# ALGEBRAIC QUANTIZATION, GOOD OPERATORS AND FRACTIONAL QUANTUM NUMBERS[1]

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## Abstract

The problems arising when quantizing systems with periodic boundary conditions are analysed, in an algebraic (group-) quantization scheme, and the "failure" of the Ehrenfest theorem is clarified in terms of the already defined notion of *good* (and *bad*) operators. The analysis of "constrained" Heisenberg-Weyl groups according to this quantization scheme reveals the possibility for new quantum (fractional) numbers extending those allowed for Chern classes in traditional Geometric Quantization. This study is illustrated with the examples of the free particle on the circumference and the charged particle in a homogeneous magnetic field on the torus, both examples featuring "anomalous" operators, non-equivalent quantization and the latter, fractional quantum numbers. These provide the rationale behind flux quantization in superconducting rings and Fractional Quantum Hall Effect, respectively.

# 1  Introduction

The need for a consistent quantization scheme which is truly suitable for systems wearing a non-trivial topology is increasing daily. Configuration spaces with non-trivial topology appear in as diverse cases as Gauge Theories, Quantum Gravity, and the more palpable ones of the superconducting ring and the Quantum Hall effect, where the measuring tools change the topology of the system in a non-trivial way.

The most common problem which appears when the configuration-space manifold possesses a non-trivial topology is the failure of the Ehrenfest theorem for certain operators,

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a problem usually referred to as an anomaly. In the sequel, we shall add the qualifier *topo-logic* to distinguish these from others directly attached to the Lie algebra of the quantum operators and characterized, roughly speaking, by the appearance of a term in a quantum commutator not present at the classical, Poisson-algebra level. We call them *algebraic* anomalies and refer the reader to [**A-N-B-L**] for a detailed analysis.

The failure of the Ehrenfest theorem for a given operator is primarily related to the non-globality of the corresponding classical function, such as is the case of the local co-ordinate on a one-dimensional closed submanifold. Geometric Quantization was intended to go further than ordinary canonical quantization does, allowing for the quantization of arbitrary symplectic manifolds. Unfortunately, Geometric Quantization only partially accomplished this task, one of the reasons being the difficulty in (or, even more, the impossibility of) finding a polarization suitable enough to quantize a given set of classical functions [**Wo**, **I-L**], or quantizing a set of operators in a way that would preserve a given polarization.

A quantization procedure based on a group structure, Group Approach to Quantiza-tion (GAQ) [**A-A**, **A-N-B-L**], improves the standard Geometric Quantization approach in that it provides two sets of mutually commuting operators, namely, the left- and right-invariant vector fields. This enables us to impose the polarization conditions by means of the left-invariant vector fields, say, while the right-invariant ones will be the quantum operators, which automatically preserve the polarization. The quantization group $\widetilde{G}$ is endowed with a $U(1)$-principal bundle structure so that generators fall into two classes according to whether or not they give rise to a term proportional to the vertical generator on the r.h.s. of a commutator. Generators which do not reproduce any $U(1)$-term close a *horizontal* subalgebra, the characteristic subalgebra, of non-dynamical generators, which should be included in the polarization subalgebra. The principal drawback of GAQ is the need for a group symmetry associated with the system to be quantized, and the apparent restriction in the number of functions which can be quantized. However, this last limi-tation is slighter than it might seem, since Canonical Quantization on a particular phase space does not quantize the entire set of functions on phase space, but rather, a restricted Poisson subalgebra. Even more, it could well happen in some cases that a more standard quantization provides only quantum operators corresponding to a finite-dimensional Lie algebra. This is the case, for instance, of the symplectic manifold $S^2$, where the quantum operators are only those of $su(2)+R$ [**G-G-H**]. Moreover, most of the interesting systems in Physics possess a symmetry group large enough to achieve a proper quantization.

To be more precise, not only the right invariant vector fields preserve the polarization, but rather the entire right enveloping algebra preserve the structure of the Hilbert space. This means that any element in the right enveloping algebra can be realized as a quantum operator, although the relation between the quantum algebra and the standard Poisson algebra on the co-adjoint orbits of the group is no longer an isomorphism; GQA provides a quantum theory rather than the quantization of a classical theory.

A reformulation of GAQ was proposed a few years ago [**A-N-R**], the Algebraic Quan-

tization on a Group (AQG) [some of the basic ideas in [**A-N-R**] have also appeared in the context of quantum systems with non-trivial topology [**L**] and in Quantum Gravity ([**A**] and references therein)] , which generalizes GAQ in two respects. Firstly, finite transformations generalize the infinitesimal ones throughout the method; that is, any concept or condition relative to Lie subalgebras is generalized by its counterpart in terms of Lie subgroups, thus allowing discrete transformations to enter the theory. Needless to say, infinitesimal objects are employed whenever possible. Secondly, it generalizes the $U(1)$ phase invariance in Quantum Mechanics (the structure group of the principal bundle fibration of the quantum symmetry) incorporating other symmetries, eventually interpreted as constraints. The new structure group $T$, which must include the traditional $U(1)$, may also contain discrete symmetries especially suitable to simulate manifold surgery as, for instance, toral compactification, by means of periodic boundary conditions.

From now on we shall call "compactified" (cylindrical or toral) Heisenberg-Weyl (H-W) group a H-W group where the structure group is $T$ rather than $U(1)$, this subgroup $T$ being the factor subgroup leading to a compactified classical (the cylinder or the torus) phase space by the quotient $\widetilde{G}/T$.

The generalization of the $U(1)$-equivariance to $T$-equivariance condition on the wave functions gives rise to two new, closely related features: a) the existence of non-equivalent quantizations associated with non-equivalent representations of the larger structural subgroup $T$, and b) the notion of *good* operators, constituting the subgroup of transformations compatible with the $T$-equivariance condition, in a sense to be specified later (see [**A-N-R**]). Furthermore, those operators not preserving the $T$-equivariance condition, the *bad* operators, may be seen as quantization-changing transformations, and exhibit topologic anomalies. Like in the $T = U(1)$ case, all the elements of the right enveloping algebra *compatible with the $T$-equivariance condition*, for arbitrary $T$, can be realized as *good* quantum operators.

It should be noted that, as mentioned above, AGQ is formulated in terms of finite objects. This means that some algebraic indices must replace the well-know Chern class $[\omega]$ of the symplectic form in Geometric Quantization. In fact, the indices characterizing the (not necessarily central) extension by $T$ of the "classical" group $G$ generalize the Chern class, providing also *fractional* values. This is precisely the case of the motion of a charged particle on a torus in the presence of a homogeneous magnetic field, closely related to the (Fractional) Quantum Hall Effect. The appearance of fractional quantum numbers generalizing the integer Chern classes reveals, once again, that the procedure of taking constraints and that of quantizing, depending at least on the specific methods employed, may not commute.

This paper is organized as follows. Section 2 illustrates the way in which AQG operates with the help of the examples of the Heisenberg-Weyl group in 1D with constraints mimicking the compactification of the coordinate $x$ (Sec 2.1) and that of the compactification of both $x$ and $p$ (Sec. 2.2). In the latter case, generalizing the quantization of a compact phase space *à la Dirac*, a not necessarily integer quantization condition is

obtained which generalizes that of Geometric Quantization, i.e. the condition $[\omega] \in Z$ (Chern class), and, associated with it, vector-valued wave functions. In solving this problem, real (versus holomorphic) polarizations have been employed, leading to a generalized $kq$-representation. This technique simplifies the treatment and is much more intuitive, even though the configuration-space wave functions contain delta functions. The results obtained in Sec. 2 are applied to the quantization of the free particle on the circumference (Sec. 3, where the failure of Ehrenfest theorem is analysed), directly related to flux quantization in superconducting rings, and to the quantization of a charged particle on a torus in the presence of an homogeneous transverse magnetic field (Sec. 4), providing the rationale behind Integer and Fractional Quantum Hall Effect.

# 2 Algebraic Quantization of "compactified" Heisenberg -Weyl groups: Fractional quantum numbers

In this section, we shall explain the AQG formalism over the example of the Heisenberg-Weyl group with one co-ordinate "compactified", i.e. with constraints associated with the compactification of one co-ordinate, and with one co-ordinate and its canonically conjugate momentum "compactified". We nevertheless recommend the reading of the Ref. [**A-N-R**]. Although explicit calculations are given for the Heisenberg-Weyl group with only one co-ordinate-momentum pair, the results can be generalized, immediately, to any finite number of them.

## 2.1 Cylindrical Heisenberg-Weyl group

Let us firstly proceed with the case of the Heisenberg-Weyl group with only one of the coordinates "compactified", i.e. with structure group $T$ such that the quotient $\widetilde{G}/T$ leads to the cylinder as the symplectic manifold. The starting point in AQG is a Lie group $\widetilde{G}$ which is a right-principal bundle with structure group $T$. $T$ is itself a principal bundle with $U(1)$ as structure group. In our case $\widetilde{G}$ is the ordinary Heisenberg-Weyl group in 1D (throughout the paper, 1D means one coordinate-momentum pair, $x$ and $p$), and $T = U(1) \times \{e_k, k \in Z\}$, where $\{e_k, k \in Z\}$ is the subgroup of $\widetilde{G}$ of finite translations in the coordinate $x$ by an amount of $kL$, $L$ being the spatial period. Note that $T$ is isomorphic to $U(1) \times Z$, so that its fibration is trivial.

