Accepted Manuscript

Moiré interferences in the map of orbits of the Mandelbrot Set

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PII: S1007-5704(16)30216-7
DOI: [10.1016/j.cnsns.2016.06.016](https://doi.org/10.1016/j.cnsns.2016.06.016)
Reference: CNSNS 3903

To appear in: *Communications in Nonlinear Science and Numerical Simulation*

Received date: 19 January 2016
Revised date: 12 April 2016
Accepted date: 9 June 2016

Please cite this article as: Pedro María Alcover Garau, Moiré interferences in the map of orbits of the Mandelbrot Set, *Communications in Nonlinear Science and Numerical Simulation* (2016), doi: [10.1016/j.cnsns.2016.06.016](https://doi.org/10.1016/j.cnsns.2016.06.016)

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Highlights

From old Zeno's paradox of Achilles and the Turtle to modern Superstring Theory or Planck's Quantum Theory, the debate over the Continuous and the Discrete has always been a disconcerting subject that has been kept alive by the Physics and Philosophy fields. At the same time, mathematics has been able to create a perfect abstraction to dominate the Continuous in the uncountable set of real numbers. The uncountable nature of real numbers is able to create mathematic wonders such as the Mandelbrot Set. Real numbers allow for expressing any magnitude with an infinite precision; however, when we use them to model the behavior of a reality considered as continuous, we then must operate on discrete samples, that is to say, we must demarcate with a finite precision. We cannot manage the Continuous, value by value: there are always infinite real numbers between two given real numbers. Then, what would happen if we studied from a discrete point of view a reality that we actually consider as continuous but it turns out that such reality is discrete instead of continuous? And, if we confront these two discrete layers (one from mathematical sampling and the other from reality) and both have similar orders of magnitude, or if we sample spaces by taking sampling intervals of the Planck length (about $1.63\text{E-}35$ meters) or even if we sample time by taking sample intervals of the Planck time (about $5.39\text{E-}44$ seconds), what would happen then? The result would be that we would find Moiré interference patterns and our perception of reality would be screened by such interferences. This is just a theory, but let's see how far it takes us.

Moiré interferences in the map of orbits of the Mandelbrot Set.

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Abstract

This article shows the presence of Moiré Interference patterns in the map of periods of the Mandelbrot Set. It describes the requirements for their appearance and shows that such interferences are highly sensitive to the original conditions that define their calculation. The specific case herein studied shows that the Moiré interference patterns appearing in a picture of a section of the map of orbits are unpredictable, even if we obtain different maps from very similar original conditions. It begins with a brief description of the Mandelbrot Set and some of the characteristics of its orbits, the Moiré Patterns, as well as a concise introduction to a description of the Discrete Wavelet Transform. In order to develop the proposed specific case, a Multi-resolution Analysis method based on the Discrete Wavelet Transform has been used. It is significant that Moiré Interference Patterns always appear when the order of magnitudes reaches a certain limit where, what is considered as hypothetically continuous, behaves as a discrete pattern. The patterns as shown by the Wavelet analysis change drastically at the slightest modification in the original calculation conditions and it does not seem possible to predict their shape beforehand. This article ends with some conclusions and suggestions.

Keywords: Mandelbrot set; Moiré Patterns; Space-time discretization; Real numbers discretization

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1. Introduction and Motivation

The Mandelbrot Set shows the behavior of the Complex Numbers in quadratic operations carried on in a recursive way. The Mandelbrot Set is well known, as well as its striking graphic representation thanks to the computation tools that any computer offers nowadays.

In the light of the results of this work, the Moiré Interference Patterns turn up in a picture, either captured or created through computation, as a consequence of its digitalization or its pixel-based presentation on screen when such captured or presented picture contains some kind of grid structure.

To the best of our knowledge, there is no mathematic relationship between the Mandelbrot Set and Moiré Patterns. They are two different mathematic realities. In fact, no scientific research literature shows references to the presence of Moiré Pattern in Mandelbrot Set. However, it is possible, as shown in this article, to find Moiré interferences in some of the Mandelbrot Sets representations. We wonder about the source of these findings: which grid structures account for them and which origin it may have.

This article tries to show how and where it is possible to find some of the Moiré Interferences Patterns within the map of orbits of the Mandelbrot Set. In order to offer a self-contained and easy to read this work, it seems convenient to present an introduction to the orbits in the Mandelbrot Set, plus another one devoted to the origin and characteristics of Moiré Interferences Patterns. It is possible to approach a deep research about the presence of Moiré in Mandelbrot in two ways: (a) from a formal mathematical point of view; and (b) through the experimentation and simulation with computer programs. In this article, the second option has been chosen, and a study about a set of interference patterns with Wavelet Techniques has been done. Accordingly, a third subsection in this introduction is devoted to present some aspects of these transforms.

The article is structured as follows. After the introductory section, the concept of map of orbits within the Mandelbrot Set follows. A list of the terms used in this article, given at the beginning in this section, makes it possible

to refer to the different computed images as well as to the several parts of them. This section is necessary since the Interference Moiré Pattern appears within these different maps of orbits. This section questions the origin of these interferences. A correlation between the precision of the floating-point type
35 data and the timing of the Moiré Interference Pattern is observed. Is it possible to infer from this observed correlation that the cause of these interferences can be found in the truncation errors in the floating-point data type? Is Moiré a consequence of the discretization in the pixel computation from the picture of the orbits map? Does Moiré appear sooner or later depending on the arithmetic
40 precision considered? Would Moiré show up if the calculations were done by any technology allowing infinite precision?

Along the following sections, an answer to these questions is offered through a study case about these interferences pictures. The fourth section deals with a study case. Furthermore, several Moiré Interference Patterns in the map of
45 orbits of the Mandelbrot Set are shown. The fifth section describes the process to analyze the different sets of computed pictures: up to a total sum of eight sets, each comprising 1,025 images of near the same area of the complex plane. This same section shows, in two different subsections, two procedures followed in the computing and analysis of the Moiré images of the Mandelbrot orbits map. Out
50 of each one of these eight sets of pictures a new image of differences is obtained; and each one of these eight images is analyzed through 40 Wavelet Transforms. Afterwards, the followed procedure and its results are equally shown.

Finally, the sixth section suggests some conclusions and tasks that might be developed in the future.

55 2. Background

The purpose of this section is to offer a brief theoretical description of Mandelbrot Set and its orbits map, as well as about Moiré Interference Patterns according to the circumstances in which they turn up, and the conditions that foster its hipper-sensitivity qualities; and finally about Wavelet Transform. Un-

avoidably, these descriptions must be short: there is no intention to offer a complete description for every topic, but only the minimal necessary information to make it possible for the article to be self-contained. Further introductions on these topics are available in the scientific literature.

