

ON TOPOLOGICAL SEQUENCE ENTROPY AND CHAOTIC MAPS ON INVERSE LIMIT SPACES

J. S. CANOVAS

ABSTRACT. The aim of this paper is to prove the following results: a continuous map $f: [0, 1] \rightarrow [0, 1]$ is chaotic iff the shift map $\sigma_f: \varprojlim([0, 1], f) \rightarrow \varprojlim([0, 1], f)$ is chaotic. However, this result fails, in general, for arbitrary compact metric spaces. $\sigma_f: \varprojlim([0, 1], f) \rightarrow \varprojlim([0, 1], f)$ is chaotic iff there exists an increasing sequence of positive integers A such that the topological sequence entropy $h_A(\sigma_f) > 0$. Finally, for any A there exists a chaotic continuous map $f_A: [0, 1] \rightarrow [0, 1]$ such that $h_A(\sigma_{f_A}) = 0$.

1. INTRODUCTION

Let (X, d) and $f: X \rightarrow X$ be a compact metric space and a continuous map respectively. Consider the space of sequences

$$\varprojlim(X, f) = \{\underline{x} = (x_0, x_1, \dots, x_n, \dots) : x_i \in X, f(x_i) = x_{i-1}, \text{ for } i = 1, 2, \dots\}.$$

This set is called the **inverse limit space** associated to X and f . Define a new metric \tilde{d} on $\varprojlim(X, f)$ as

$$\tilde{d}(\underline{x}, \underline{y}) = \sum_{i=0}^{\infty} \frac{d(x_i, y_i)}{2^i},$$

where $\underline{x} = (x_0, x_1, \dots, x_n, \dots)$ and $\underline{y} = (y_0, y_1, \dots, y_n, \dots)$. Then $(\varprojlim(X, f), \tilde{d})$ is a compact metric space. Consider the natural projection $\pi: \varprojlim(X, f) \rightarrow X$ defined by $\pi(x_0, x_1, \dots, x_n, \dots) = x_0$. Note that $\tilde{d}(\underline{x}, \underline{y}) \geq d(\pi(\underline{x}), \pi(\underline{y}))$ for all $\underline{x}, \underline{y} \in \varprojlim(X, f)$. The **shift map** is a homeomorphism $\sigma_f: \varprojlim(X, f) \rightarrow \varprojlim(X, f)$ defined by

$$\sigma_f(\underline{x}) = \sigma_f(x_0, x_1, \dots, x_n, \dots) = (f(x_0), x_0, x_1, \dots, x_n, \dots).$$

It is clear that $\pi \circ \sigma_f = f \circ \pi$.

Received December 14, 1998; received March 25, 1999.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 58F03, 26A18.

Inverse limit spaces have been studied in the setting of dynamical systems in a large number of papers. In [6], Shihai Li proved that some dynamical properties hold at the same time for f and σ_f . In particular, he showed that f is chaotic in Devaney's sense iff σ_f is also like that. He also proved a similar result for a suitable definition of w -chaos. In this paper, a similar result is studied in case of the Li-Yorke's chaos. Recall briefly this definition of chaos.

A point $p \in X$ is **periodic** if there exists a positive integer n such that $f^n(p) = p$. The smallest positive integer satisfying this condition is called the **period of p** . Denote by $\text{Per}(f)$ the set of **periodic points of f** . A point $x \in X$ is said to be **asymptotically periodic** if there exists a $p \in \text{Per}(f)$ such that $\limsup_{n \rightarrow \infty} d(f^n(x), f^n(p)) = 0$. A map $f: X \rightarrow X$ is said to be **chaotic in the sense of Li-Yorke** or simply **chaotic** if there exists an uncountable set $D \subset X \setminus \text{Per}(f)$ such that

$$\begin{aligned} \limsup_{n \rightarrow \infty} d(f^n(x), f^n(y)) &> 0, \\ \liminf_{n \rightarrow \infty} d(f^n(x), f^n(y)) &= 0, \end{aligned}$$

hold for all $x, y \in D$, $x \neq y$. D is called a **scrambled set** of f .

The Li-Yorke's chaos on inverse limit spaces has been studied by Gu Rongbao in [3]. In that paper the author attempts to prove that a continuous map f is chaotic iff the shift map σ_f is chaotic. However, in the proof he uses implicitly that f is surjective. As we will see later, this hypothesis on f cannot be removed in the following theorem essentially proved in [3].