The group law $g'' = g' * g$ for $\widetilde{G}$ is:

$$
\begin{aligned}
x'' &= x' + x \\
p'' &= p' + p \\
\zeta'' &= \zeta'\zeta \, e^{\frac{i}{\hbar}[(1+\lambda)x'p + \lambda xp']}
\end{aligned}
\tag{1}
$$

where the first two lines correspond to the group law of $G$, and the third to that of $U(1)$. The real parameter $\lambda$ has been introduced to account for a complete class of central

extensions differing in a coboundary [coboundaries have the form $\xi(g', g) = \eta(g' * g) - \eta(g') - \eta(g)$, where $\eta : G \to R$ is called the generating function of the coboundary] generated by the function $\eta(x, p) = \lambda x p$ (In particular, for $\lambda = -\frac{1}{2}$ we have Bargmann's cocycle).

From this group law we can read immediately the right and left translations, $R_{g'}g = g' * g = L_g g'$. In particular, the left- and right-invariant vector fields (generating the finite translations) become:

$$
\begin{array}{llll}
\tilde{X}_x^L & = & \frac{\partial}{\partial x} + \frac{\lambda}{\hbar} p \, \Xi & \qquad \tilde{X}_x^R = \frac{\partial}{\partial x} + \frac{1+\lambda}{\hbar} p \, \Xi \\
\tilde{X}_p^L & = & \frac{\partial}{\partial p} + \frac{1+\lambda}{\hbar} x \, \Xi & \qquad \tilde{X}_p^R = \frac{\partial}{\partial p} + \frac{\lambda}{\hbar} x \, \Xi \\
\tilde{X}_\zeta^L & = & i\zeta \frac{\partial}{\partial \zeta} \equiv \Xi & \qquad \tilde{X}_\zeta^R = i\zeta \frac{\partial}{\partial \zeta} \equiv \Xi
\end{array}
\tag{2}
$$

The quantization 1-form (the left-invariant 1-form associated with the parameter $\zeta$) can also be obtained:

$$
\Theta = -\lambda p \, dx - (1 + \lambda) x \, dp + \hbar \frac{d\zeta}{i\zeta}
\tag{3}
$$

Since we are not considering time evolution, the quantization 1-form has no characteristic subalgebra, there exists no discrete characteristic subgroup $G_C$, and any combination of the two generators $\tilde{X}_x^L$ and $\tilde{X}_p^L$ constitutes a first-order full polarization (with the time evolution added, as in the free particle in 1D, things are a bit more complicated, see Sec. 3). Two polarizations are singled out, $\mathcal{P}_p = < \tilde{X}_x^L >$ and $\mathcal{P}_x = < \tilde{X}_p^L >$, or their finite (versus infinitesimal) counterparts $G_{\mathcal{P}_p} = \{$Space translations$\}$ and $G_{\mathcal{P}_x} = \{$Boosts transformations$\}$, leading to momentum and configuration space representations, respectively. It should be borne in mind that the polarization conditions are needed to reduce the group representation which otherwise would provide only the Bohr-Sommerfeld quantization. These polarization conditions read, in general, $\tilde{X}^L \Psi = 0, \forall \tilde{X}^L \in \mathcal{P}$ or $\Psi(g * G_{\mathcal{P}}) = \Psi(g)$ in finite terms.

The $T$-function condition generalizes ordinary phase invariance ($U(1)$-equivariance) in Quantum Mechanics, which is written $\Psi(\zeta * g) = \rho(\zeta)\Psi(g)$, where $\rho(\zeta)$ is the natural representation of $U(1)$ on the complex numbers, $\rho(\zeta) = \zeta$. The generalization to a bigger group $T$ involves the use of a general representation $\mathcal{D}$ of $T$ (or, to be precise, of $T_B \equiv U(1) \cup T_p$, where $T_p$ is a maximal polarization subgroup of $T$; see [**A-N-R**]) on a complex vector space $E$, where the wave functions themselves take their values. In the formalism of AQG, the representation of $T_B$ is constructed from the very representation of $\tilde{G}$, i.e. the vector space $E$ on which the constrained functions are evaluated is made out of the unconstrained wave functions by properly choosing their arguments. This is the reason why the group $T_B$ is interpreted as constraints: the representation of $T_B$ is not an abstract representation, but rather built with the same functions of the representation of $\tilde{G}$.

The $T_B$-function condition then reads $\Psi(g_{T_B} * g) = \mathcal{D}(g_{T_B})\Psi(g), \forall g_{T_B} \in T_B, \forall g \in \tilde{G}$. In the present case $T_B = T$ and $\mathcal{D}(\zeta, e_k) = \zeta D(e_k)$, where $D(e_k)$ is a representation of

$\{e_k, \ k \in Z\} \ (\approx Z)$ in the complex numbers, and there is an infinity of non-equivalent irreducible representations, of the form $D^\epsilon(e_k) = e^{\frac{i}{\hbar}\epsilon kL}$, with $\epsilon \in [0, \frac{2\pi\hbar}{L})$ (the first Brillouin zone, in Solid State nomenclature). Therefore, *there is a non-equivalent quantization associated with each choice of non-equivalent representation of $T$, parameterized by $\epsilon$.* The $T$-function condition for the wave function implies the restriction:

$$e^{\frac{i}{\hbar}(1+\lambda)kLp}\Psi^\epsilon(x + kL, p, \zeta) = e^{\frac{i}{\hbar}\epsilon kL}\Psi^\epsilon(x, p, \zeta) \qquad (4)$$

Note that the constrained wave functions can be identified with the space of sections of a $U(1)$-bundle on the cylinder, with connection given by (3).

We now impose the polarization conditions in order to reduce the representation. Firstly, we shall consider the momentum space representation, where the polarization conditions (either in finite or infinitesimal form) lead to the following form of the wave functions:

$$\Psi^\epsilon(x, p, \zeta) = \zeta e^{-\frac{i}{\hbar}\lambda xp}\Phi^\epsilon(p) \qquad (5)$$

where the fact that $\Psi(\zeta g) = \zeta\Psi(g)$ (by the $T$-function property) has been used.
Both conditions (4) and (5) together imply for the wave function $\Phi^\epsilon(p)$ a form like:

$$\Phi^\epsilon(p) = \sum_{k \in Z} \alpha_k \phi_k^\epsilon(p) \qquad (6)$$

where $\phi_k^\epsilon(p) \equiv \delta(p - \epsilon - \frac{2\pi\hbar}{L}k)$, i.e. the wave function is peaked at the values of the momentum $p_k^\epsilon = \epsilon + \frac{2\pi\hbar}{L}k, \ k \in Z$. The Hilbert space $\mathcal{H}^\epsilon(\widetilde{G})$ is made from the wave functions defined by (5) and (6).

The quantum operators, defined as $\hat{P} \equiv -i\hbar\tilde{X}_x^R$ and $\hat{X} \equiv i\hbar\tilde{X}_p^R$, act on the wave functions as:

$$\begin{aligned} \hat{P}\Psi^\epsilon &= p\Psi^\epsilon \\ \hat{X}\Psi^\epsilon &= \zeta e^{-\frac{i}{\hbar}\lambda xp}\left[i\hbar\frac{\partial}{\partial p}\right]\Phi^\epsilon \end{aligned} \qquad (7)$$

One of the main consequences of having generalized the structure group in AQG is the classification of the operators (actually left translations) as good and bad operators according to whether or not they are compatible with the $T$-function condition. More precisely, the subgroup of good operators, $G_\mathcal{H}$, is characterized by the condition (see [**A-N-R**])

$$Ad(\widetilde{G})\,[g_T, G_\mathcal{H}] \subset G_\mathcal{P}, \ \ \forall g_T \in T \qquad (8)$$

In the present case, and due to the discrete character of the "physical" momenta, the position operator $\hat{X}$ is expected to be problematic, since the subgroup of good transformations compatible with (4) and (5) is the subgroup of $\widetilde{G}$, in which the continuous variable $p$ is substituted by the discrete variable $p_k \equiv p_k^0 = \frac{2\pi\hbar}{L}k, \ k \in Z$, as can be deduced from

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$Ad(\widetilde{G})[e_n, g] = (0, 0, e^{\frac{i}{\hbar}nLp}) \subset G_{\mathcal{P}} \ \forall n \in Z$. Therefore, the good operators are $\hat{P}$ and the finite boosts transformations by the amount of $p_k$. Position is not a good operator in the sense that it does not preserve the structure of the wave functions, i.e. it does not leave the Hilbert space (for fixed $\epsilon$) $\mathcal{H}^\epsilon(\widetilde{G})$ stable. This fact will be further discussed in Sec. 2.1.1.