2.1. The Mandelbrot Set

There is already a long way from the appearance of Fractals. Fractals may be described as mathematical objects, the final product of the iteration of a geometrical process that is normally very simple. Such iteration, which may be infinitely recursive in theory, provides the final result (fractal) with an extraordinary apparent complexity. The most representative fractals are the Mandelbrot Set and the infinite Julia sets (Mandelbrot's predecessors) [1, 2]. The appearance of fractal objects, as well as of their proliferation thanks to the computing and graphics representation power of modern computers, has contributed to the spread of a new type of geometry and to an increasing interest for it.

As is well known, to determine whether a point belongs to the M-Set or not, we take such point, square it and finally add it. Then, we square the resulting point and add the original point again repeating this same operation over and over again... If the resulting points or the intermediate values obtained from this infinite iteration are always bounded in the 2 radius circumference, then the original point belongs to the M-Set. If these results diverge towards infinity, which happens whenever a point or intermediate value is not bounded within the said 2 radius circumference, then the original point does not belong to the M-Set. Obviously, infinite iteration is not possible, and this process determines whether such point belongs to the M-Set or not only if, after a preset number of iterations, the subsequent values remain bounded in the 2 radius circumference.

The series of points resulting from the subsequent iterations performed after every original point are called Orbits. Mandelbrot's expression defines an orbit for each point in the complex plane. What we have here is a discrete dynamical system of the form $z_{n+1} = f(z_n) = z_n^2 + z_0$, where, for each point $z_0 \in \mathbb{C}$ the orbit is $\{z_0, z_1 = f^1(z_0), z_2 = f^2(z_0), z_3 = f^3(z_0), \dots\}$ and where point

90 z_0 belongs to the M-Set when its orbit remains bounded. This means that $f^n(z_0)$ does not tend to infinity when n tends to infinity.

Unlike traditional geometrical figures, such as circumferences, ellipses or parabolas, the M-Set, and in general any fractal object, does not use shortcuts to determine the points that are part of it. There is no analytic equation for
95 defining the geometry of the M-set whereas the geometry of a circumference of radius r and center (x_0, y_0) actually has one. In fractal geometry, equations express an iteration, a process. One point belongs to the set defined by an equation, not when it provides a solution for such equation, but when it meets a certain behavior during the iterative process defined by the said equation [3].

100 The most outstanding feature of fractals is self-similarity. A self-similar object is similar to a part of itself. The M-Set contains infinite copies or replicas similar to the whole set; however, none of these infinite replicas are identical [4]. This is an infinite complexity resulting from an extremely simple equation that is iterated an infinite number of times. Its infinite complexity is the result of
105 its infinite iteration. Such infinite capacity to iterate is due to the fact that it operates in the complex plane, where each of its two dimensions is defined by an uncountable set. The M-set is a numerical set that, due to its nondimensional nature, expands in the same way and infinitely at any scale we wish to work with.

110 When a fractal is the image of the strange attractor of a dynamical system (and Mandelbrot's equation acts as such), the said system is also very sensitive to any change made in the original conditions: the Lorenz attractor [5] and the well-known Butterfly effect are good examples. This hypersensitivity is obviously reflected in the graphic image of the attractor and it is only feasible
115 when we work on uncountable numerical sets. As we approach the boundaries of M-Set, orbits became more and more chaotic. In this regard, it is possible to consult the paper of Ragulskis et al. [6], where a computer-algebraic criterion based on the autocorrelation function and Laplace-Borel transformation for the onset of chaos in nonlinear dynamical systems is proposed. In its fifth section
120 the authors show that the orbits became chaotic as the points get closer to the

boundaries of M-Set. It is only in the boundary of the M-Set where we can find two points as close to each other as we wish: one point has a bounded orbit while the orbit of the second point diverges after a finite sequence of iterations. The boundaries, not the inside or outside, of the M-set has a fractal structure.

125 Between two points close to the boundaries, one belonging to the M-Set and the other laying out of it, there are always infinite points that do not belong to the M-set and others that actually belong to it. The boundary value cannot be obtained because it is not analytically defined.

The fact that the orbit of any point of the M-Set may be bounded within the circumference of radius 2 raise questions on the nature of such orbit. There is a wealth of information on the different potential orbits present at the M-Set [7]. This potential orbits will verify one of the following behaviors:

- Orbit attracted by a fixed point (formed by just one point). The point in the M-Set whose orbit is formed by just a single point is: $z_0 = (0, 0)$.
- 135 • p -Periodic Orbit: It is formed by different p points. It confirms that $f^p(z_0) = z_0$. Obviously, this is a periodic orbit too with period $k \times p$, where k is a positive integer.
- Fixed point orbit with pre-period m : It reaches to a fixed point after m previous iterations.
- 140 • Period p orbit with pre-period m : It reaches a sub-orbit of period p after m initial iterations.
- Asymptotically fixed point orbit: Subsequent iterations set the orbit increasingly closer to a specific fixed point.
- 145 • Asymptotically periodic orbit with period p : Subsequent iterations set the orbit increasingly closer to a p -periodic orbit.
- Aperiodic orbit: The orbit is none of the aforementioned orbits.

The M-Set is formed by a main cardioid to which an infinite number of disk-like bulbs are attached. As the M-Set is a fractal, such structure is infinitely

iterated at different scales. Thus, there are infinite cardioids and each cardioid
 150 has an infinite number of disk-like bulbs attached to it that, at the same time,
 are attached to other disc-like bulbs, and so ad infinitum. All these cardioids
 and bulbs have a common property: the final period of their orbits (with or
 without pre-period) is the same for all their points. For instance, all the points
 of the main cardioid of the M-Set have a fixed point orbit, or a fixed point
 155 orbit with its pre-period, or they form asymptotically fixed point orbits. Each
 of the bulbs attached to the left side of the cardioid, one after the other and
 increasingly smaller, is formed by points of period 2, 4, 8, 16 and so on until they
 reach period 3 (the Sarkovskii Theorem [8]) and other periods (which are usually
 orbits with a pre-period or asymptotically periodic orbits). Each attached bulb,
 160 located both up and down the main cardioid, has its own period, and each bulb
 has its own bulbs attached with periods that are a multiple of the period of the
 primary bulb.

It is very interesting to visualize the values of the periods of each orbit for
 the different points of the M-Set. We can assign different colors to such points
 165 depending on the period of their orbit. The given name to the resulting image
 is "Map of Orbits".

2.2. *Moiré Patterns*

It is well known that when we superimpose two grids/periodic structures,
 we create a new periodic structure/grid that was not present in the two original
 170 ones. These structures are known as Moirés structures, patterns or effects and
 they may also appear when we superimpose two grids formed by discrete patterns.
 Sometimes, we can create Moiré patterns by overlying two grids with the same
 lines pattern while we displace or rotate a small amount, or with a slight change
 in the scale, from one another [9]. The said grids may be formed from dots,
 175 straight lines or curves that could be either parabolas or any other shape. Moiré
 Patterns have been comprehensively studied and their mathematical bases are
 already well known [10]. A Moiré Pattern is the interference of two or more
 similar superimposed patterns; therefore, they are considered as interference

patterns and this article refers to them as “Moiré Interference Patterns”.

