1} p_{i+(m-1)}(j\omega) K^i = 0. \end{cases} \tag{2}$$

We will now write this system of equations in compact matrix form. For this, we define the vector $l_m(K)$ as the vector with successive powers from K to $m-1$, i.e.

$$l_m(K) = \begin{pmatrix} 1 \\ K \\ \vdots \\ K^{m-1} \end{pmatrix},$$

and the reflection operator $*$ of a vector $x \in \mathbb{R}^n$ as the one that reverses the order of the elements x:

$$(x_1, x_2, \ldots, x_{n-1}, x_n)^* = (x_n, x_{n-1}, \ldots, x_2, x_1).$$

Then the system of equations in (2) is rewritten as

$$L_m(\omega)\, l_m\left(K^{-1}\right)^* + K U_m(\omega)\, l_m(K) = 0, \tag{3}$$

where

$$U_m(\omega) = \begin{pmatrix} p_1(j\omega) & p_2(j\omega) & p_3(j\omega) & \cdots & p_m(j\omega) \\ p_2(j\omega) & p_3(j\omega) & p_4(j\omega) & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ p_{m-1}(j\omega) & p_m(j\omega) & 0 & \cdots & 0 \\ p_m(j\omega) & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$L_m(\omega) = \begin{pmatrix} 0 & \cdots & 0 & 0 & p_0(j\omega) \\ 0 & \cdots & 0 & p_0(j\omega) & p_1(j\omega) \\ \vdots & & & & \\ 0 & p_0(j\omega) & p_1(j\omega) & \cdots & p_{m-2}(j\omega) \\ p_0(j\omega) & p_1(j\omega) & p_2(j\omega) & \cdots & p_{m-1}(j\omega) \end{pmatrix}.$$

In a similar way to what was done above, we will multiply $p^*\left(-j\omega;\frac{1}{K}\right)$ successively by $K^m, \ldots, K^1$, which leads to a system of polynomial equations given by

$$\overline{L_m(\omega)} l_m\left(K^{-1}\right)^* + K\overline{U_m(\omega)} l_m(K) = 0. \tag{4}$$

We will write Eqs. (3) and (4) as a single matrix equation:

$$\begin{pmatrix} L_m(\omega) & U_m(\omega) \\ \overline{L_m(\omega)} & \overline{U_m(\omega)} \end{pmatrix} \begin{pmatrix} l_m\left(K^{-1}\right)^* \\ K l_m(K) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5}$$

Let $q(\omega)$ be the polynomial defined by the determinant of the matrix of the system (5):

$$q(\omega) = \det \begin{pmatrix} L_m(\omega) & U_m(\omega) \\ \overline{L_m(\omega)} & \overline{U_m(\omega)} \end{pmatrix}. \tag{6}$$

For (5) to have a non-trivial solution it is necessary that $q(\omega) = 0$. The crossover frequencies ω are the roots of $q(\omega)$. In the following lemma, we see that $q(\omega)$ is an even/odd polynomial depending on m.

Lemma 3. *The polynomial q is even for even m, and odd for odd m.*

Proof. The proof is straightforward since $p_i(-j\omega) = \overline{p_i(j\omega)}$, and in particular,

$$q(-\omega) = \det \begin{pmatrix} \overline{L_m(\omega)} & \overline{U_m(\omega)} \\ L_m(\omega) & U_m(\omega) \end{pmatrix} = (-1)^m q(\omega).$$

Note that the exchange of rows in the matrix changes the sign of the determinant, hence the term $(-1)^m$. $\blacksquare$

The nonzero crossover frequencies are the non-negative real roots of $q\left(\sqrt{\omega}\right)$ for even m, or of $\frac{q(\sqrt{\omega})}{\sqrt{\omega}}$ for odd m. Let $\mathcal{W}$ be the set $\left\{\omega \in \mathbb{R}^+ \cup \{0\} : q(\omega) = 0\right\}$. The elements of $\mathcal{W}$ are candidate frequencies for crossing points with the imaginary axis. For each of them, we determine the gain K and check if $K \geq 0$. This can be done by decomposing the polynomial p into its even part $p_e\left(s^2, K\right)$, and odd $sp_o\left(s^2, K\right)$:

$$p(s) = p_e\left(s^2, K\right) + sp_o\left(s^2, K\right).$$

The polynomials $p_e\left(s^2, K\right)$ and $p_o\left(s^2\right)$ are determined from the even and odd part, respectively, of the coefficients of $p(s)$:

$$p(s) = \sum p_i^e\left(s^2\right) K^i + s \sum p_i^o\left(s^2\right) K^i. \tag{7}$$