With regard to the configuration space representation given by the polarization $\mathcal{P}_x$ or the polarization subgroup $G_{\mathcal{P}x}$, the solutions to this polarization are:

$$\Psi(x, p, \zeta) = \zeta e^{-\frac{i}{\hbar}(1+\lambda)px}\Phi(x) \tag{9}$$

Applying the condition of $T$-function (4) to this wave functions in configuration space, we obtain:

$$e^{\frac{i}{\hbar}(1+\lambda)kLp}e^{-\frac{i}{\hbar}(1+\lambda)(x+kL)p}\Phi^\epsilon(x + kL) = e^{\frac{i}{\hbar}\epsilon kL}e^{-\frac{i}{\hbar}(1+\lambda)xp}\Phi^\epsilon(x) \tag{10}$$

$\forall k \in Z$, where the quasi-periodicity condition for $\Phi^\epsilon(x)$ immediately follows:

$$\Phi^\epsilon(x + L) = e^{\frac{i}{\hbar}\epsilon L}\Phi^\epsilon(x) \tag{11}$$

It should be stressed that this result is independent of the chosen cocycle, since it does not depend on $\lambda$, as expected.

The quantum operators are:

$$\begin{aligned}
\hat{P}\Psi^\epsilon &= \zeta e^{-\frac{i}{\hbar}(1+\lambda)px}\left[-i\hbar\nabla\right]\Phi^\epsilon \\
\hat{X}\Psi^\epsilon &= \zeta x\Psi^\epsilon
\end{aligned} \tag{12}$$

Again, the position operator $\hat{X}$ is not a good operator, for the same reason as in the momentum-space case, and the subgroup of (left) transformations leaving the structure of the wave functions (9) and (11) stable is the same $G_{\mathcal{H}}$ as before, containing only $\hat{P}$ and the finite boosts in $p_k$, $k \in Z$. Therefore, the standard position has no meaning for any (Galilean) system with the circumference as configuration space (see Sec. 3).

### 2.1.1 Is there any good position-like operator?

The position operator $\hat{X}$ is not a good operator because the variable $x$ is not periodic: if $\phi(x)$ is a quasi-periodic function, $x\phi(x)$ is no longer quasi-periodic. However, the function $\eta = e^{i\frac{2\pi}{L}x}$ is periodic, so that we could define the operator $\hat{\eta} \equiv e^{i\frac{2\pi}{L}\hat{X}}$, and verify that $\hat{\eta}\Psi^\epsilon = e^{i\frac{2\pi}{L}x}\Psi^\epsilon$ satisfies the same quasi-periodicity condition as $\Psi^\epsilon$. We can then say that $\hat{\eta}$ is a good operator.

The reason why $\hat{\eta}$ is a good operator is precisely that it generates a good finite boost. We know that the only good boosts are indexed by $p_k = \frac{2\pi\hbar}{L}k$, i.e.

$$\Psi^\epsilon(p_k * g) = e^{\frac{2\pi\hbar}{L}k\tilde{X}_p^R}\Psi^\epsilon(g) = \left(e^{i\frac{2\pi}{L}\hat{X}}\right)^{-k}\Psi^\epsilon(g) = \hat{\eta}^{-k}\Psi^\epsilon(g) \tag{13}$$

This means that $\hat{\eta}^k$, $k \in Z$ are the only good position operators.

The finite operator $\hat{\eta}$ is obviously not Hermitian; rather, it is unitary as it should be. However $\hat{\eta}$ can be written as $\hat{\eta} = \cos(\frac{2\pi}{L}\hat{X}) + i\sin(\frac{2\pi}{L}\hat{X})$, the good operators $\cos(\frac{2\pi}{L}\hat{X})$ and $\sin(\frac{2\pi}{L}\hat{X})$ being Hermitian. These are good operators, given that they are periodic functions of the operator $\hat{X}$. Since the set of functions $\{e^{i\frac{2\pi}{L}mx}, m \in Z\}$ constitutes a basis for the periodic functions of $x$ in the interval $[0, L]$, any operator which is a periodic function of the position operator $\hat{X}$ is a good operator.

In any case, we might wonder about the finite boosts transformations for $\tilde{p} \neq p_k$, i.e. about transformations of the form $\Phi'(x) = e^{\frac{i}{\hbar}\tilde{p}\hat{X}}\Phi^{\epsilon}(x) = e^{\frac{i}{\hbar}\tilde{p}x}\Phi^{\epsilon}(x)$. This new function verifies the boundary conditions $\Phi'(x + L) = e^{\frac{i}{\hbar}\tilde{p}(x+L)}\Phi^{\epsilon}(x + L) = e^{\frac{i}{\hbar}(\epsilon+\tilde{p})L}\Phi'(x)$, and therefore belongs to the Hilbert space $\mathcal{H}^{\epsilon+\tilde{p}}(\widetilde{G})$. In fact, since the representations parameterized by $\epsilon$ and $\epsilon + \frac{2\pi\hbar}{L}k$, $k \in Z$ are equivalent, the transformed wave functions lie in the representation $(\epsilon+\tilde{p})$ mod $\frac{2\pi\hbar}{L}$. Of course, if $\tilde{p} = p_k$ for some $k$, the transformed wave function lies in the same Hilbert space as before and we recover the result that the finite boosts in $p_k$ are good operators. In particular, $\Phi^{\epsilon}(x) = e^{\frac{i}{\hbar}\epsilon\hat{X}}\Phi^0(x) = e^{\frac{i}{\hbar}\epsilon x}\Phi^0(x) \equiv e^{\frac{i}{\hbar}\epsilon x}\Phi(x)$, with $\Phi(x)$ satisfying (the usual) periodic boundary conditions. This means that all the Hilbert spaces $\mathcal{H}^{\epsilon}(\widetilde{G})$, although yielding non-equivalent representations, are related to each other by means of finite boosts transformations, which are unitary transformations considered in the union of all these Hilbert spaces $\cup_{\epsilon \in [0, 2\pi\hbar/L)}\mathcal{H}^{\epsilon}(\widetilde{G})$. We could say that $\mathcal{H}^{\epsilon}(\widetilde{G})$ for a fixed $\epsilon$ is too small for the boosts operator to live in. The momentum operator, however, preserves (and is Hermitian in) each one of these Hilbert spaces, but it is not Hermitian in the union of all of them.

It is worth mentioning that the set of operators $\hat{P}$, $\hat{\eta}$ and $\hat{\eta}^{\dagger}$ close a Lie algebra under ordinary commutation which is isomorphic to the non-extended harmonic oscillator algebra. The operators $\hat{\eta}$ and $\hat{\eta}^{\dagger}$ act as ladder operators on the eigenfunctions of $\hat{P}$ (this fact has been used in [O-K] to study Quantum Mechanics on the circumference).

## 2.2 Toral Heisenberg-Weyl group

Let us now proceed with the case of the Heisenberg-Weyl group with both the coordinate and the momentum "compactified", i.e. with a structure group $T$ such that $\widetilde{G}/T$ leads to the torus as the symplectic manifold. We shall parameterize the plane with coordinates $(x_1, x_2)$ because in physical applications the coordinates play the double rôle of coordinate and momentum (see Sec. 4.1).

We also apply AQG to this system. Here again, $\widetilde{G}$ is the Heisenberg-Weyl group in 1D (now parameterized by $\vec{x} = (x_1, x_2)$ and $\zeta$). Given $L_1$ and $L_2$, we introduce the lattice points $\vec{L}_{\vec{k}} \equiv (k_1L_1, k_2L_2)$, $k_1$ and $k_2$ being integers (thus defining a rectangular torus if we would take quotient by them). The structure group $T$ will be a principal bundle with base $\{e_{\vec{k}}, \vec{k} \in Z \times Z\}$ and fiber $U(1)$, where $\{e_{\vec{k}}, \vec{k} \in Z \times Z\} \subset \widetilde{G}$ is the set of finite translations in the coordinates $\vec{x}$ by an amount of $\vec{L}_{\vec{k}}$. The fibration of $T$ is non-trivial in

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general.

The group law for $\tilde{G}$ now reads:

$$
\begin{aligned}
\vec{x}'' &= \vec{x}' + \vec{x} \\
\zeta'' &= \zeta'\zeta e^{\frac{i}{\hbar}m\omega[(1+\lambda)x_1' x_2 + \lambda x_1 x_2']}
\end{aligned}
\tag{14}
$$

where a new numerical constant $\omega$ (with dimensions of $T^{-1}$), besides the mass $m$, which was implicit in the momentum $p = mv$, has been introduced to accommodate the dimensions in the exponent above.