180 According to Amidror [10], the theory of the Moiré phenomena is an interdisciplinary subject with an extremely wide scope of application: physics, optics, mechanics, mathematics, images reproduction, human vision, and so on. It can be also applied to very different fields, such as the detection of slight object deviations or deformations, the measurement of angles of a very small
185 size or the detection of very slight displacements or movements.

Moiré Interference patterns may also appear when two superimposed grids or patterns are not periodic. The figures resulting from the superposition of random dot patterns are known as Glass Patterns [11], which are named after Leon Glass, the first scientist to describe them [12].

190 Moiré Interference Patterns are present in any discrete physical phenomenon in nature, but more frequently in the creation of digital images. Sometimes their appearance is sought intentionally, or different patterns are designed or created to cause their appearance on the plane, frequently for artistic reasons. Moiré Patterns can be very useful in fields like Interferometry, where such patterns are
195 used to measure slight displacements that take place in thin surfaces and materials [13]. However, most of the time, Moiré Patterns are found inadvertently, for instance, when a digital camera captures the content of a pixel screen [14].

In general, it is assumed that all computer processes, where it is inevitable to discretize any amounts set, are liable to produce Moiré interference patterns.
200 It is open to question whether this phenomenon takes also place in the real world, where there is always a dimension that can be no longer divided, and a discrete and net-like structure is obtained in any case. In an abstract way, it is always possible to work with the vertiginous unaccountability of real numbers. However, it is common that the reality of our physic world ends up frequently,
205 even always, by presenting discrete values, like quantic dimensions for instance. If mathematical procedures are used with values in such dimensions, the appearance of Moiré interference patterns will perhaps be inevitable, as it is in the case of the Mandelbrot Set.

It is widely known that, under certain conditions, Moiré interference pat-

terms may be very sensitive to a slight change in the original conditions [14, 10].
 The aforementioned simple case corresponds to the superimposition of two periodic grids (grid 1 and grid 2) made of parallel straight black lines with different periods, which we shall call p_1 and p_2 ($p_1 \neq p_2$ and assuming $p_1 < p_2$). If p_1 and p_2 are too similar, even though their values are low, the result could be an interference pattern with a very wide period. It all depends on the difference between the periods of the initial grids, not on the size of such periods. When very small displacements take place (between 0 and p_1 inclusive) in the relative position of the original grids, the appearance of the resulting grid change drastically. What we have here, then, is a system that may prove to be highly sensitive to its original conditions.

We can find similar behaviors in different original grids. There are always certain conditions under which the resulting interference pattern is extremely sensitive to the slightest change in the position, direction or size of the original grids. Many of the technical applications of Moiré interference patterns rest precisely on their hypersensitivity with regard to their original grids.

2.3. What is a Wavelet Transform?

The performance of a multiresolution analysis by means of the Wavelet Transform is extensively documented [15, 16, 17, 18]. The success of this resource is justified on the ground that it analyzes any signal in the frequency and time domains at the same time. In other words, it is able to perform a time-frequency analysis either with high frequencies components in small fragments of time or with low frequencies components in both periods of time.

The Discrete Wavelet Transform (DWT) allows for analyzing digital signals; the algorithm on which it is based is simple and quick to develop. The process can be reversed and it is elegant, from a mathematical point of view. It is based on the structure of filter banks designed as a Finite Impulse Response, which are defined by means of a finite set of coefficients. The filter banks of a Wavelet Transform are formed by two filters: low-pass filter and band-pass filter. These filters are applied on the discrete signal that is going to be analyzed by means

240 of a circular or symmetric convolution operation. The output signal of the first filter (low-pass) is similar to the input signal, but softened, without abrupt discontinuities. Later, by means of a down-sampling (consisting in taking only the values placed in even position), a signal reduced by half is finally obtained, which is known as the approximation signal or scale signal. The second filter 245 helps to detect discontinuities and to get the so-called detail signal, which is also obtained by means of down-sampling. The result is an output signal with the same size as half the size of the input signal. Both signals complement each other.

The synthesis process is carried out in reverse order to the order of the 250 analysis process.

The coefficients of the filters define two bound wave signals in a univocal way: one is the so-called Scaling function and the other is the Wavelet function. Both functions may be encoded by means of a finite set of coefficients, which are used to define filters. The scaling function (φ) provides the coefficients of 255 the low-pass filters, while the Wavelet function (ψ) provides the coefficients of the band-pass filters.

The algorithm used for the analysis process, as well as for the synthesis process, may be applied recursively. This allows for performing the analysis at different scales or resolution levels. Two new approximation and detail signals 260 are obtained in each scale. Thus, from an original signal, that can be considered as the first approximation signal of size N , a detail signal of size $N/2$ is obtained and a new approximation signal (also of size $N/2$). When this new approximation signal is processed again in the recursive algorithm, provides a new detail signal of size $N/4$ and a new approximation signal (also of size $N/4$), which will 265 be processed again in order to get a new detail signal of size $N/8$... and so on. If the input signal is dyadic, for instance $N = 2^J$, then the process may be iterated J times until getting an approximation signal of size $N/2^{j=J} = 1$, as well as subsequent detail signals of size $N/2^j$ for values of $j = J - 1, J - 2, \dots, 2, 1$. This way, it is possible to carry out a complete multi-resolution analysis with as 270 many values as the original signal analyzed, where discontinuities are detected

at different sequences in each level. It is always desirable to know how to determine the level of decomposition to which it is appropriate to reduce the signal and the point from where we no longer get results after applying the algorithm in a recursive way.

275 We can apply this algorithm also on discrete signals with more than one dimension, such as images, for instance, which are two-dimensional signals. The Discrete Wavelet Transform in two dimensions (DWT-2D) has proven to be a relevant tool for texture analysis. This algorithm is a generalization of the algorithm developed for 1D images. Taking the coefficients of the high level
280 scale, which are now distributed in a matrix $(M \times N)$, we first apply algorithm DWT-1D on rows to obtain two intermediate matrices whose size is $M \times N/2$: one matrix is the output signal of the scaling filter (low-pass) and the other is the output signal of the Wavelet filter (band-pass). Then, we apply algorithm DWT-1D again, this time on the columns of the said two matrices, to obtain 4
285 matrices of size $M/2 \times N/2$.

The image we obtain after applying the scaling filter twice is the approximation signal. The three remaining matrices, which result from applying the pass-band filter (or Wavelet filter) at least once, are the detail signals. The result of applying the pass-band filter on rows and the low-pass filter on columns
290 is the so-called *horizontal detail signal or image*, which contains horizontal discontinuities; the result of applying the low-pass filter on rows and the band-pass filter on columns is the so-called *vertical detail signal or image*, which contains vertical discontinuities. Finally, the signal resulting from applying the band-pass filter on rows and columns is the diagonal detail signal or image, which
295 includes diagonal discontinuities. As in the case of the unidimensional signals, this case also requires dyadic signals, whose number of rows and columns are raised to the power of 2. The resolution of the signal/image can be reduced as many levels as the exponent of the said power indicates. Likewise, it is advisable to determine in each process the level of decomposition to which it is
300 appropriate to reduce the signal/image.