Theorem 1.1. *Suppose f that is surjective. Then it is chaotic in the sense of Li-Yorke if and only if the map σ_f is chaotic in the sense of Li-Yorke.*

When continuous maps $f: [0, 1] \rightarrow [0, 1]$ are concerned, the Li-Yorke's chaos is connected with the notion of topological sequence entropy. Let us recall the definition (see [2]). Let $A = \{a_i\}_{i=1}^{\infty}$ be an increasing sequence of positive integers. Given $\epsilon > 0$, we say that $E \subset X$ is an (A, ϵ, n, f) -separated set if for any $x, y \in E$ with $x \neq y$ there exists $1 \leq k \leq n$ such that $d(f^{a_k}(x), f^{a_k}(y)) > \epsilon$. Denote by $s_n(A, \epsilon, f)$ the cardinality of any maximal (A, ϵ, n, f) -separated set. The **topological sequence entropy** of f is given by

$$h_A(f) = \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log s_n(A, \epsilon, f).$$

In general,

$$(1) \quad h_A(f) \geq h_A(\sigma_f)$$

for every A . When f is surjective we obtain the equality (see [2])

$$(2) \quad h_A(f) = h_A(\sigma_f).$$

The connection between the Li-Yorke's chaos and the topological sequence entropy is established in the following result (see [1] and [5]).

Theorem 1.2. *Let $c, d \in \mathbb{R}$ and let $f: [c, d] \rightarrow [c, d]$ be continuous. Then*

- (a) *f is chaotic iff there exists an increasing sequence of positive integers A such that $h_A(f) > 0$.*
- (b) *For any increasing sequence A there exists a chaotic map $f_A: [c, d] \rightarrow [c, d]$ such that $h_A(f_A) = 0$.*

Theorem 1.2(a) does not hold in general for continuous maps on arbitrary compact metric spaces as it can be seen in [8]. In that paper, on $[0, 1] \times [0, 1]$ a chaotic map f with $\sup_A h_A(f) = 0$ and a non-chaotic map g with $\sup_A h_A(g) > 0$ are constructed.

The aim of this paper is to prove the following results: $f: [0, 1] \rightarrow [0, 1]$ is chaotic iff σ_f is chaotic. Theorem 1.2 holds for maps $\sigma_f: \varprojlim([0, 1], f) \rightarrow \varprojlim([0, 1], f)$. Moreover, an example of a chaotic map f for which σ_f is not chaotic is given.

2. POSITIVE RESULTS FOR ONE-DIMENSIONAL MAPS

Let $f: [0, 1] \rightarrow [0, 1]$ be continuous. Consider $[a, b] = \bigcap_{n \geq 0} f^n[0, 1]$. Then $f|_{[a,b]}: [a, b] \rightarrow [a, b]$ is obviously surjective.

Proposition 2.1. *Under the above conditions f is chaotic iff $f|_{[a,b]}$ is chaotic.*

Proof. It is clear that if $f|_{[a,b]}$ is chaotic then f is chaotic. Suppose that f is chaotic and let D be a scrambled of f . It is easy to see that $f^n(D)$ is also a scrambled set of f . Let $D_n = f^n(D) \cap [a, b]$. If $\{f^n(x) : n \geq 0\} \cap [a, b] = \emptyset$, then x is asymptotically periodic and then $x \notin D_n$ for all $n \in \mathbb{N}$. So, it must exist a positive integer n_0 such that D_{n_0} is uncountable. Then, $f|_{[a,b]}$ is chaotic. \square

Theorem 2.2. *Let $f: [0, 1] \rightarrow [0, 1]$ be continuous. Then:*

- (a) *f is chaotic if and only if σ_f is chaotic.*
- (b) *$\sigma_f: \varprojlim([0, 1], f) \rightarrow \varprojlim([0, 1], f)$ is chaotic if and only if there exists an increasing sequence of positive integers A such that $h_A(\sigma_f) > 0$.*
- (c) *For any increasing sequence of positive integers A there exists a chaotic map $f_A: [0, 1] \rightarrow [0, 1]$ such that $h_A(\sigma_{f_A}) = 0$.*

Proof. It is clear that

$$\varprojlim([0, 1], f) = \{(x_0, x_1, \dots, x_n, \dots) : x_i \in [a, b], f(x_i) = x_{i-1}\} = \varprojlim([a, b], f).$$

First of all we prove (a). Assume that f is chaotic. By Proposition 2.1, $f|_{[a,b]}$ is also chaotic. Applying Theorem 1.1 it follows that σ_f is chaotic. Conversely, suppose that σ_f is chaotic. Applying Theorem 1.1 it follows that $f|_{[a,b]}$ is chaotic. Proposition 2.1 proves that f is chaotic.