The left and right invariant vector fields can be obtained:

$$
\begin{array}{llll}
\tilde{X}_{x_1}^L &= \frac{\partial}{\partial x_1} + \frac{\lambda}{\hbar}m\omega x_2\, \Xi & \tilde{X}_{x_1}^R &= \frac{\partial}{\partial x_1} + \frac{1+\lambda}{\hbar}m\omega x_2\, \Xi \\
\tilde{X}_{x_2}^L &= \frac{\partial}{\partial x_2} + \frac{1+\lambda}{\hbar}m\omega x_1\, \Xi & \tilde{X}_{x_2}^R &= \frac{\partial}{\partial x_2} + \frac{\lambda}{\hbar}m\omega x_1\, \Xi \\
\tilde{X}_\zeta^L &= i\zeta\frac{\partial}{\partial\zeta} \equiv \Xi & \tilde{X}_\zeta^R &= i\zeta\frac{\partial}{\partial\zeta} \equiv \Xi,
\end{array}
\tag{15}
$$

and the quantization 1-form is:

$$
\Theta = -\lambda m\omega x_2 dx_1 - (1+\lambda)m\omega x_1 dx_2 + \hbar\frac{d\zeta}{i\zeta}
\tag{16}
$$

As before, the quantization 1-form has no characteristic module, and any combination of the two generators $\tilde{X}_{x_1}^L$ and $\tilde{X}_{x_2}^L$ constitutes a first-order full polarization. These can be written as $\mathcal{P}_{\vec{n}} = \langle \vec{n}\cdot\tilde{X}_{\vec{x}}^L \rangle$, where $\vec{n} = (n_1, n_2)$ is an arbitrary unit vector. The choice of an $\vec{n}$ corresponds to the selection of a particular direction in the plane. [All directions are indistinguishable, but on the mimicked torus, there are geodesics (directions) which close, as happens with the lines $x_2 = 0$ and $x_1 = 0$, and others which are open and fill the torus densely. It can be easily checked that the condition for a geodesic with direction given by $\vec{n}$ to close is either that $\frac{n_2}{n_1} = \frac{k_{02}}{k_{01}}\frac{L_2}{L_1}$, $k_{01}, k_{02} \in Z$ or that $\vec{n} = (1,0)$ or $\vec{n} = (0,1)$, i.e. $\vec{n}$ is of the form $\vec{n} = \vec{L}_{\vec{k}_0}/|\vec{L}_{\vec{k}_0}|$, with $\vec{k}_0 \in Z \times Z$. Also, for a geodesic and its orthogonal one to close, it is necessary and sufficient that $\frac{L_2^2}{L_1^2}$ be a rational, except for the case $\vec{n} = (1,0)$ and $\vec{n} = (0,1)$, which are always orthogonal and closed. This condition is similar to the condition of commensurability of the frequencies for a Lissajoux figure to be closed].

The polarization condition $\mathcal{P}_{\vec{n}}$ leads to the following wave functions:

$$
\Psi = \zeta e^{-\frac{i}{\hbar}m\omega[(\lambda n_1^2 - (1+\lambda)n_2^2)y_1 y_2 + (\lambda+\frac{1}{2})n_1 n_2 y_1^2]}\Phi(y_2)
\tag{17}
$$

where $y_1 \equiv \vec{n}\cdot\vec{x}$, $y_2 \equiv \vec{n}\cdot\hat{\mathrm{J}}\cdot\vec{x}$, and $(\hat{\mathrm{J}})_{ij} = \epsilon_{ij}$, with $\epsilon_{12} = 1$. The action of the right operators on these wave functions is:

$$
\begin{aligned}
\vec{n}\cdot\tilde{X}_{\vec{x}}^R\Psi &= \frac{i}{\hbar}m\omega y_2\Psi \\
\vec{n}\cdot\hat{\mathrm{J}}\cdot\tilde{X}_{\vec{x}}^R\Psi &= \zeta e^{-\frac{i}{\hbar}m\omega[(\lambda n_1^2 - (1+\lambda)n_2^2)y_1 y_2 + (\lambda+\frac{1}{2})n_1 n_2 y_1^2]} \times \\
&\qquad \left[\frac{\partial}{\partial y_2} - \frac{i}{\hbar}m\omega n_1 n_2(1+2\lambda)y_2\right]\Phi(y_2)
\end{aligned}
\tag{18}
$$

9

Before imposing the constraints, we have to determine the structure of the group $T$. It must be done by means of finite transformations, since it is basically a discrete group (times $U(1)$). We then compute the group commutator of two elements of $\left\{e_{\vec{k}}\right\}_{\vec{k} \in Z \times Z}$, with the result $[e_{\vec{k}'}, e_{\vec{k}}] = (0, 0, e^{\frac{i}{\hbar} m \omega L_1 L_2 (k_1' k_2 - k_2' k_1)})$. Two cases have to be considered:

$i)$ $e^{\frac{i}{\hbar} m \omega L_1 L_2 (k_1' k_2 - k_2' k_1)} = 1 \, \forall \vec{k}, \vec{k}' \in Z \times Z \ \Rightarrow \ [e_{\vec{k}'}, e_{\vec{k}}] = \mathbf{1}_{\widetilde{G}}$ ,

$ii)$ $\exists \vec{k}$ and $\vec{k}'$ $/$ $e^{\frac{i}{\hbar} m \omega L_1 L_2 (k_1' k_2 - k_2' k_1)} \neq 1 \ \Rightarrow \ \mathbf{1}_{\widetilde{G}} \neq [e_{\vec{k}'}, e_{\vec{k}}] \in U(1)$ ,

For the case $i)$, $T$ is the direct product $T = \left\{e_{\vec{k}}, \vec{k} \in Z \times Z\right\} \times U(1)$ and the whole group $T$ can be imposed as constraints. For the case $ii)$, when $e^{\frac{i}{\hbar} m \omega L_1 L_2 (k_1' k_2 - k_2' k_1)} \neq 1$ for some values of $\vec{k}$ and $\vec{k}'$ (an infinite discrete set of values, in fact), there are two possibilities, depending on whether $\frac{m \omega L_1 L_2}{2 \pi \hbar}$ is rational or irrational. In neither case can we impose the entire group $T$ as a constraint group and we have to choose a polarization subgroup $T_p$ of $T$ (see [**A-N-R**]).

### 2.2.1  Integer quantum numbers

For the condition $i)$ to hold, it is necessary that

$$\frac{m \omega L_1 L_2}{2 \pi \hbar} = n \in N \,, \tag{19}$$

which implies a *quantization* of the "frequency" $\omega$. As we shall see in Sec. 4, this condition will imply the quantization of the magnetic flux through the torus surface. This quantization condition is of the same nature as that of the Dirac monopole case. Concerning this case, AQG simply reproduces the quantization condition of the standard Geometric Quantization: the symplectic form must be of integer class, defining the Chern class of the quantum manifold.

The rest of the procedure follows the same lines as in the case of the cylindrical H-W group: the condition of $T$-function is $\Psi(g_T * g) = \mathcal{D}(g_T)\Psi(g)$, with $\mathcal{D}(e_{\vec{k}}, \zeta) = \zeta D(e_{\vec{k}})$, where $D(e_{\vec{k}})$ is a representation of the group $\left\{e_{\vec{k}}, \vec{k} \in Z \times Z\right\} \approx Z \times Z$ on the complex numbers. For the moment, we shall use the trivial representation $D^0(e_{\vec{k}}) = 1$, and later the rest of non-equivalent representations (leading to non-equivalent quantization of $\widetilde{G}$ ) will be computed with the help of the bad operators, as was shown in Sec. 2.1.1 for the cylindrical H-W group. The $T$-function condition then reads:

$$e^{\frac{i}{\hbar} m \omega [(1+\lambda) k_1 L_1 x_2 + \lambda k_2 L_2 x_1]} \Psi^0(\vec{x} + \vec{L}_{\vec{k}}, \zeta) = \Psi^0(\vec{x}, \zeta) \tag{20}$$

$\forall \vec{k} \in Z \times Z$. Note that the space of constrained wave functions can be identified with the space of sections of a $U(1)$-bundle on the torus, with Chern class $n$ and connection given by (16).

Applying this constraint to the polarized wave functions (17) the following restriction is obtained:

$$
\begin{aligned}
e^{\frac{i}{\hbar}m\omega\left\{y_2+(1+2\lambda)n_1n_2y_1+[(1+\lambda)n_2^2-\lambda n_1^2](\vec{n}\cdot\hat{J}\cdot\vec{L}_{\vec{k}})\right\}(\vec{n}\cdot\vec{L}_{\vec{k}})}\times \\
e^{-\frac{i}{\hbar}m\omega[(1+2\lambda)n_1n_2(y_1+y_2)](\vec{n}\cdot\hat{J}\cdot\vec{L}_{\vec{k}})}\Phi^0(y_2+\vec{n}\cdot\hat{J}\cdot\vec{L}_{\vec{k}}) &= \Phi^0(y_2)
\end{aligned}
\tag{21}
$$

$\forall\vec{k}\in Z\times Z$. This restriction has important consequences: $a)$ the possible polarizations are only those given by $\vec{n}=(1,0)$ and $\vec{n}=(0,1)$; $b)$ the wave function is peaked at certain equally spaced values of $y_2$; and $c)$ the parameter $\lambda$ is also quantized. From these facts it can also be deduced that the dimension of the representation is $n$, i.e. the representations of the toral Heisenberg-Weyl group are finite-dimensional, having dimension $n$, where $n$ is given by (19).