Evaluating (7) at $s = j\omega$ where $\omega \in \mathcal{W}$, we obtain two equations in K:

$$q_e(K;\omega) = \sum p_i^e\left(-\omega^2\right) K^i = 0,$$

and

$$q_o(K;\omega) = \sum p_i^o\left(-\omega^2\right) K^i = 0.$$

These equations must be satisfied simultaneously. To solve them, we make the following observations: (i) if q_e and q_o are proportional, we would have a single equation of degree m to solve, (ii) if q_e and q_o are not proportional, we calculate the greatest common divisor of these polynomials:

$$r(K;\omega) = \gcd\left(q_e(K;\omega), q_o(K;\omega)\right). \tag{8}$$

The set of candidate gains will then be,

$$\mathcal{K}=\left\{K\in\mathbb{R}^{+}:q_{e}\left(K;\omega\right)=q_{o}\left(K;\omega\right)=0,\text{ with }\omega\in\mathcal{W}\right\}.$$

The PRL allows us to find the gain margin graphically or analytically. The gain margin is defined as the gain factor by which the design value can be multiplied to make the system unstable. It depends on the critical gain $K_{crit}=\min\mathcal{K}$ that is achieved at the point where the PRL crosses the imaginary axis in the complex plane. Mathematically the gain margin ρ is given by the formula $\rho=\frac{K_{crit}}{K}$, where K is the design gain.

The crossing direction of the root locus at $s=j\omega_0$ as the gain K_0 increases can be determined by an expression for that trend:

$$T_{\omega_0,K_0}=sgn\left(\mathrm{Re}\left\{\left.\frac{\partial s}{\partial K}\right|_{(j\omega_0,K_0)}\right\}\right), \tag{9}$$

where $sgn(\cdot)$ denotes the sign function. The following lemma provides a simple method for calculating the trend of roots:

Lemma 4. *Let $p(s;K)$ be the characteristic polynomial as defined in (1) and let K_0 be the gain for which the system crosses the imaginary axis at $j\omega_0$, for some $\omega_0\in\mathcal{W}$, i.e. $p(j\omega_0;K_0)=0$. Let us assume that $\left.\frac{\partial p(s;K)}{\partial s}\right|_{(j\omega_0,K_0)}\neq 0$. Then, the root tendency of $j\omega_0$ is given by*

$$T_{\omega_0,K_0}=-sgn\left(\mathrm{Re}\left\{\frac{\sum_{i=1}^{m}ip_i(j\omega_0)K_0^{i-1}}{\sum_{i=0}^{m}\left.\frac{\partial p_i(s)}{\partial s}\right|_{s=j\omega_0}K_0^{i}}\right\}\right). \tag{10}$$

Proof. According to the implicit function theorem, there exist a $\delta>0$, a neighborhood $U_0\subset\mathbb{C}$ of $j\omega_0$ and a continuous function $s:(K_0-\delta,K_0+\delta)\to U_0$ such that $p(s(K);K)=0$ for all $K\in(K_0-\delta,K_0+\delta)$. Since p is differentiable and since $\left.\frac{\partial p(s;K)}{\partial s}\right|_{(j\omega_0,K_0)}\neq 0$, it follows that

$$\left.\frac{\partial s}{\partial K}\right|_{(j\omega_0,K_0)}=-\frac{\left.\frac{\partial p(s;K)}{\partial K}\right|_{(j\omega_0,K_0)}}{\left.\frac{\partial p(s;K)}{\partial s}\right|_{(j\omega_0,K_0)}}, \tag{11}$$

where

$$\left.\frac{\partial p_c(s;K)}{\partial s}\right|_{(j\omega_0,K_0)}=\sum_{i=0}^{m}\left.\frac{\partial p_i(s)}{\partial s}\right|_{s=j\omega_0}K_0^{i}, \tag{12}$$

$$\left.\frac{\partial p(s;K)}{\partial K}\right|_{(j\omega_0,K_0)}=\sum_{i=1}^{m}ip_i(j\omega_0)K_0^{i-1}.\quad\blacksquare$$

A branch $r_i(s;K)$ of the PRL can cross the imaginary axis several times, so the trend of the crossing is crucial to determine the minimum gain $K_{\min}$ so that all the closed-loop poles are in $\mathbb{C}^-$ when $K>K_{\min}$.

3. Illustrative examples

In this section, we will apply the calculation method presented in the previous section to two case studies.