Part (b). If σ_f is chaotic, then it follows by (a) that f is chaotic. Hence, by Proposition 2.1, $f|_{[a,b]}$ is chaotic. Applying Theorem 1.2 (a), there exists an

increasing sequence of positive integers such that $h_A(f|_{[a,b]}) > 0$. Since $f|_{[a,b]}$ is surjective, by (2), $h_A(\sigma_f) = h_A(f|_{[a,b]}) > 0$. Now suppose that σ_f is non-chaotic. Assertion (a) states that f is non-chaotic. Applying Theorem 1.2 and (1), we conclude that $h_A(\sigma_f) \leq h_A(f) = 0$ for any increasing sequence of positive integers A .

Part (c). Let A be an arbitrary sequence of positive integers. By Theorem 1.2(b), there exists a chaotic map $f_A: [0, 1] \rightarrow [0, 1]$ such that $h_A(f_A) = 0$. Since f_A is chaotic, by (a), σ_{f_A} is also chaotic. By (2), $h_A(\sigma_{f_A}) \leq h_A(f_A) = 0$, and the proof ends. \square

3. A COUNTEREXAMPLE

As usual, \mathbb{Z} will stand for the set of integers, while if $Z \subset \mathbb{Z}$ then Z^n (resp. Z^∞) will denote the set of finite sequences of length n (resp. infinite sequences) of elements from Z . If $\theta \in \mathbb{Z}^n$ or $\alpha \in \mathbb{Z}^\infty$ then we will often describe them through their components as $(\theta_1, \theta_2, \dots, \theta_n)$ or $(\alpha_i)_{i=1}^\infty$, respectively. The **shift map** $\sigma: \mathbb{Z}^\infty \rightarrow \mathbb{Z}^\infty$ is defined by $\sigma((\alpha_i)_{i=1}^\infty) = (\alpha_{i+1})_{i=1}^\infty$. If $\theta \in \mathbb{Z}^n$ and $\vartheta \in \mathbb{Z}^m$ (with $m \leq \infty$) then $\theta * \vartheta \in \mathbb{Z}^{n+m}$ (where $n + \infty$ means ∞) will denote the sequence λ defined by $\lambda_i = \theta_i$ if $1 \leq i \leq n$ and $\lambda_i = \vartheta_{i-n}$ if $i > n$. In what follows we will denote $\mathbf{0} = (0, 0, \dots, 0, \dots)$ and $\mathbf{1} = (1, 1, \dots, 1, \dots)$, while if $\alpha \in \mathbb{Z}^\infty$, then $\alpha|_n \in \mathbb{Z}^n$ is defined by $\alpha|_n = (\alpha_1, \alpha_2, \dots, \alpha_n)$.

This section is devoted to construct a chaotic map f on a compact metric space for which σ_f is non-chaotic. In order to do this we will need some information concerning so-called **weakly unimodal maps of type 2^∞** . Recall briefly the definition. We say that a continuous map $f: [0, 1] \rightarrow [0, 1]$ is **weakly unimodal** if $f(0) = f(1) = 0$, it is non-constant and there is $c \in (0, 1)$ such that $f|_{(0,c)}$ and $f|_{(c,1)}$ are monotone. The map f is said to be of **type 2^∞** if it has periodic points of period 2^n for any $n \geq 0$ but no other periods.