Explicitly, the "allowed" values for the coordinates are $x_2=\frac{k}{n}L_2$, $k\in Z$ for $\vec{n}=(1,0)$ and $x_1=\frac{k}{n}L_1$, $k\in Z$ for $\vec{n}=(0,1)$. The wave functions then turn out to be, respectively:

$$
\Phi^0(x_2)=\sum_{k\in Z}a_k\delta(x_2-\frac{k}{n}L_2)\quad\text{for}\quad\vec{n}=(1,0)
\tag{22}
$$

$$
\Phi^0(x_1)=\sum_{k\in Z}b_k\delta(x_1-\frac{k}{n}L_1)\quad\text{for}\quad\vec{n}=(0,1)
\tag{23}
$$

The coefficients $a_k$ and $b_k$ are not completely arbitrary; due to the $T$-function condition, which now reads $\Phi^0(x_2+k_2L_2)=\Phi^0(x_2)$ (for $\vec{n}=(1,0)$) and $\Phi^0(x_1+k_1L_1)=\Phi^0(x_1)$ (for $\vec{n}=(0,1)$) $\forall\vec{k}\in Z\times Z$, they satisfy $a_{k+n}=a_k$, and $b_{k+n}=b_k$, $\forall k\in Z$. Then, there are only $n$ independent coefficients, so that the dimension of the representation is $n$. The allowed values for $\lambda$ are given by $\lambda=\frac{k}{n}$, $k\in Z$, i.e. the possible (equivalent) cocycles, or that which is the same, the possible coboundaries are quantized. This fact can be easily understood in terms of the generating function of the coboundary parameterized by $\lambda$, which has the form $\lambda x_1x_2$, or better, $e^{\frac{i}{\hbar}m\omega\lambda x_1x_2}$. For this function to be quasi-periodic, i.e. $e^{\frac{i}{\hbar}m\omega\lambda(x_1+k_1L_1)(x_2+k_2L_2)}=e^{i\epsilon\cdot\vec{L}_{\vec{k}}}e^{\frac{i}{\hbar}m\omega\lambda x_1x_2}$, $\forall\vec{k}\in Z\times Z$, the quantization condition for $\lambda$ is necessary, besides the quantization condition for $x_1$ and $x_2$.

Let us focus on the case $\vec{n}=(1,0)$ for concreteness (the case $\vec{n}=(0,1)$ is completely analogous and in fact equivalent). Using the expression (22) and the fact that $a_{k+n}=a_k,\forall k\in Z$, the summatory can be regrouped, and we arrive at a rather compact form for the wave functions:

$$
\begin{aligned}
\Phi^0(x_2) &= \sum_{k\in Z}a_k\delta(x_2-\frac{k}{n}L_2)=\sum_{k=0}^{n-1}\sum_{k_2\in Z}a_{k+nk_2}\delta(x_2-\frac{k+nk_2}{n}L_2) \\
&= \sum_{k=0}^{n-1}a_k\sum_{k_2\in Z}\delta(x_2-\frac{k+nk_2}{n}L_2) \\
&= \sum_{k=0}^{n-1}a_k\Lambda_k^0(x_2)
\end{aligned}
\tag{24}
$$

where $(x_2^{(k)} \equiv x_2 - \frac{k}{n} L_2)$

$$\Lambda_k^0(x_2) \equiv \sum_{k_2 \in Z} \delta(x_2^{(k)} - k_2 L_2) = \frac{1}{L_2} \sum_{q \in Z} e^{i2\pi q x_2^{(k)}/L_2} \qquad (25)$$

Therefore, the dimension of the Hilbert space $\mathcal{H}^0(\widetilde{G})$ is $n$, since it is spanned by the functions $\Lambda_k^0(x_2)$, $k = 0, 1, ..., n-1$.

Next, we determine the subgroup $G_{\mathcal{H}}$ of good transformations (those preserving the structure of the wave function). As in the case of the cylindrical H-W group, it is deduced from $Ad(\widetilde{G}) \left[e_{\vec{k}'}, g\right] = \left(0, 0, e^{\frac{i}{\hbar} m\omega(k_1 L_1 x_2 - k_2 L_2 x_1)}\right) \subset G_{\mathcal{P}} \; \forall \vec{k} \in Z \times Z$, which implies $\frac{m\omega}{\hbar}(k_1 L_1 x_2 - k_2 L_2 x_1) = 2\pi k$, $k \in Z$, and, together with the quantization condition (19) for $\omega$, leads to $\vec{x} = \frac{1}{n}\vec{L}_{\vec{k}}$. Therefore, the subgroup $G_{\mathcal{H}}$ of good transformations is the subgroup of $\widetilde{G}$ in which the parameters $\vec{x}$ are restricted to be $\vec{x} = \frac{1}{n}\vec{L}_{\vec{k}}$, although only a finite number of them corresponding to $\{\vec{x} = (\frac{k_1}{n}L_1, \frac{k_2}{n}L_2), \; k_1, k_2 = 0, 1, ..., n-1\}$ are actually different, due to the $T$-function condition. Consequently, no infinitesimal transformation (apart from that of $U(1)$) preserves the structure of the wave function.

If we introduce the (finite) operators $\hat{\eta}_i \equiv e^{L_i \tilde{X}_{x_i}^R}$, $i = 1, 2$, in a similar way as in Sec. 2.1.1 (although here they represent finite translations), we can write the elements of $T$ as $e_{\vec{k}} \equiv (\hat{\eta}_1)^{k_1}(\hat{\eta}_2)^{k_2}$, and the subgroup of good operators is:

$$G_{\mathcal{H}} = \left\{\zeta(\hat{\eta}_1)^{\frac{k_1}{n}}(\hat{\eta}_2)^{\frac{k_2}{n}}, \;\; k_1, k_2 \in Z, \; \zeta \in U(1)\right\} \qquad (26)$$

As in Sec. 2.1.1, the set of bad operators can be interpreted as quantization-changing operators, sweeping the space of all non-equivalent quantizations. As was proven there, the action of a bad operator takes the wave function out of our Hilbert space $\mathcal{H}^0(\widetilde{G})$ and puts it into a different Hilbert space $\mathcal{H}^{\vec{\alpha}}(\widetilde{G})$ corresponding to a non-equivalent representation of $T_B (= T)$ parameterized by $\vec{\alpha}$. Thus, we define the new functions (we restrict ourselves to the $\Phi(x_2)$ part of the wave function):

$$\begin{aligned}
\Phi^{\vec{\alpha}}(x_2) &\equiv e^{\alpha_1 \tilde{X}_{x_1}^R + \alpha_2 \tilde{X}_{x_2}^R} \Phi^0(x_2) = e^{i2\pi n \frac{x_2}{L_2} \frac{\alpha_1}{L_1}} \Phi^0(x_2 + \alpha_2) \\
&= \sum_{k=0}^{n-1} a_k e^{i2\pi n \frac{x_2}{L_2} \frac{\alpha_1}{L_1}} \Lambda_k^0(x_2 + \alpha_2) = \sum_{k=0}^{n-1} a_k \Lambda_k^{\vec{\alpha}}(x_2)
\end{aligned} \qquad (27)$$

where

$$\Lambda_k^{\vec{\alpha}}(x_2) \equiv e^{i2\pi n \frac{x_2}{L_2} \frac{\alpha_1}{L_1}} \Lambda_k^0(x_2 + \alpha_2) = e^{i2\pi n \frac{x_2}{L_2} \frac{\alpha_1}{L_1}} \frac{1}{L_2} \sum_{q \in Z} e^{i2\pi q(x_2^{(k)} + \alpha_2)/L_2} \qquad (28)$$

and the values of $\vec{\alpha}$ are different from $\frac{1}{n}\vec{L}_{\vec{k}}$ (good transformations).

To determine the non-equivalent quantizations (i.e. the minimum range of values of the parameters $\alpha_1$ and $\alpha_2$ that sweeps the whole set of non-equivalent quantizations)

we let the transformations of $T$ act on these new functions and then we determine the quasi-periodicity conditions:

$$
\begin{align}
(\hat{\eta}_1)^{k_1} \Phi^{\vec{\alpha}}(x_2) &= e^{-i2\pi n \frac{\alpha_2}{L_2} k_1} \Phi^{\vec{\alpha}}(x_2) \tag{29} \\
(\hat{\eta}_2)^{k_2} \Phi^{\vec{\alpha}}(x_2) &= e^{i2\pi n \frac{\alpha_1}{L_1} k_2} \Phi^{\vec{\alpha}}(x_2) \tag{30}
\end{align}
$$

from which it can be deduced that $\alpha_1 \in [0, \frac{L_1}{n})$ and $\alpha_2 \in [0, \frac{L_2}{n})$. This range of values is associated with the first Brillouin zone of the reciprocal lattice, as can be checked if we define the parameters $\vec{\epsilon} \equiv m\omega \hat{\mathrm{J}} \cdot \vec{\alpha}$. It is easy to verify that the wave functions $\{\Lambda_k^{\vec{\alpha}}(x_2),\ k = 0, 1, ..., n-1\}$ constitute the carrier space (they span $\mathcal{H}^{\vec{\alpha}}(\widetilde{G})$) for unitary irreducible representations (parameterized by $\vec{\alpha}$) of the subgroup of good operators. Under these operators the wave functions transform as:

$$
\begin{align}
(\hat{\eta}_1)^{k_1/n} \Lambda_k^{\vec{\alpha}}(x_2) &= e^{i2\pi(\frac{k}{n} - \frac{\alpha_2}{L_2})k_1} \Lambda_k^{\vec{\alpha}}(x_2) \tag{31} \\
(\hat{\eta}_2)^{k_2/n} \Lambda_k^{\vec{\alpha}}(x_2) &= e^{i2\pi \frac{\alpha_1}{L_1} k_2} \Lambda_{k-k_2 \bmod n}^{\vec{\alpha}}(x_2) \tag{32}
\end{align}
$$

In a recent paper, [**G**], it is shown that, for the case $n = 1$, the symplectic manifold defined by the torus can be fully quantized, i.e. the entire Poisson algebra on the torus can be irreducibily represented by self-adjoint operators acting on a Hilbert space. Here, the same result is obtained for arbitrary integer $n$. Even more, more operators than those associated with classical functions (those of $T$) can be irreducibily represented, namely $(\hat{\eta}_i)^{\frac{k_i}{n}}$, $k_i \in Z, i = 1, 2$. To be precise, the $n^{th}$'s roots of the classical functions can be quantized according to our scheme. This is possible thanks to the fact that the representation defined by the equations above is a vector representation, i.e. wave functions are really sections of an associated vector bundle of dimension $n$ over the torus.