3.1. Example 1

For the plant

$$G_p(s)=\frac{s+10}{(s+30)^4},$$

let $G_c(s)$ be the quadratic controller at gain K given by

$$G_c(s)=K^2\frac{(s+30)^4}{(s+40)^4K+(s+20)^7}.$$

The characteristic polynomial for the closed-loop control system with unity feedback is quadratic in K over the ring of polynomials $\mathbb{R}[s]$:

$$p(s;K)=(s+10)K^2+(s+40)^4K+(s+20)^7.$$

Note that $p_0(s)=(s+20)^7$, $p_1(s)=(s+40)^4$, $p_2(s)=(s+10)$. For this polynomial, Eq. (5) is written as follows

$$\begin{pmatrix}0 & p_0(j\omega) & p_1(j\omega) & p_2(j\omega)\\ p_0(j\omega) & p_1(j\omega) & p_2(j\omega) & 0\\ 0 & \overline{p_0(j\omega)} & \overline{p_1(j\omega)} & \overline{p_2(j\omega)}\\ \overline{p_0(j\omega)} & \overline{p_1(j\omega)} & \overline{p_2(j\omega)} & 0\end{pmatrix}\begin{pmatrix}K^{-1}\\1\\K\\K^2\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}. \tag{13}$$

We construct the polynomial $q(\omega)$,

$$q(\omega)=\det\begin{pmatrix}0 & (j\omega+20)^7 & (j\omega+40)^4 & (j\omega+10)\\ (j\omega+20)^7 & (j\omega+40)^4 & (j\omega+10) & 0\\ 0 & (-j\omega+20)^7 & (-j\omega+40)^4 & (-j\omega+10)\\ (-j\omega+20)^7 & (-j\omega+40)^4 & (-j\omega+10) & 0\end{pmatrix},$$

whose positive real roots are $\omega=139.8$, $\omega=1.2584$, $\omega=15.502$ and $\omega=0$. These frequencies are candidates for crossover points. For each one of them, we determine the gain K and verify if $K\in\mathbb{R}^+$. According to Eq. (7), the polynomial $p(s)$ is divided into its odd and even part, which results in the following polynomials:

$$p_2^e(s^2)=10,$$

$$p_1^e(s^2)=\sum_{k=0}^{2}\binom{4}{2k}s^{4-2k}40^{2k}=s^4+6\cdot 40^2s^2+40^4,$$

$$p_0^e(s^2)=\sum_{k=0}^{3}\binom{7}{2k+1}s^{7-(2k+1)}20^{2k+1}$$
$$=7\cdot 20s^6+35\cdot 20^3s^4+21\cdot 20^5s^2+20^7,$$

$$p_2^o(s^2)=1,$$

$$p_1^o(s^2)=\sum_{k=0}^{1}\binom{4}{2k+1}s^{4-(2k+1)-1}40^{2k+1}$$
$$=4\cdot 40s^2+4\cdot 40^3,$$

$$p_0^o(s^2)=\sum_{k=0}^{3}\binom{7}{2k}s^{7-2k-1}20^{2k}$$
$$=s^6+21\cdot 20^2s^4+35\cdot 20^4s^2+7\cdot 20^6.$$

And for each candidate frequency, we determine the polynomial $r(s;K)$ according to (8): $r(K;139.8)=K-3.97062\times 10^6$, $r(K;15.502)=K-1943.33$, $r(K;1.2584)=K+254018$, and $r(K;0)=K+8000\times\left(16+\sqrt{254}\right)$.

From this, it follows that

$$K_{crit}=\min\left\{3.97062\times 10^6,1943.33\right\}=1943.33.$$

In Fig. 1, the quadratic root locus and the crossing points with the imaginary axis have been plotted using the derivative with respect to K of the implicit equation $p(s;K)=0$, that is, solving numerically the equation $\frac{\partial p(s;K)}{\partial s}\frac{\partial s}{\partial K}+\frac{\partial p(s;K)}{\partial K}=0$. Since the characteristic polynomial $p(s;K)$ is of degree 7, by the fundamental theorem of algebra the root locus will present seven branches. These branches begin at the root $s=-20$ of multiplicity seven (open-loop pole of the direct function $G_c(s;K)G_p(s)$ for $K=0$). This point is marked in Fig. 1 with an "x". If we divide the