Weakly unimodal maps of type 2^∞ (briefly, **w-maps**) were studied in [4]. In that paper it was proved that for any w-map f it is possible to construct a family $\{K_\alpha(f)\}_{\alpha \in \mathbb{Z}^\infty}$ (or simply $\{K_\alpha\}_{\alpha \in \mathbb{Z}^\infty}$ if there is no ambiguity on f) of pairwise disjoint (possibly degenerate) compact subintervals of $[0, 1]$ satisfying the following key properties (P1)–(P4):

- (P1) The interval $K_{\mathbf{0}}$ contains all absolute maxima of f .
- (P2) Define in \mathbb{Z}^∞ the following total ordering: if $\alpha, \beta \in \mathbb{Z}^\infty$, $\alpha \neq \beta$ and k is the first integer such that $\alpha_k \neq \beta_k$, then $\alpha < \beta$ if either $\text{Card} \{1 \leq i < k : \alpha_i \leq 0\}$ is even and $\alpha_k < \beta_k$ or $\text{Card} \{1 \leq i < k : \alpha_i \leq 0\}$ is odd and $\beta_k < \alpha_k$. Then $\alpha < \beta$ if and only if $K_\alpha < K_\beta$ (that is, $x < y$ for all $x \in K_\alpha$, $y \in K_\beta$).
- (P3) Let $\alpha \in \mathbb{Z}^\infty$, $\alpha \neq \mathbf{0}$, and let k be the first integer such that $\alpha_k \neq 0$. Define $\beta \in \mathbb{Z}^\infty$ by $\beta_i = 1$ for $1 \leq i \leq k - 1$, $\beta_k = 1 - |\alpha_k|$ and $\beta_i = \alpha_i$ for $i > k$

Then $f(K_\alpha) = K_\beta$ and $f(K_0) \subset K_1$.

For any n and $\alpha \in \mathbb{Z}^\infty$, let $K_{\alpha|_n}(f)$ (or just $K_{\alpha|_n}$) be the least interval including all intervals K_β , $\beta \in \mathbb{Z}^\infty$, such that $\alpha|_n = \beta|_n$. Then

(P4) For any $\alpha \in \mathbb{Z}^\infty$, $K_\alpha = \bigcap_{n=1}^\infty K_{\alpha|_n}$.

Additionally, for any fixed n it can be easily checked that the intervals K_θ , $\theta \in \mathbb{Z}^n$, are open and pairwise disjoint and (after replacing ∞ by n , $\mathbf{0}$ by $(0, 0, \dots, 0)$ and $\mathbf{1}$ by $(1, 1, \dots, 1)$), they also satisfy (P1)–(P3). Observe that if $\theta \in \{-1, 0, 1\}^n$ and we put $|\theta| := (|\theta_1|, |\theta_2|, \dots, |\theta_n|)$ then $f^{2^n}(K_\theta) \subset K_{|\theta|}$; in particular, $f^{2^n}(K_\theta) \subset K_\theta$ if $\theta \in \{0, 1\}^n$.

In the rest of this section \tilde{f} will denote a fixed w-map with the additional property that $\alpha \in \mathbb{Z}^\infty$ implies $K_\alpha(\tilde{f})$ is non-degenerate if and only if there is an $n \geq 0$ such that $\sigma^n(\alpha) = \mathbf{0}$. An example of such a map is constructed in [4]; it is possible to show that the stunted tent map $\tilde{f}(x) = \min\{1 - |2x - 1|, \mu\}$ ($\mu \approx 0.8249\dots$) from [7] is also a w-map with this property.

$\text{Bd}(Z)$, $\text{Cl}(Z)$ and $\text{Int}(Z)$ will respectively denote the boundary, the closure and the interior of Z .

Now, we are ready to construct our counterexample. Consider

$$X = \bigcup_{\alpha \in \{-1, 0, 1\}^\infty} \text{Bd}(K_\alpha).$$

Let us emphasize that $\text{Bd}(K_\alpha)$ consists of both endpoints of K_α if it is non-degenerate and of its only point if it is degenerate. Let $f = \tilde{f}|_X : X \rightarrow X$ be the restriction of the above-mentioned w-map \tilde{f} to the set X . The following lemma shows that the above choices make sense.

Lemma 3.1. *X is a compact set and $f : X \rightarrow X$ is a well-defined continuous map.*

Proof. Since

$$X = \left(\bigcap_{n=1}^\infty \bigcup_{\theta \in \{-1, 0, 1\}^n} \text{Cl}(K_\theta) \right) \setminus \bigcup_{\alpha \in \{-1, 0, 1\}^\infty} \text{Int}(K_\alpha),$$

by (P2) and (P4), X is compact.