As in Sec. 2.1.1, we could consider the union of all the Hilbert spaces $\cup_{\vec{\alpha}} \mathcal{H}^{\vec{\alpha}}(\widetilde{G})$. In this Hilbert space, the bad operators $\tilde{X}_{\vec{x}}^R$ are Hermitian and act irreducibly, carrying a unitary irreducible representation of the toral H-W group, turning out to be a generalization, for arbitrary integer $n$, of that called $kq$-representation in Solid State Physics [**Z**], where only the case $n = 1$ is considered.

*Summarizing* the integer case, there is a continuum of non-equivalent quantizations, corresponding to non-equivalent representations of $T$ parameterized by $\vec{\alpha}$, giving rise to different quasi-periodic boundary conditions. The value $\vec{\alpha} = 0$, corresponding to the trivial representation $D^0(e_{\vec{k}}) = 1$ of $\{e_{\vec{k}};\ \vec{k} \in Z \times Z\}$, reproduces the standard periodic boundary conditions. The wave functions are (27-28) with quasi-periodicity conditions given by (29-30) and the subgroup of good operators is (26).

It should be stressed the difference between the two representations here obtained. On the one hand, for a fixed $\vec{\alpha}$, the Hilbert space $\mathcal{H}^{\vec{\alpha}}(\widetilde{G})$ carries an irreducible representation of the subgroup of good operators. In this representation the operators $\tilde{X}_{\vec{x}}^R$ do not preserve the Hilbert space; they are bad operators. On the other hand, the union of all the Hilbert spaces $\cup_{\vec{\alpha}} \mathcal{H}^{\vec{\alpha}}(\widetilde{G})$ carries an irreducible representation of the entire toral H-W group, in

such a way that the operators $\tilde{X}^R_{\vec{x}}$ are Hermitian and the good operators act in a diagonal form.

A brief comment is now in order. Let us consider the discrete (infinite) subgroup generated by $\left\{(\hat{\eta}_1)^{\frac{k_1}{n}}(\hat{\eta}_2)^{\frac{k_2}{n}}, \; k_1, k_2 \in Z\right\}$, which constitutes a principal fibre bundle with base $Z \times Z$ and fibre $Z_n \subset U(1)$. The group algebra of this discrete group can be proven to be (in a suitable basis) an infinite-dimensional trigonometric algebra [**F-F-Z**]. Since $n$ is an integer, this discrete group has a centre, which can be removed by means of the $T$-function condition. The quotient group is the finite group generated by $\left\{(\hat{\eta}_1)^{\frac{k_1}{n}}(\hat{\eta}_2)^{\frac{k_2}{n}}, \right.$ $\left. k_1, k_2 = 0, ..., n-1\right\}$. This finite group (which can be seen as a finite version of the Heisenberg-Weyl group) constitutes a principal fibre bundle with base $Z_n \times Z_n$ and fibre $Z_n \subset U(1)$, and admits a simple matrix representation given in Ref. [**W**] (see also [**F-F-Z**] and [**F**]). The corresponding group algebra is the algebra of $SU(n) \times U(1)$ for $n$ odd or $U(n/2)$ for $n$ even, in a trigonometric basis [**F-F-Z**]. By means of this representation, the limit $n \to \infty$ (the "classical" limit) is particularly simple, leading to the algebra of infinitesimal area-preserving diffeomorphisms of a 2D-surface (the torus, in this case). This algebra, referred to as $\omega_\infty$ in the literature, is the classical version of a variety of infinite-dimensional algebras called collectively $W_\infty$, of increasing interest nowadays (see [**S**] for a review). In this sense, the subgroup of good operators $G_{\mathcal{H}}$ can be seen as the quantum version of the area-preserving diffeomorphisms of the torus, thus constituting a realization of the $W_\infty$ algebras on the torus.

### 2.2.2 Fractional quantum numbers

We now consider the rational case, in which $\frac{m\omega L_1 L_2}{2\pi\hbar} = \frac{n}{r}$. In this case $T$ has a non-trivial characteristic subgroup, i.e. there are non-trivial elements commuting with the whole group $T$. This is $G_C = \{r\vec{L}_{\vec{k}}, \vec{k} \in Z^2\}$, and the polarization subgroup, which must contain $G_C$, is $T_p = G_C \cup \{k\vec{L}_{\vec{k}_p}, k \in Z\}$, where $\vec{k}_p$ is a vector the components of which are either relative prime integers, (1,0) or (0,1). This condition is required for maximality of the polarization subgroup, and therefore for the irreducibility of the representation of $T$.

The $T$-function condition now reads $\Psi(g_{T_B} * g) = \mathcal{D}(g_{T_B})\Psi(g)$, where $T_B \equiv T_p \cup U(1)$ is the maximal subgroup of compatible constraints that can be applied to the wave function, and $\mathcal{D}(g_{T_B})$ is a representation of $T_B$ on the complex numbers. For the moment, we shall use the representation $\mathcal{D}^0(e_{T_p}, \zeta) = \zeta$, which is trivial for the elements in $T_p$. Later, the non-equivalent representations of $T_B$ will be straightforwardly computed, as in Sec 2.2.1. The $T_B$-function condition on the polarized wave functions (17) is then:

$$e^{\frac{i}{\hbar}m\omega\left\{y_2+(1+2\lambda)n_1n_2y_1+[(1+\lambda)n_2^2-\lambda n_1^2](\vec{n}\cdot\hat{J}\cdot\vec{L}_{r\vec{k}+k\vec{k}_p})\right\}(\vec{n}\cdot\vec{L}_{r\vec{k}+k\vec{k}_p})} \times$$
$$e^{-\frac{i}{\hbar}m\omega[(1+2\lambda)n_1n_2(y_1+y_2)](\vec{n}\cdot\hat{J}\cdot\vec{L}_{r\vec{k}+k\vec{k}_p})}\Phi^0\left(y_2 + \vec{n}\cdot\hat{J}\cdot(\vec{L}_{r\vec{k}+k\vec{k}_p})\right) \quad = \quad \Phi^0(y_2) \qquad (33)$$

$\forall k \in Z$, and $\forall \vec{k} \in Z \times Z$. As in the integer case, the only polarization vectors $\vec{n}$ consistent

with these restrictions are $\vec{n} = (1,0)$ and $\vec{n} = (0,1)$, and the same for $\vec{k}_p$, for which the only possible values are $\vec{k}_p = (1,0)$ and $\vec{k}_p = (0,1)$.

Let us fix the polarization to $\vec{n} = (1,0)$ for concreteness (the case $\vec{n} = (0,1)$ is completely analogous and in fact leads to an equivalent representation). The two different choices of $\vec{k}_p$, perpendicular and parallel to $\vec{n}$, lead to *non-equivalent* representations [this is a general feature in AQG: for a given polarization in $\widetilde{G}$, different choices of polarization subgroups $T_p$ in $T$ can lead to non-equivalent quantizations, even though the polarization subgroups were equivalent from the point of view of the subgroup $T$ itself (see [**A-N-R**])], both with dimension $n$ and with $\lambda$ restricted to be $\lambda = k/n$, $k \in Z$:

a) $\vec{L}_{\vec{k}_p} \perp \vec{n}$  i.e. $\vec{k}_p = (0,1)$, then the wave function is peaked at the values $y_2 = x_2 = \frac{k}{n}L_2$, $k \in Z$, satisfies $\Phi_\perp^0(x_2 + k_2 L_2) = \Phi_\perp^0(x_2)$, and has the form

$$\Phi_\perp^0(x_2) = \sum_{k=0}^{n-1} a_k \Lambda_k^0(x_2) \tag{34}$$

where $\Lambda_k^0(x_2)$ is defined as in Sec 2.2.1, and the subgroup of good transformations is $G_{\mathcal{H}} = \{\frac{r}{n}\vec{L}_{\vec{k}}, \vec{k} \in Z \times Z\} \cup \{\frac{k}{n}L_2, k \in Z\}$, although only a finite subgroup of them are distinct:

$$G_{\mathcal{H}}^\perp = \left\{ (\hat{\eta}_1)^{r\frac{k_1}{n}}, (\hat{\eta}_2)^{\frac{k_2}{n}}, \quad k_1, k_2 = 0, ..., n-1 \right\} \tag{35}$$

b) $\vec{L}_{\vec{k}_p} \parallel \vec{n}$  i.e. $\vec{k}_p = (1,0)$, then the wave function is peaked at the values $y_2 = x_2 = k\frac{r}{n}L_2$, $k = 0, 1, ...n-1$, satisfies $\Phi_\parallel^0(x_2 + rk_2 L_2) = \Phi_\parallel^0(x_2)$, and has the form

$$\Phi_\parallel^0(x_2) = \sum_{k=0}^{n-1} a_k \Lambda_k^{r,0}(x_2) \tag{36}$$

where $\Lambda_k^{r,0}(x_2) \equiv \frac{1}{rL_2} \sum_{q \in Z} e^{i2\pi q x_2^{r,(k)}/(rL_2)}$, with $x_2^{r,(k)} \equiv x_2 - \frac{k}{n}rL_2$, and the subgroup of good transformations is $G_{\mathcal{H}} = \{\frac{r}{n}\vec{L}_{\vec{k}}, \vec{k} \in Z \times Z\} \cup \{\frac{k}{n}L_1, k \in Z\}$. Again, only a finite subgroup of them are distinct:

$$G_{\mathcal{H}}^\parallel = \left\{ (\hat{\eta}_1)^{\frac{k_1}{n}}, (\hat{\eta}_2)^{r\frac{k_2}{n}}, \quad k_1, k_2 = 0, ..., n-1 \right\} \tag{37}$$