We can analyze the original image/signal at a new scale or dimension in

each level of decomposition. This allows us for finding periodical structures at different temporary scales. For every level of decomposition j , we can gather information on the frequencies that are present at resolution level $J - j$. We should count on mathematical methods that allow for measuring the information level contained in each approximation image or detail image for every new level of decomposition. The analysis of the images has required calculating the Shannon Entropy [19] for each of the four sub-images, in each decomposition level. Thus, we obtain four entropy values for each level of decomposition j .

“The four entropy values that are calculated in each level of decomposition correspond to the values of normalized entropy; in other words, they correspond to the values resulting from dividing the entropy values between N (the total number of pixels in the image).”

For each level of decomposition, we calculated four values R_{ab}^j in accordance expression 1

$$R_{ab}^j = \frac{S_{ab}^j}{S_{LL}^j + S_{LH}^j + S_{HL}^j + S_{HH}^j} \quad (1)$$

where a and b take values H and L .

And, as the algorithm presented by Fernández et al. [20] explains, the ideal level of decomposition is that which offers a greater and most significant variation between two consecutive values of R_{ab}^j . This way, we calculate a collection of values ADR_{ab}^j (4 for each decomposition level) in accordance with expression 2.

$$ADR_{ab}^j = \begin{cases} 0 & : \text{if } j = 0 \\ R_{ab}^j - R_{ab}^{j-1} & : \text{if } j \neq 0 \end{cases} \quad (2)$$

This algorithm is already explained by Fernández et al. [20]. If the maximum value of ADR_{ab}^j verifies that it is considerably higher than the rest, then, we could say that we have found a pattern in resolution Jj . Otherwise, if the maximum value is similar to the rest of the calculated values, then we may conclude that such image does not contain a texture pattern. This algorithm

allows for detecting textures that, in this case, we could treat as periodicities that occur in the distribution of pixels inside the image.

3. Map of Orbits of the M-Set when a truncation error takes place

330 In order to describe the computed images displayed in this article, the following vocabulary may be necessary:

1. **ULV**: This abbreviation refers to the upper-left vertex of any given image and corresponds to the two coordinates of a point.
2. **Size**: This term refers to the size of the image; it is usually described
335 as the product of two integers that indicate the number of pixels in each row and column of the image. Each image has as many pixels as its Size indicates.
3. **Increment**: Starting from the ULV, a grid of points is created in the Complex plane with as many points as indicated with Size parameter.
340 The grid cell size is named Increment.
4. **LRV**: This abbreviation refers to the lower-right vertex, which is the last point resulting from using the ULV to apply the increment on each row and column of the image.
Then, it is obvious that ULV, Size, Increment and LRV are non-independent
345 values and any of them is clearly determined by the other three.
5. The section of the complex plane that is inside the quadrilateral between the ULV and the LRV has been named as the **Section of the Complex Plane**. It is evident that, whatever the section, it will always require a discrete sampling: as many singular points as the Size of the image
350 indicates.
6. Each pixel resumes, with just one value, the information of an infinite set of points: all the points that are located between the point represented by the said pixel and the point represented by the diagonally adjacent pixel. The surface of the complex plane that is represented by just one pixel, has

355 been named **Pixel Surface**. Obviously, the Pixel Surfaces size equals the square of Increment value.

In order to calculate images of the Mandelbrot Set, it is necessary to consider a series of points located in a square grid of the complex plane. As all the points of the M-Set are inside the radius 2 circle, which is centered at the origin of the complex plane, such grid is always defined inside the said circle. All the
360 points in the grid have been obtained from the said increment and from ULV coordinates. The color value for each one of them, which indicates whether that point belongs to the M-set or not, as well as the period of the orbit for such point, is represented by one pixel.

365 The set of points of the M-Set can be displayed with color in certain areas and indicating the period of the points inside each of these areas. Theory suggests that all the points in each disc-like bulb and/or cardioid share the same period. However, when we calculate the periods of points that have been randomly selected in the different areas of the M-Set, we not always, not even usually,
370 obtain the period predicted by Mathematics. This is due to the fact that the value domain of a specific type of data, for instance, the `double` type data, is not a set that can be clearly considered as uncountable. These types of data, whose range is necessarily finite, always bring a truncation error along. Maybe, the magic of fractals, and especially of the M-Set, lies in the lack of countability
375 of the domain of real and complex numbers. Nevertheless, such calculation will never be able to represent the infinity of the complex plane, but always just a finite selection of points.

Figure 1 shows some examples of orbits. Every one of them represents the orbit of different points z_0 , which are “close” to the tangency area between the
380 main cardioid and a period 5 bulb. The Figure shows the way the orbits of different points z_0 change: the two first images correspond to points z_0 located inside the bulb; the next six images correspond to points located inside the main cardioid. The last image represents a point that is outside the boundary of the M-Set but close to it. Points z_0 get close and away progressively from

the boundary between the bulb and the cardioid. Table 1 shows information on points z_0 and the resulting orbits. Figure 1. (c) represents an orbit that, after 100,000 iterations, has not been able to converge to a periodic orbit yet. The calculated period for all points was 5, except for the aforementioned point and the last point, which has a divergent orbit.

Figure 1: Images of different orbits drawn from different points of the Mandelbrot set (except for Image i). The point of origin of the orbit is highlighted in green color in each of the figures. For all the points shown, the period of the calculated orbit was 5, although the orbits of image (c) to image (h) belong to the main cardioid. The image displayed in Figure (i) corresponds to a point that does not belong to the M-Set; its orbit diverges after 4050 iterations.