Recall that if $\mathbf{0} \neq \alpha \in \{-1, 0, 1\}^\infty$ then \tilde{f} carries the interval K_α onto K_β with β defined as in (P3) (and hence belonging to $\{-1, 0, 1\}^\infty$). Moreover, \tilde{f} is monotone on K_α because of (P1). So it maps the endpoints of K_α onto the endpoints of K_β . Similarly, since K_1 is degenerate both endpoints of K_0 are mapped onto its only point. The conclusion is that $f(X) \subset X$ and the map $f : X \rightarrow X$ is well-defined (and it is clearly continuous). \square

Let $X_1 = \bigcup_{\alpha \in \{0, 1\}^\infty} \text{Bd}(K_\alpha)$. Note that, by (P3), $\bigcap_{n \geq 0} f^n(X) = X_1$. Let us see that f is chaotic while $f|_{X_1}$ is non-chaotic.

Theorem 3.2. $f|_{X_1}$ is non-chaotic and hence σ_f is non-chaotic.

Proof. Let $x, y \in X_1$ with $x \in K_\alpha$ and $y \in K_\beta$ for some $\alpha \neq \beta$, $\alpha, \beta \in \{0, 1\}^\infty$. We will see that there exists a positive real number M satisfying

$$\liminf_{i \rightarrow \infty} |f^i(x) - f^i(y)| \geq M,$$

and hence x and y cannot belong to the same scrambled set D . This proves that $\text{Card}(D) \leq 2$ for each scrambled set D of $f|_{X_1}$ and so $f|_{X_1}$ is non-chaotic.

Let j be the first positive integer satisfying $\alpha_j \neq \beta_j$. Suppose, for example, that $\alpha_j = 0$ and $\beta_j = 1$. For any $\theta \in \{0, 1\}^{j-1}$ consider the closed interval $A_{\theta*0}$ satisfying $K_{\theta*1} < A_{\theta*0} < K_{\theta*0}$ or $K_{\theta*0} < A_{\theta*0} < K_{\theta*1}$, and let $M = \min\{|A_{\theta*0}| : \theta \in \{0, 1\}^{j-1}\} > 0$. By (P3) and (P2), $f^i(x) \in K_{\theta*1}$ and $f^i(y) \in K_{\theta*0}$ or viceversa for all $i \in \mathbb{N}$ and for all $\theta \in \{0, 1\}^{j-1}$. This shows that

$$|f^i(x) - f^i(y)| \geq M$$

which concludes the proof. \square

For any $\alpha \in \{-1, 0, 1\}^\infty$ let $\tau(\alpha|_n) = \sum_{i=1}^n |\alpha_i| 2^{i-1}$ for all $n \in \mathbb{N}$.

Theorem 3.3. $f : X \rightarrow X$ is chaotic.

Proof. Define on $\{-1, 1\}^\infty$ the following relation: $\alpha \sim \beta$ if and only if there exists a positive integer k such that $\sigma^k(\alpha) = \sigma^k(\beta)$. Obviously \sim is an equivalence relation. Moreover, for $\alpha \in \{-1, 1\}^\infty$ the class of α is given by

$$[\alpha] = \bigcup_{k=0}^{\infty} \{\sigma^{-k}(\sigma^k(\alpha))\} \cap \{-1, 1\}^\infty.$$

Since $\{-1, 1\}^\infty$ is uncountable and $[\alpha]$ is countable for all $\alpha \in \{-1, 1\}^\infty$, the set containing all the equivalence classes $\{-1, 1\}^\infty / \sim$ is uncountable.

Let \mathcal{A} be a set containing one and only one representative $\alpha \in [\alpha]$ for all $[\alpha] \in \{-1, 1\}^\infty / \sim$, and let D be the set containing exactly one $x \in X \cap K_\alpha$ for all $\alpha \in \mathcal{A}$. We claim that D is a scrambled set for f . In order to see this take $x, y \in D$, $x \in K_\alpha$ and $y \in K_\beta$ with $\alpha, \beta \in \mathcal{A}$, $\alpha \neq \beta$. Then there exists an increasing sequence of positive integers $(k_i)_{i=0}^\infty$ satisfying $\alpha_{k_i} \neq \beta_{k_i}$ and $\alpha_j = \beta_j$ if $j \neq k_i$ for all $i \in \mathbb{N}$. Note that $\tau(\alpha|_n) = \tau(\beta|_n)$ for all $n \in \mathbb{N}$. Suppose, for example, that $\alpha_{k_i} = 1$ and $\beta_{k_i} = -1$ for some i . Then, by (P3),

$$f^{\tau(\alpha|_{k_i})}(x) \in K_{(0,0,\dots, \overset{k_i-1}{0}, 1)*\sigma^{k_i}(\alpha)} \quad \text{and} \quad f^{\tau(\alpha|_{k_i})}(y) \in K_{(0,0,\dots, \overset{k_i-1}{0}, -1)*\sigma^{k_i}(\beta)}.$$