As in Sec. 2.1.1, we can compute the non-equivalent representations by applying the whole set of bad operators to the wave functions. We proceed as in the integer case (Sec 2.2.1) and obtain:

a) $\vec{L}_{\vec{k}_p} \perp \vec{n}$   the wave functions have the form

$$\Phi_\perp^{\vec{\alpha}_p}(x_2) = \sum_{k=0}^{n-1} a_k \Lambda_k^{\vec{\alpha}_p}(x_2) \tag{38}$$

15

with $\Lambda_k^{\vec{\alpha}_p}(x_2) \equiv e^{i2\pi \frac{n}{r} \frac{\alpha_{p1}}{L_1} \frac{x_2}{L_2}} \Lambda_k^0(x_2)$. They satisfy

$$(\hat{\eta}_1)^{rk_1} \Phi_\perp^{\vec{\alpha}_p}(x_2) = e^{-i2\pi n \frac{\alpha_{p2}}{L_2} k_1} \Phi_\perp^{\vec{\alpha}_p}(x_2) \tag{39}$$

$$(\hat{\eta}_2)^{k_2} \Phi_\perp^{\vec{\alpha}_p}(x_2) = e^{i2\pi n \frac{\alpha_{p1}}{rL_1} k_2} \Phi_\perp^{\vec{\alpha}_p}(x_2) \tag{40}$$

with $\alpha_{p1} \in [0, r\frac{L_1}{n})$, $\alpha_{p2} \in [0, \frac{L_2}{n})$.

b) $\vec{L}_{\vec{k}_p} \parallel \vec{n}$   the wave functions have the form

$$\Phi_\parallel^{\vec{\beta}_p}(x_2) = \sum_{k=0}^{n-1} a_k \Lambda_k^{r,\vec{\beta}_p}(x_2) \tag{41}$$

with $\Lambda_k^{r,\vec{\beta}_p}(x_2) \equiv e^{i2\pi \frac{n}{r} \frac{\beta_{p1}}{L_1} \frac{x_2}{L_2}} \Lambda_k^{r,0}(x_2)$. They satisfy

$$(\hat{\eta}_1)^{k_1} \Phi_\parallel^{\vec{\beta}_p}(x_2) = e^{-i2\pi n \frac{\beta_{p2}}{rL_2} k_1} \Phi_\parallel^{\vec{\alpha}_p}(x_2) \tag{42}$$

$$(\hat{\eta}_2)^{rk_2} \Phi_\parallel^{\vec{\beta}_p}(x_2) = e^{i2\pi n \frac{\beta_{p1}}{L_1} k_2} \Phi_\parallel^{\vec{\beta}_p}(x_2) \tag{43}$$

with $\beta_{p1} \in [0, \frac{L_1}{n})$, $\beta_{p2} \in [0, r\frac{L_2}{n})$.

It is easy to verify that the wave functions $\{\Lambda_k^{\vec{\alpha}_p}(x_2), k = 0, 1, ..., n-1\}$ and $\{\Lambda_k^{r,\vec{\beta}_p}(x_2), k = 0, 1, ..., n-1\}$ constitute the carrier spaces for unitary irreducible representations (parameterized by $\vec{\alpha}$ and $\vec{\beta}$, respectively) of the subgroup of good operators. Under these operators they transform as:

a) $\vec{L}_{\vec{k}_p} \perp \vec{n}$

$$(\hat{\eta}_1)^{rk_1/n} \Lambda_k^{\vec{\alpha}_p}(x_2) = e^{i2\pi(\frac{k}{n} - \frac{\alpha_{p2}}{L_2})} \Lambda_k^{\vec{\alpha}_p}(x_2) \tag{44}$$

$$(\hat{\eta}_2)^{k_2/n} \Lambda_k^{\vec{\alpha}_p}(x_2) = e^{i2\pi \frac{\alpha_1}{rL_1} k_2} \Lambda_{k-k_2 \bmod n}^{\vec{\alpha}_p}(x_2) \tag{45}$$

b) $\vec{L}_{\vec{k}_p} \parallel \vec{n}$

$$(\hat{\eta}_1)^{k_1/n} \Lambda_k^{r,\vec{\beta}_p}(x_2) = e^{i2\pi(\frac{k}{n} - \frac{\beta_{p2}}{rL_2})} \Lambda_k^{r,\vec{\beta}_p}(x_2) \tag{46}$$

$$(\hat{\eta}_2)^{rk_2/n} \Lambda_k^{r,\vec{\beta}_p}(x_2) = e^{i2\pi \frac{\beta_{p1}}{L_1} k_2} \Lambda_{k-k_2 \bmod n}^{r,\vec{\beta}_p}(x_2) \tag{47}$$

It should be noted that although $\frac{m\omega L_1 L_2}{2\pi\hbar} = \frac{n}{r}$, the dimension of the representations is $n$, and $\lambda = k/n$, $k \in Z$, as in the integer case (even more, in the case $\vec{L}_{\vec{k}_p} \perp \vec{n}$ the wave functions coincide); the difference is found in the subgroups of good operators, which, although isomorphic, differ in the specific values of the transformations. This

representation can be reinterpreted as mimicking a torus $r$ times greater in one direction (determined by the orthogonal vector to $\vec{k}_p$), i.e., the area of the effective torus is $rL_1L_2$, and therefore $\frac{m\omega(rL_1L_2)}{2\pi\hbar} = n$. Thus, the same results as in the integer case now apply, although changing $L_2$ by $rL_2$ if $\vec{k}_p = (1,0)$ or $L_1$ by $rL_1$ if $\vec{k}_p = (0,1)$.

*Summarizing* the fractional case, there are two continua of non-equivalent quantizations, according to the choices $\vec{L}_{\vec{k}_p} \perp \vec{n}$ and $\vec{L}_{\vec{k}_p} \parallel \vec{n}$, parameterized by $\vec{\alpha}_p$ and $\vec{\beta}_p$, respectively. The wave functions are given by (38) and (41), satisfying quasi-periodicity conditions given by (39-40) and (42-43), respectively. The subgroups of good operators are given by (35) and (37), respectively.

*Associated $r$-vector bundle:* If we act on the wave functions with the bad operators of $T$ (i.e. those operators of $T$ which are not in $T_B$) the resulting wave functions lie in a different Hilbert space belonging to a different quantization. However, as these operators are finite and their $r^{th}$ power are good operators, these new wave functions transform among each other under the action of the subgroup $T_{\mathrm{bad}}$, defined as the set of bad operators of $T$ and the identity. Therefore, constructing the vector space spanned by these $r$ functions ($T_{\mathrm{bad}}$ has $r$ elements), we obtain an $r$-dimensional, unitary irreducible representation of the group $T$ as a whole, including the bad operators. Explicitly:

a) $\vec{L}_{\vec{k}_p} \perp \vec{n}$    we define

$$\Lambda_{k,j}^{\vec{\alpha}_p}(x_2) \equiv (\hat{\eta}_1)^j \Lambda_k^{\vec{\alpha}_p}(x_2) = e^{i2\pi \frac{n}{r} \frac{x_2}{L_2} j} \Lambda_k^{\vec{\alpha}_p}(x_2) \tag{48}$$

for $j = 0, 1, ..., r-1$, where they satisfy:

$$(\hat{\eta}_1)^{j'} \Lambda_{k,j}^{\vec{\alpha}_p}(x_2) = e^{-i2\pi n \frac{\alpha_{p2}}{L_2}(j+j' \operatorname{div} r)} \Lambda_{k,j+j' \bmod r}^{\vec{\alpha}_p}(x_2) \tag{49}$$

for $j, j' = 0, 1, ..., r-1$.

b) $\vec{L}_{\vec{k}_p} \parallel \vec{n}$    we define

$$\Lambda_{k,j}^{r,\vec{\alpha}_p}(x_2) \equiv (\hat{\eta}_2)^j \Lambda_k^{r,\vec{\alpha}_p}(x_2) = \Lambda_k^{r,\vec{\alpha}_p}(x_2 + jL_2) \tag{50}$$

for $j = 0, 1, ..., r-1$, satisfying:

$$(\hat{\eta}_2)^{j'} \Lambda_{k,j}^{r,\vec{\alpha}_p}(x_2) = e^{i2\pi n \frac{\alpha_{p1}}{L_1}(j+j' \operatorname{div} r)} \Lambda_{k,j+j' \bmod r}^{r,\vec{\alpha}_p}(x_2) \tag{51}$$

for $j, j' = 0, 1, ..., r-1$.