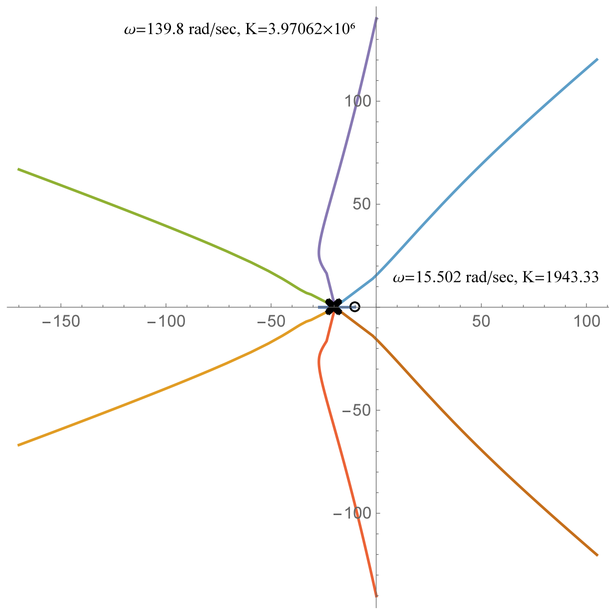

Fig. 1. Quadratic root locus defined by the polynomials $p_0(s) = (s+20)^7$, $p_1(s) = (s+40)^4$, $p_2(s) = (s+10)$. The crossing points with the imaginary axis and the value of gain K are indicated.

Table 1
Root tendency for the crossing frequencies of the root locus in the example 1.

ω_0	K_0	$\frac{p_1(j\omega_0)+2p_2(j\omega_0)K_0}{\sum_{i=0}^{2}\frac{\partial p_i(s)}{\partial s}\big\vert_{s=j\omega_0}K_0^i}$	T_{ω_0,K_0}
15.502	1943.33	$\frac{(3.4962+j34.327)\times10^5}{(-9.8379-j7.8743)\times10^8}$	$+1$
139.8	3.97062×10^6	$\frac{(1.9695-j4.0083)\times10^8}{(-3.6559+j4.1798)\times10^{13}}$	$+1$

characteristic equation by K^2, and we make K tend towards ∞, one of the roots will tend towards the zero located at $s = -10$ through the real axis (this zero has been represented in Fig. 1 with an "o"). The remaining six branches follow asymptotes centered at $\sigma = -21.7$ with angles $\pm\frac{2\pi k}{9}$, $k = 1, 2, 4$. The PRL branches following asymptotes with angles $\pm\frac{2\pi}{9}$ and $\pm\frac{4\pi}{9}$, cross the imaginary axis at $\pm j15.502$ and $\pm j139.8$ respectively (with gains 1943.33 and 3.97062×10^6 respectively).

Now we analyze the tendency for the cross frequencies at $\omega_0 = 15.502$ rad/sec and $\omega_0 = 139.8$ rad/sec (with gains 1943.33 and 3.97062×10^6 respectively). The root locus crosses the imaginary axis at these frequencies from the left half-plane to the right half-plane (positive tendency) as shown in Fig. 1. The computation of the tendency are collected in Table 1.

3.2. Osprey Tiltrotor aircraft

In this section, we analyze the crossover gains for a control system of a Bell-Boeing V-22 Osprey Tiltrotor. This artifact is both an airplane and a helicopter. We consider the attitude control in the helicopter mode, (Dorf & Bishop, 2011). The transfer function from the elevator input to the pitch attitude is given by

$$G_p(s)\,\frac{1}{(20s+1)(10s+1)\left(\frac{1}{2}s+1\right)}.$$

This plant has a relative degree of 3 so it has three asymptotes forming angles of 180° and ±60°. The asymptotes of ±60° make the system not high-gain stabilizable since there is a critical gain value from which two closed-loop poles are confined in the right half-plane. We will use a controller with a transfer function that depends quadratically on the gain K,

$$G_c(s) = \frac{\gamma_2 K^2 (s+z)^2}{(\gamma_0(s+p)+\gamma_1 K)}.$$

The characteristic equation for the closed-loop control system with unity feedback is of the form

$$p_2(s)K^2 + p_1(s)K + p_0(s) = 0, \quad (14)$$

where $p_1(s) = \gamma_1\left(s+\frac{1}{20}\right)\left(s+\frac{1}{10}\right)(s+2)$, $p_2(s) = \gamma_2(s+z)^2$ and $p_0(s) = \gamma_0\gamma_1^{-1}p_1(s)(s+p)$. The root locus starts from the poles in open loop, i.e. from the roots of $p_0(s)$, so it will have four branches. Dividing (14) by K^2 and making K tend to infinity we have that two closed-loop poles move towards double zero at $s = -z$, while the other two poles follow asymptotes that form angles