By (P2), $|f^{\tau(\alpha|_{k_i})}(x) - f^{\tau(\alpha|_{k_i})}(y)| \geq |K_0|$ and then

$$\limsup_{n \rightarrow \infty} |f^n(x) - f^n(y)| \geq \limsup_{i \rightarrow \infty} |f^{\tau(\alpha|_{k_i})}(x) - f^{\tau(\alpha|_{k_i})}(y)| \geq |K_0|.$$

Let now $j \neq k_i$ for all $i \in \mathbb{N}$, and suppose that $\alpha_j = \beta_j = 1$ (the case $\alpha_j = -1$ is analogous). Then $f^{\tau(\alpha|_j)}(y), f^{\tau(\alpha|_j)}(x) \in K_{(0,0,\dots,0,1)}^j$. For any $n \in \mathbb{N}$ let $K_{\mathbf{0}|_n}^+$ and $K_{\mathbf{0}|_n}^-$ be the right and left side components of $K_{\mathbf{0}} \setminus K_{\mathbf{0}|_n}$. Applying (P4), for any $\varepsilon > 0$ there exists a positive integer n_ε such that $\max\{|K_{\mathbf{0}|_n}^+|, |K_{\mathbf{0}|_n}^-|\} < \varepsilon$ for all $n \geq n_\varepsilon$. By (P2) and (P3), $K_{(0,0,\dots,0,1)}^j \subset K_{\mathbf{0}|_j}^-$ or $K_{(0,0,\dots,0,1)}^j \subset K_{\mathbf{0}|_j}^+$, and so for $j \geq n_\varepsilon$ we conclude that $|f^{\tau(\alpha|_j)}(y) - f^{\tau(\alpha|_j)}(x)| < \varepsilon$. This proves that

$$\liminf_{n \rightarrow \infty} |f^n(y) - f^n(x)| = 0,$$

and the proof concludes. \square

Acknowledgement. This paper has been partially supported by the grant PB/2/FS/97 (Fundación Séneca, Comunidad Autónoma de Murcia).

I wish to thank the referee for the proof of Proposition 2.1 and the comments that helped me to improve the paper.

References

1. Franzová N. and Smítal J., *Positive sequence topological entropy characterizes chaotic maps*, Proc. Amer. Math. Soc. **112** (1991), 1083–1086.
2. Goodman T. N. T., *Topological sequence entropy*, Proc. London Math. Soc. **29** (1974), 331–350.
3. Gu Rongbao, *Topological entropy and chaos of shift maps on the inverse limits spaces*, J. Wuhan Univ. (Natural Science Edition) **41** (1995), 22–26. (Chinese)
4. Jiménez López V., *An explicit description of all scrambled sets of weakly unimodal functions of type 2^∞* , Real. Anal. Exch. **21** (1995/1996), 1–26.
5. Hric R., *Topological sequence entropy for maps of the interval*, to appear in Proc. Amer. Math. Soc.
6. Li Shihai, *Dynamical properties of the shift maps on the inverse limit spaces*, Ergod. Th. and Dynam. Sys **12** (1992) 95–108.
7. Misiurewicz M. and Smítal J., *Smooth chaotic functions with zero topological entropy*, Ergod. Th. and Dynam. Sys. **8** (1988), 421–424.
8. Paganoni L. and Santambrogio P., *Chaos and sequence topological entropy for triangular maps*, quaderno n. 59/1996, Universtá degli studi di Milano.

J. S. Canovas, Departamento de Matemática Aplicada, Universidad Politécnica de Cartagena, Paseo de Alfonso XIII, 34–36, 30203 Cartagena, Spain; *e-mail*: canovas@plc.um.es