The map of orbits of the M-Set points calculated with `double` or `decimal` Data Type can be represented. As it is usual in the M-Set displays, such map of orbits can be enlarged. A code made up of 20 different colors has been used to represent periods 1 (fixed point) to 20 in all the figures displayed throughout this article. Any point of the M-Set whose period is higher than 20, or whose

	X_0	Pre-period	Period	M-Set	Orbit points
(a)	$-0.47908855357415,$ $+0.536363867296652$	2835	5		$-0.433744143358848,$ $-0.550876756081173,$ $-0.184477428335134,$ $-0.632280006020782,$ $-0.221228221958436,$ $+0.509825641180516$ $+0.094096095304215$ $+0.432693163814470$ $+0.376719623059279$ $+0.0599792962245163$
(b)	$0.47908855357415,$ $+0.534463750444546$	10730	5		-0.422667850045817 -0.550160275148574 -0.188963573929833 -0.612459981651181 -0.247670251335955 $+0.499719754499236$ $+0.112032801925316$ $+0.411191756178752$ $+0.379063222808503$ $+0.0701416414726795$
(c)	$-0.47908855357415,$ $+0.533513692018493$	Unknown period after 100000 iterations
(d)	$-0.477188436722044,$ $+0.534463750444546$	70671	5		$-0.402679283763734,$ $-0.402679283763694,$ $-0.402679283763686,$ $-0.402679283763739,$ $-0.402679283763661,$ $+0.296042991158588$ $+0.296042991158519$ $+0.296042991158598$ $+0.296042991158539$ $+0.296042991158555$
(e)	$-0.474338261443886,$ $+0.534463750444546$	13846	5		$-0.401253179534977,$ $-0.401253179535010,$ $-0.401253179535006,$ $-0.401253179534980,$ $-0.401253179535026,$ $+0.296511436841908$ $+0.296511436841946$ $+0.296511436841896$ $+0.296511436841939$ $+0.296511436841920$
(f)	$-0.477188436722044,$ $+0.53256363359244$	14319	5		$-0.402367936947877,$ $-0.402367936947848,$ $-0.402367936947919,$ $-0.402367936947833,$ $-0.402367936947900,$ $+0.295092285411689$ $+0.295092285411770$ $+0.295092285411722$ $+0.295092285411719$ $+0.295092285411771$
(g)	$-0.474338261443886,$ $+0.529713458314281$	4781	5		$-0.400472744259578,$ $-0.400472744259573,$ $-0.400472744259604,$ $-0.400472744259559,$ $-0.400472744259601,$ $+0.294130756145094$ $+0.294130756145141$ $+0.294130756145106$ $+0.294130756145115$ $+0.294130756145135$
(h)	$-0.468637910887568,$ $+0.521162932479805$	1781	5		$-0.396203218508696,$ $-0.396203218508690,$ $-0.396203218508683,$ $-0.396203218508700,$ $-0.396203218508679,$ $+0.290761582706126$ $+0.290761582706105$ $+0.290761582706126$ $+0.290761582706114$ $+0.290761582706113$
(i)	$-0.477188436722044,$ $+0.535413808870599$	Divergent orbit $ x_i > 2$ from iteration 4050	2

Table 1: Coordinates of those points whose orbits are shown in Figure 1

	Upper-left Vertex (ULV)	Lower-right Vertex (LRV)	Increment by Pixel
(a)	-0.687146268353401, +0.261151755321851	-0.495234508105739, +0.140336838896510	0.00016572690867673800
(b)	-1.053686706534060, +0.228759636378782	-1.049906163159920, +0.226379657000763	0.00000326471793967020
(c)	-1.192857205081890, +0.135524618704407	-1.188664910601110, +0.132885428137488	0.00000362028884351047
(d)	+0.110260572460328, +0.566404705947289	+0.114136243234845, +0.563964840666958	0.00000334686595381500

Table 2: Data from images shown in Figure 2. All images have been calculated taking an 1158×729 pixels Size

orbit could not be determined through the corresponding calculation process, is herein represented in white color. All points not belonging to the M-Set are represented in black color. To determine the periods, 15,900 iterations on each point in order to exceed a potential preperiod m for each orbit were originally performed. After this first process, if the orbit was not divergent, then 100 more iterations, to try to determine the period of the orbit of the points, were performed. The necessary code to obtain all the images and perform their subsequent processing has been implemented in C# by using the Microsoft Visual Studio Professional 2013 tool. The `double` Data Type by default has been used. The images displayed in Figure 2 show areas that are clearly defined and perfectly delimited by straight lines. Table 2 shows the coordinates of the ULV and LRV, as well as the Increment. All images have been calculated taking an 1158×729 pixels Size.

If we enlarge several areas of the images, we will see that the distribution of the different periods is apparently random. If we keep on enlarging the said areas, when the value of the increment is about $1.0E - 8$ and we use `double` Data Type variables, some classic structures of the Moiré interference patterns appear in the image. Figure 3 shows some of these images and Table 3 provides information on them. By taking certain values for the Increment, Moiré interference patterns appear, in one way or another, in all the map of orbits made at a greater scale.

If we work with changeable-precision data types, we can vary the moment in which Moiré interference patterns appear, but we won't be able to prevent them in any case. Regardless of the extension of the mantissas and exponents

Figure 2: Images of the map of periods that have been calculated on areas of the Mandelbrot set. They show perfectly delimited areas of behavior and simple geometric structures.

	Upper-left Vertex (ULV)	Lower-right Vertex (LRV)	Increment by Pixel
(a)	-1.191845462624830, +0.134480685753174	-1.191832894234620, +0.134472773528247	$1.085353213575000E - 08$
(b)	-1.190569141131450, +0.135151528962825	-1.190563743611390, +0.135148131042164	$4.661070865470000E - 09$
(c)	-1.191801908265370, +0.134445022747539	-1.191798489897310, +0.134442870769717	$2.951958603480000E - 09$
(d)	-0.354370698459464, +0.567481446969675	-0.354369728073515, +0.567480836079039	$8.379844114800000E - 10$

Table 3: Data from images shown in Figure 3.

of our representations, we will always be working with limited precision values.

420 Figure 4 shows the map of periods resulting from working with a data type that has a 4 decimal digits mantissa. This figure shows the map of periods of the upper side of the main cardioid. For data types of this size, the Moiré interference patterns in the Mandelbrot Set already appear with increment values of about $1.0E - 3$. Is the presence of these patterns in the map of orbits a consequence of the limited mantissas and the truncation error that such limitation adds to the calculation?

425

Figure 3: Images of the map of periods that have been calculated on areas of the Mandelbrot set. They show different Moiré patterns. Table 3 shows the data to locate these patterns.

4. Study on a specific example of Moiré Interference Patterns

Figure 3 shows images that cannot be easily computed again if the original data used to create them are not comprehensively known. Table 3 shows the coordinates of the ULVs and LRVs of these images on the complex plane, and their Increment values. Just a slight variation in one of the decimals of the ULV (upper-left vertex) coordinates, from which the image is calculated, gives rise to a significant change in the resulting image. The effects could be even greater if the value of the increment is slightly modified.

A study on one of the Moiré patterns shown in Figure 3, image Img644 (c), has been conducted. The Size has been reduced because the said study required the creation of thousands of images from the aforementioned image, and the computation time required for calculating these images is not exactly short. Figure 5 (a) shows a section of the said image, which Size is 400×300 pixels. The study herein presented was based on such section. A total of 10 different values of period define this image: 255 pixels of period 2; 45,812 pixels of period 4 (38.18% displayed in green color); 206 pixels of period 6; 209 pixels of period

Figure 4: Map of periods of the Mandelbrot set that have been calculated on a type of data with a precision limited to 4 decimal digits. Each period has been calculated after having performed 100,000 original iterations on the point. For a mantissa of this size, the Moiré interference patterns appear almost immediately when the value of the increment between the pixel rows or columns is about $1.0E - 3$.