$$\theta = \arg\left(-\gamma_1 + \sqrt{\gamma_1^2 - 4\gamma_2\gamma_0}\right),$$

and $\varphi = 2\pi - \theta$. Increasing z attracts the root locus to positions further to the left in the complex plane. With a sufficiently large gain, it is possible to increase the response speed of the system with a high gain selection (the system is highly stabilizable). We will consider the following parameter values: $z = 10$, $p = 1$, $\gamma_2 = 50$, $\gamma_1 = 100$, and $\gamma_0 = 100$. For these parameters the asymptotes form angles of $\theta = 135.0°$ and $\varphi = 225°$ (cutting the real axis at $\alpha = 1874.5$ and $\beta = 8.4726$ respectively). The system of equations for the characteristic polynomial in K is given by (13). In particular, the determinant of the matrix of this system of equations gives us the auxiliary polynomial $q(\omega)$ whose non-negative real roots are $\omega = 0.33262$, $\omega = 0.35121$, $\omega = 14.749$ and $\omega = 0$ (double). Each of these frequencies has an associated crossover gain that will have to be checked for positivity. For this we must separate the polynomial $p(s;K)$ into its odd and even part, which leads to the following polynomials: $p_2^e(s^2) = 50(s^2+100)$, $p_1^e(s^2) = 215s^2+1$, $p_0^e(s^2) = 100s^4 + \frac{491}{2}s^2 + 1$, $p_2^o(s^2) = 1000$, $p_1^o(s^2) = 100s^2 + \frac{61}{2}$, and $p_0^o(s^2) = 315s^2 + \frac{63}{2}$. For each candidate crossover frequency $\omega \in \mathcal{W}$ we calculate the polynomial $r(s;K)$ according to equation (8): $r(K;0) = 1$, $r(K;0.33262) = K + 0.0684171$, $r(K;0.35121) = K - 0.0771575$, and $r(K;14.749) = K - 24.5159$.

Table 2
Root tendency for the crossing frequencies of the root locus in the control system of the Osprey Tiltrotor aircraft.

ω_0	K_0	$\frac{p_1(j\omega_0)+2p_2(j\omega_0)K_0}{\sum_{i=0}^{2}\frac{\partial p_i(s)}{\partial s}\big\vert_{s=j\omega_0}K_0^i}$	T_{ω_0,K_0}
0.35121	0.0771575	$\frac{745.1+j60.577}{-79.613+j166.98}$	$+1$
14.74	24.5159	$\frac{-3.342+j4.0293}{(-12.015-j2.3247)}$	-1

As shown in Fig. 2, the system crosses the imaginary axis at $s = \pm j0.35121$ with a gain $K = 0.0771575$ and at $s = \pm j14.749$ with $K = 24.5159$.

Using the root tendency lemma we can compute the crossing direction for each crossing frequency as shown in the last column in Table 2; $T_{\omega_0,K_0} = -1$ indicates that the crossing is from $\mathbb{C}^+$ to $\mathbb{C}^-$ and $T_{\omega_0,K_0} = +1$ implies a crossing from $\mathbb{C}^-$ to $\mathbb{C}^+$. As a result of this analysis, the minimum gain for all the closed-loop poles to be within the stable region is $K = 24.5159$.

4. Conclusions

In this technical note, the crossing of the imaginary axis has been analyzed in root loci defined by non-affine polynomials

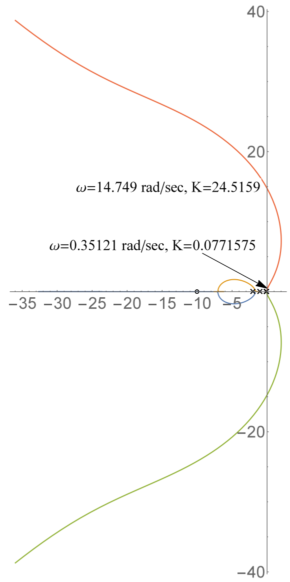

Fig. 2. Quadratic Root Locus for the Control System of an Osprey Tiltrotor aircraft. The PRL is determined by the polynomials $p_2(s) = 50(s+10)^2$, $p_1(s) = 100\left(s+\frac{1}{20}\right)\left(s+\frac{1}{10}\right)(s+2)$, and $p_0(s) = p_1(s)(s+1)$. The crossing frequencies with the imaginary axis at $\omega = 0.35121$ rad/sec and $\omega = 14.749$ rad/sec are also marked.

with respect to the gain. This analysis is important to decide the margin of stability as well as the minimum gain for all the poles to be in the stable region. For this, a simple algebraic method based on reflected polynomials has been developed and the tendency of the roots at the crossing points with the imaginary axis has been studied.

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