8; 71,574 pixels of period 10 (59.65% displayed in red color); 616 pixels of period 14; 1 pixel of period 16; 1,030 pixels of period 18; 12 pixels of period 20; and 285
445 pixels of period 24. Total sum: 120,000 pixels. The two most frequent periods are period 4 (green color) and period 10 (red color). Figures 5 (b) and (c) show the images of these two main periods in white and black color: Figure 5 (b) represents points of period 4 in white color, or otherwise, in black color; Figure 5 (c) represents points of period 10 in white color, or otherwise, in black
450 color. Row (a) of Table 4 shows the coordinates of the ULVs and LRVs of this image on the complex plane. The resulting value of the Increment is $3.0E - 9$, as shown in Table 4.

Figure 5: 400×300 pixels Moiré interference patterns. Image (a) shows the interference pattern, where up to 10 different Mandelbrot periods take part. The values of presence of period 4 are represented by the white dots shown in Image (b) and the values of presence of period 10 are represented by the white dots shown in Image (c). Both periods have the strongest presence and are used to carry out the analysis.

	Upper-left Vertex (ULV)	Lower-right Vertex (LRV)	Increment by Pixel
(a)	-1.191800609285510, +0.134444678431087	-1.191799428502070, +0.134443792843506	$2.95195860348E - 09$
(b)	-1.191800609285520, +0.134444678431097	-1.191799428502060, +0.134443792843516	$2.95195860348E - 09$
(c)	-1.191800609285530, +0.134444678431107	-1.191799428502050, +0.134443792843526	$2.95195860348E - 09$
(d)	-1.191800609285540, +0.134444678431117	-1.191799428502040, +0.134443792843536	$2.95195860348E - 09$
(e)	-1.191800609275270, +0.134444678441319	-1.191799428491770, +0.134443792853738	$2.95195860348E - 09$
(f)	-1.191800609285510, +0.134444678431097	-1.191799428502070, +0.134443792843516	$2.95195860348E - 09$
(g)	-1.191800609285510, +0.134444678431107	-1.191799428502070, +0.134443792843526	$2.95195860348E - 09$
(h)	-1.191800609285510, +0.134444678431117	-1.191799428502070, +0.134443792843536	$2.95195860348E - 09$
(i)	-1.191800609285510, +0.134444678441319	-1.191799428502070, +0.134443792853738	$2.95195860348E - 09$
(j)	-1.191800609285520, +0.134444678431087	-1.191799428502060, +0.134443792843506	$2.95195860348E - 09$
(k)	-1.191800609285530, +0.134444678431087	-1.191799428502050, +0.134443792843506	$2.95195860348E - 09$
(l)	-1.191800609285540, +0.134444678431087	-1.191799428502040, +0.134443792843506	$2.95195860348E - 09$
(m)	-1.191800609275270, +0.134444678431087	-1.191799428491770, +0.134443792843506	$2.95195860348E - 09$

Table 4: Values that identify the images shown in Figures 5 and 6. Image (a) is the first image calculated; it is shown in Figure 5. The rest of images are obtained by readjusting the real and/or imaginary axis of their Upper-Left Vertex in $1.0E - 14$. The first 4 values correspond to the first three images and the last image calculated by carrying out a diagonal displacement of the ULV. The second group of values correspond to those images obtained from a displacement with a constant real value. Finally, the third group of values correspond to those images obtained from a displacement of the ULV in the imaginary axis.

Figure 6: 400×300 pixels Moiré interference patterns. Table 4 shows the coordinates of their ULVs and URV.

A simple study on this Moiré interference pattern and its variation by making a very slight change in its ULV has been conducted. To this end, 1,025 images similar to the image shown in Figure 5 has been calculated by increasing the real and imaginary ULV coordinates up to $1.0E - 014$ for each image. Besides, another 1,025 images has been calculated by increasing only their real coordinate while keeping the imaginary coordinate constant. Finally, another 1,025 images resulted after applying the same increment, but only in their imaginary coordinates. Figure 6 shows the 3 first images of the aforementioned three sequences (b, c, d, f, g, h, j, k, l), as well as the last image of each one of them (e, i, m). Table 4, from row (b) on, shows the ULV and LRV coordinates for each one of the said images. The value of the increment is the same for all of them.

Each and every coordinate of the 3,075 ULV of the images calculated, are located on a minimum section of the pixel surface of the ULV of the first image. As it is mentioned before, the displacement of the successive ULVs is $1.0E - 14$, while the value of the increment is about $3.0E - 9$. The distance between the ULVs of the first and last calculated images is about $1.0E - 11$, which is around $1.0E - 2$ times the value of the increment. Even though the changes made on the

coordinates are significantly smaller than the changes made on the size of the resulting image, such modification eventually gives rise to interference patterns that are considerably different from each other.

The following operations in order to be able to analyze such images have
475 been carried out:

1. In first place, each image was separated into two sections. One section displayed period 4 as white on black and the other section displayed period 10. By using this method a collection of 1,025 images was obtained. All of them had the same interference pattern but a different distribution of
480 frequencies.
2. Then, each of the images of the said collections was subtracted from the remaining images: all images were compared pixel by pixel and then, the percentage of coincident points for each one of them was calculated. The result for each collection was a square matrix of 1024×1024 values
485 between 0 and 1 that represents the degree of similarity between each one of the images. Values close to 1, indicated a high similarity between images; values close to 0, indicated highly complementary images; and values close to 0.5 indicated that both images are randomly different. Obviously, each image is never compared to itself. Had that been the case,
490 the value for each of the elements of the main diagonal of the said matrix would have been 1.0. Figure 7 shows the distribution of the differences on a supposed miniature where, instead of 1,025 images, there would be only 5. Considering the way the distribution of the values of the matrix was defined, we could say that it is not symmetric; however, the difference
495 between the values of image I1 and image I2 was the same than the difference between the values of image I2 and image I1.
3. After obtaining the said six matrices, they were turned into 1024×1024 pixels images on a gray scale and their values were extrapolated between 0 and 1 to the interval of integers between 0 and 255. The result was a
500 visual description of the differences where, images with more similarities

d1_2	d1_3	d1_4	d1_5
d2_1	d2_3	d2_4	d2_5
d3_1	d3_2	d3_4	d3_5
d4_1	d4_2	d4_3	d4_5

Figure 7: This figure shows the way values have been assigned to the matrix of differences between images of collections. $d_{i,j}$ corresponds to the value of the difference between Image i and Image j .

were represented with whiter pixels, complementary images were represented with darker pixels and, finally, images with less similarities were represented in gray pixels close to value 128. We called each of the said six new images “Images of Differences”. Each of these six interference images is referred to as: **XcteP04** and **XcteP10** for those images obtained from the different Moiré interference patterns resulting from displacing the ULV vertically, while keeping their real coordinates constant; **YcteP04** and **YcteP10** for those images obtained from the different Moiré interference patterns resulting from displacing the ULV horizontally, while keeping their imaginary coordinate constant; and finally, **DiagP04** and **DiagP10** for those images obtained from the different Moiré interference patterns resulting from displacing the ULV diagonally, while increasing the same value for both the real coordinate and the imaginary coordinate.

Figure 8 shows two images obtained through the aforementioned procedure: **XcteP04** (a) and **YcteP10** (b). By enlarging these images, a diagonal distribution is observed, which means that the difference between one image displaced N times and the other image displaced M times (applying a horizontal or vertical $1.0E - 14$ displacement on their ULV, respectively) is similar to the difference between images $N + d$ and $M + d$.

All these images of differences enabled us to conduct a study to find periodicities, or some other type of pattern, through the use of a multiresolution analysis technique by different Wavelet Transforms. The last stage of this study consisted in carrying out a Wavelet analysis of the six resulting im-

Figure 8: View of the 1024×1024 matrices of differences between the interference patterns obtained from a displacement (about 10^{-14}) in the coordinates of its ULV.

ages. This analysis is shown in the epigraphs below. The purpose of this
 525 analysis was to look for periodicities that may add a certain predictability
 in the Moiré interference patterns resulting from displacements.

The next step was to conduct a second study on a fourth set of interference
 patterns that had been obtained by increasing the value of the Increment in
 each new image instead of making slight variations in their ULV coordinates.
 530 If we analyze these images of differences, we will be able to verify that the
 stability of Interference Patterns is particularly sensitive against any change
 made on the value of the increment. While the ULV displacement allowed for

an apparent similarity between the different images obtained, by changing the value of increment between pixels, even though very slightly, we observed that the resulting images were completely different with no spatial similarity between them at all.

Two images of differences of such images have been created and a study conducted on them by means of a multiresolution analysis. As shown below, the Wavelet Transform study did not offer useful information on patterns. Images of Differences are really special and surprisingly unpredictable.

5. Wavelet Analysis on Interference Images

As explained in the previous section, two image interferences sets have been analyzed: the first one, attained by successive displacement of its ULV; the second one by variations in the value of Increment between two consecutive pixels.

5.1. Analysis on interference images calculated by the displacement of the Upper-Left Vertex (ULV)

Images displayed in Figure 8 show a periodical structure, specially pattern on image (c), which shows an enlargement of the matrix of differences among the 1,025 images obtained from the horizontal displacement (constant imaginary part) of its ULV. Image (d) shows a periodical structure too, although it mainly consists of an apparent diagonal structure that is not necessarily periodical. The Wavelet analysis of these images and the calculation of entropy among the 4 images of each resolution level provides us with information on the Image of Differences and, therefore, on the evolution of the subsequent interference patterns, as the position of the view within the complex plane changes.

Different wavelet transforms have been carried out; and different families of the mother Wavelet have been used: ten Daubechies wavelets, from Daubechies 01 to Daubechies 10; nineteen Symlets wavelets, from Symlet 02 to Symlet 20; five Coiflets wavelets, from Coiflets 01 to Coiflets 05; the Discrete Meyer wavelets family and fifteen Biorthogonal wavelets: 1.1, 1.3, 1.5, 2.2, 2.4, 2.6, 2.8, 3.1, 3.3, 3.5, 3.7, 3.9, 4.4, 5.5 and 6.8. In conclusion, a total of 50 transforms (actually

49, since wavelet Daubechies 01 and wavelet Biorthogonal 1.1 are the same Wavelet: the so-called Haar wavelet) have been carried out. The coefficients of
565 these waves can be taken from different references [21].

As described in the aforementioned section, 6 images of differences of $1,024 \times 1,024$ pixels have been obtained (since the number of images in each of the analyzed sets is 1,025), which could be considered as square and dyadic data matrices. The Wavelet decomposition has been performed, up to reaching an
570 approximation signal of one single pixel. For each of the four sub-images that we got in each new level of decomposition, we calculated the corresponding value of the Shannon entropy. In conclusion, for each overview image (6 in total) and for each mother Wavelet used (49 in total), 10 levels of decomposition have been obtained, that is, 40 entropy values. The tables of values are too large to
575 include them in this article; however, according to the aforementioned algorithm presented in Fernández-Isla et al. [20], it is possible to stress the following facts:

Significant patterns in any of the resolution levels of images **DiagP04** and **DiagP10** are not found. Visual inspection was not enough to detect periodical distributions; however, it is obvious that, a diagonal distribution of gray levels
580 is observed, as in the rest of images. The results of the *ADR* equation were not significant. It is not possible to find a known interference pattern after displacing the ULV diagonally.

From the analysis conducted on images **XCteP04** and **XCteP10**, the presence of a gap in the entropy values is deduced, especially from level $j = 2$ to $j = 3$
585 ($J - j = 7$: size 128×128 pixels); for these levels, and for almost all the mother Wavelet functions, the value $ADR_{HH}^{j=3}$ was about 0.21, while the average value was about 0.03. It could be seen that taking one out of eight pixels in each of the two dimensions of the image, a periodicity at the third level of decomposition took place. For every 8 or 9 pixels, the structure of these two images repeated
590 itself; especially when the Symlets wavelets family was used. In the case of Wavelet mother with few coefficients (Haar, Daubechies 02, Symlets 02, Coiflets 01 and Biorthogonal 2.2 and 2.3 wavelets), the calculation failed and the results of the algorithm were not useful. All these wavelets let detecting a period for

the seventh level of decomposition when the approximation and detail images
 595 had only 64 pixels. They also led to detecting a period that appeared every 128
 pixels approximately.

A similar result for images YCteP04 and YCteP10 was achieved. The gap
 value is between $j = 3$ and $j = 4$ ($J - j = 6$: 64×64 pixels resolution); An
 iteration was interpreted as a texture that appeared every 16 pixels approxi-
 600 mately. Generally speaking, values are as decisive as the aforementioned case:
 value $ADR_{HH}^{j=3}$ is about 0.23 against an average value of about 0.03. However,
 for most wavelets these results were not so decisive, especially for Biorthogonal
 wavelets and, in general, for wavelets with few coefficients; the result could be
 sometimes in a fuzzy position between the aforementioned result and the gap
 605 value between $j = 1$ and $j = 2$. In other words, these mother wavelet are able
 to detect a structure of texture that iterates every 4 pixels, approximately.

Never two images with a similarity greater than 90% nor lower than 20%
 could be found.

5.2. Analysis on Interference Images calculated by subsequent increases of the 610 increment between pixels

As pointed out in previous section, a set of 1,025 images has been created
 by making changes in the value of the increment while keeping the value of the
 ULV constant. If the value of the Increment was $2.95195860348E - 09$ in the
 images shown before, now each new image takes a value of the Increment that
 615 has increased in $1.0E - 14$ with regard to the previous value of the Increment.
 The interference patterns obtained for this case have proven to be completely
 unpredictable, at least for a visual inspection, as shown in some of the sample
 images displayed in Figure 9.

The said unpredictable images show almost the same complex plane seg-
 620 ment. The value of the Increment of the first interference pattern of the series
 is $2.95195860348E - 09$, while the value of the increment of the last interfer-
 ence pattern (image no 1,025) is $2.96219860348E - 09$; therefore, the difference
 between the last and the first values is: $1024 \times 1.0E - 14 = 1.024E - 11$. The
 difference between the LRVs of both images is $4.09E - 09$ for the real axis and

Figure 9: Images calculated with a constant ULV (X: -1.191800609285510 , Y: $+0.134444678431087$) and with changes of about $1.0E-14$ made on the value of the increment, starting by 0.00000000295195860348 as an original value of the increment. Table 5 shows the values of the increments of each of these ten images.

Image	Step Value
(a) Img953	$951 \times I0 + 1.0E.14 = 0.00000000296146860348$
(b) Img814	$812 \times I0 + 1.0E.14 = 0.00000000296007860348$
(c) Img483	$481 \times I0 + 1.0E.14 = 0.00000000295676860348$
(d) Img704	$702 \times I0 + 1.0E.14 = 0.00000000295897860348$
(e) Img362	$360 \times I0 + 1.0E.14 = 0.0000000029555860348$
(f) Img145	$143 \times I0 + 1.0E.14 = 0.00000000295338860348$
(g) Img994	$992 \times I0 + 1.0E.14 = 0.00000000296187860348$
(h) Img349	$347 \times I0 + 1.0E.14 = 0.00000000295542860348$
(i) Img210	$208 \times I0 + 1.0E.14 = 0.00000000295403860348$
(j) Img542	$540 \times I0 + 1.0E.14 = 0.00000000295735860348$

Table 5: Step values from images shown in Figure 9. The value of $I0$, which is the increment of the first original interference image, is 0.00000000295195860348 .

about $3.07E-09$ for the imaginary axis; thus, between the first and the last image, the segment of the complex plane that is shown in a surface of four and three pixels in each Size has increased, which means only a 1% increase.

The two images of differences resulting from all the differences of such 1,025 images have been calculated (one image of differences for each of the two periods with more presence: **IncrP04** and **IncrP10** for periods 4 and 10). Then, the aforementioned analysis on the said images yielded no periodical property or grid structure in either image. In fact, there was no trace of their diagonal distribution, which was actually present at the images of differences obtained among the series of images of interference produced by the displacement of the ULV. Figure 10 shows the image of differences for period 4 and an enlarged and pixelated image of the said image of differences. The fact that the maximum and minimum values of these two matrices of differences are close to 0.5 is of

great significance: 0.53371 and 0.51868 in image **IncrP04**; 0.52398 and 0.51008 in image **IncrP10**. These values indicate that the degree of similarity between any two images of the calculated series is completely random and it does not exceed the 50% of similarity against any change made in one single digit of the mantissa of the value of the increment.

Figure 10: View of the 1024×1024 matrices of differences between the interference patterns obtained from a gradual variation (about $1.0E - 14$) in the value of the increment. Figure on the right is an enlargement of one of the areas of the image on the left.

The resulting property is that, for each point of the plane and for a prefixed value of the increment, we obtain an image that presumably defines unequivocally and unambiguously the ULV point and the Increment. A minimum modification in these values produces a significant change in the calculated image, especially when such modification in the original values is made on the value of the Increment. Besides, this system would be interesting because it is not possible to intentionally find two pairs of original values (each pair formed by the values of the ULV and the value of the Increment) with the same interference pattern. Such behavior would be similar to that used by a hash function on an input sequence, which results in an unequivocal and unambiguous output sequence where, in principle, it is impossible from a computational point of view to find two input sequences with the same Message Digest output. Other proposals in this sense can be found; for instance the one published by Ragulskis y Navickas [22], that offers an algorithm for the implementation of hash function based on the optical time average moir experimental technique.

6. Conclusion and Future Research

This article has shown the appearance of Moiré Interference patterns inside the map of orbits of the Mandelbrot Set. Such interference patterns seem to become present as a consequence of the fact that the process of calculation requires to discretize the points of the complex plane, thereby reducing that infinite set of points to a network structure. This means that the calculated image must have as many points as pixels. This article conjectures that the second grid frame in superposition, as the Moiré Pattern demands, can be generated by the truncation of the floating point data type values during the iterated operations defined for the Mandelbrot calculation: square and sum. If this conjecture proves to be true, the second network or grid on the calculation of the orbits would be caused by the fact that the variables encoding the coordinate values are limited to a finite number of digits. A future task would be either proving this conjecture or postulating a different origin for the second grid accounting, together with the pixilation grid, for the Moiré Interference Pattern.

The properties that, under certain conditions, verify many of the figures of the Moiré interference patterns, allow us for using the images of the map of orbits that are inside the Mandelbrot set. The said images are “extremely” sensitive to any change that may be made on the values used to calculate them. For any minor change made in the original position of the ULV, or for any minimum change made on the value of the Increment, an essential change is produced in the final images. Therefore, the result is a system that is hypersensitive to any change on its original conditions produced by squaring operations. Besides (and this is a fact based on the experience hereby presented), each time that a floating point data type is designed or configured, there is a change on the range of values from which the Moiré interference patterns start appearing in the said map of orbits.

The appearance of the Moiré interference patterns provides us with a system that allows for identifying different magnitudes by means of a Moiré interference pattern (whether they are encoded by decimal coordinates representing the

vertices of a rectangle or by decimal values representing the increase of displacement in the coordinates of such rectangle). Given that these patterns are highly sensitive to any change made on the original conditions, the result is a visual representation (a draw on a rectangle) that considerably varies at the slightest change made on the identified magnitudes.

The existence of Moiré interference patterns, which, as revealed throughout this article, can behave unpredictably at the slightest change in their input values, may allow engineers developing new technical solutions in the fields of computer vision, image recognition, etc. Another future task would be developing possible applications of the hypersensitivity of the Moiré Interference Patterns, as they appear in a deterministic process like the period orbits calculation for the M-Set points. For instance, an algorithm able to map a Moiré Picture to some integer values, as a hash function does to an arbitrary size to data. This opens new possibilities for cryptographic facilities such as authentication, message integrity, data corruption detection, etc.

Acknowledgments

Counting on the publisher's kindness, I would dedicate this piece of work to our greatly missed professor and researcher Mr. Pedro José García Laencina, who passed away unexpectedly on the evening of August 28th, 2015 at the age of 34. This article is the result of the activity carried out under the "*Research Programme for Groups of Scientific Excellence at Region of Murcia*" of the Seneca Foundation (Agency for Science and Technology of the Region of Murcia - 19895/GERM/15). The work has been partially supported and funded by the Spanish Ministerio de Economía y Competitividad (MINECO) under the project ViSeITR (TIN2012-39279).

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