This construction can be viewed as the $r$-dimensional vector bundle associated with the principal bundle $\widetilde{G}$, which has structure group $T$. The $r$-component wave functions are sections of this associated vector bundle.

As stated before, AQG generalizes Geometric Quantization in some respects, in particular in that which concerns (topologic) quantum numbers. The fractional value $\frac{m\omega L_1 L_2}{2\pi\hbar} =$

$\frac{n}{r}$ generalizes the integer class of the standard symplectic form (the Chern class of the line bundle). The geometric quantization of a symplectic manifold with "fractional class" $\frac{n}{r}$ would have led to r-valued wave functions (as opposed to single-valued). Eventually, this trouble could have been circumvented by replacing the usual line bundle by a complex vector bundle $E$ of rank $r$ and Chern class $n$, as constructed before.

The comments at the end of Sec. 2.2.2 concernig the generalized $kq$-representation can be translated to the $r$-bundle structure associated with the fractional case.

### 2.2.3 Irrational case

Finally, and for the sake of thoroughness, let us briefly comment on the case in which $\rho \equiv \frac{m\omega L_1 L_2}{2\pi\hbar}$ is an irrational number. In this case the characteristic group is trivial, and $T_B = T_p \cup U(1)$, with $T_p = \{k\vec{L}_{\vec{k}_p}, k \in Z\}$ only. As before, it can be proven that the only possible polarization vectors are $\vec{n} = (1,0)$ and $\vec{n} = (0,1)$. Moreover, the only consistent choice of polarizations $T_p$ in $T$ are also $\vec{k}_p = (1,0)$ and $\vec{k}_p = (0,1)$. No restriction for $\lambda$ appears in this case, and the structure of $T_B$-function condition closely resembles that of the case of the cylindrical H-W group: the wave functions are either peaked at an infinite series of equally spaced values of $y_2$ if $\vec{k}_p \parallel \vec{n}$ (as in the momentum space representation in the cylindrical H-W group), or quasi-periodic if $\vec{k}_p \perp \vec{n}$ (as in the configuration space representation in the cylindrical H-W group). In both cases the non-equivalent representations are labelled by $\epsilon \in [0, \frac{2\pi\hbar}{|\vec{L}_{\vec{k}_p}|})$. The representations are therefore infinite dimensional, and the subgroup of good operators is given by $G_{\mathcal{H}} = \{\frac{1}{\rho}\vec{L}_{\vec{k}}, \vec{k} \in Z \times Z\} \cup \{\alpha\vec{L}_{\vec{k}_p}, \alpha \in R\}$. Consequently, besides the discrete transformations in $\vec{x} = \frac{1}{\rho}\vec{L}_{\vec{k}}$, the infinitesimal operator $\vec{L}_{\vec{k}_p} \cdot \tilde{X}^R_{\vec{x}}$ is also a good operator, that is, arbitrary translations in the direction of $\vec{L}_{\vec{k}_p}$ are good transformations. Note that, $\rho$ being an irrational number, $\frac{1}{\rho}\vec{L}_{\vec{k}}$ never reaches a point of the lattice defined by $\vec{L}_{\vec{k}}$, although it fills the corresponding torus densely when varying $\vec{k} \in Z \times Z$.

Therefore, in this case, the subgroups $x_1 = 0$ and $x_2 = 0$ (the classical circumferences), are represented faithfully, as in the case of the cylindrical H-W group, but the rest of the group is not faithfully represented, nor are even the points of the lattice (the group $T$). In particular, for the infinite-order operators $\hat{\eta}_1, \hat{\eta}_2$, defined as in Sec 2.1.1 for the directions $x_1, x_2$, only one is a good operator (the one in the direction of $\vec{L}_{\vec{k}_p}$), the other being a bad operator. Consequently, we cannot represent the toral H-W group faithfully for irrational values of $\rho$.

# 3  Free Galilean particle on the circumference

Let us apply the results obtained in the last section to the simple example of the free particle moving on the circumference.

We can study this problem easily by simply adding the temporal evolution to the results obtained in Sec. 2.1 (for the group law, vector fields, polarizations, Schrödinger equation, etc., see [**A-B-G-N**] and references therein), without affecting the main conclusions of that section. The main new features are the introduction of a new operator $\hat{\text{E}}$ associated with the temporal evolution and the fact that, by using the Schrödinger equation, this operator can be written in terms of the momentum operator as $\frac{1}{2m}\hat{\text{P}}^2$. Since $\hat{\text{P}}$ is a good operator, $\hat{\text{E}}$ proves also to be a good operator. A common set of eigenfunctions is given by

$$
\begin{array}{rcl}
\phi_n^\epsilon(x,t) & = & e^{-\frac{i}{\hbar}\frac{1}{2m}(\epsilon+\frac{2\pi\hbar}{L}n)^2 t} e^{\frac{i}{\hbar}(\epsilon+\frac{2\pi\hbar}{L}n)x} \\[2mm]
\hat{\text{E}}\phi_n^\epsilon & = & \frac{1}{2m}(\epsilon + \frac{2\pi\hbar}{L}n)^2 \phi_n^\epsilon \\[2mm]
\hat{\text{P}}\phi_n^\epsilon & = & (\epsilon + \frac{2\pi\hbar}{L}n)\phi_n^\epsilon
\end{array}
\tag{52}
$$

where $n \in Z$. Note that for $\epsilon = 0$ the states $n$ and $-n$ have the same energy, which means that all the energy eigenstates except for the vacuum are degenerate. For any other value of $\epsilon$, the states $n$ and $-(n + 2\epsilon\frac{L}{2\pi\hbar})$ have the same energy, but $-(n + 2\epsilon\frac{L}{2\pi\hbar})$ is an eigenstate only if $2\epsilon\frac{L}{2\pi\hbar} \in Z$, i.e. $\epsilon\frac{L}{2\pi\hbar}$ is integer or half-integer. This means that, in general, there is no degeneracy for any value of $\epsilon$ except for the integer values, in which case all the eigenstates are doubly degenerate except for the vacuum, and half-integers, for which all the eigenstates, including the vacuum, are doubly degenerate. The phenomenon of degenerate ground state in this simple model parallels $\theta$-vacuum phenomenon in Yang-Mills field theories [**A-E-P**].

The feature of non-equivalent quantizations can be reproduced (in an equivalent way, indeed) by the introduction of an extra coboundary in (1) (more precisely, in its counterpart when the temporal evolution is added; see [**A-B-G-N**]) generated by the function $\epsilon x$, i.e. a multiplicative factor of the form $e^{\frac{i}{\hbar}\epsilon\frac{p'}{m}t}$ in the $\zeta \in U(1)$ composition law [We remind that $x'' = x' + x + \frac{p'}{m}t$ is the composition law for $x$ when the temporal evolution is added]. In the case of the free Galilean particle on the real line, the only consequence of this term is the appearance of a total derivative in the quantization 1-form $\Theta$ (or, what is the same, in the Lagrangian), leading thus to equivalent (classical and quantum) theories, as expected from the fact that $\epsilon\frac{p'}{m}t$ is a coboundary. The situation is quite different when the particle is on the circumference: the generating function $\epsilon x$, or better $e^{\frac{i}{\hbar}\epsilon x}$, is not single-valuate on the circumference unless $\epsilon = \frac{2\pi\hbar}{L}k$, $k \in Z$. As a consequence, two cocycles differing in a coboundary generated by $\epsilon x$ (and therefore leading to equivalent theories on the real line) lead to non-equivalent theories on the circumference if $\epsilon \neq \frac{2\pi\hbar}{L}k$, $k \in Z$. This process of creation of non-trivial cohomology closely resembles the appearance of

cohomology under the process of group contraction, as in the case of the Poincaré group, in which a certain class of coboundaries (generated by a linear function in time) become true cocycles in the $c \rightarrow \infty$ limit since their generating function goes to infinity in this limit.

Another interesting way of interpreting the feature of non-equivalent quantizations parameterized by $\epsilon$, at least in the case of charged particles, is as an Aharonov-Bohm-like effect. The different quantizations can be carried out physically by producing (externally, with the help of a solenoid) a magnetic flux $\Phi$ through the circumference, in a way that the particle does not feel the magnetic field, but rather the vector potential only. Under these circumstances, the effect of the vector potential is the same as that of a boost, leading to non-equivalent quantizations depending on the flux through the circumference, in such a way that $\epsilon = e\Phi/c$. An interesting physical application is that of a superconducting ring threaded by a magnetic flux, where by Meissner effect the magnetic flux is pulled out of the interior region of the superconducting ring, and therefore the magnetic field is effectively zero and only the vector potential is relevant (Aharonov-Bohm effect). If the flux is [in this case the effective electric charge is $e^* = 2e$ because electrons form Cooper pairs] $k\Phi_0$, $k \in Z$, where $\Phi_0 \equiv hc/e^*$ is the quantum unit of flux, there is no net current in the superconducting ring, but for any other value of the flux there is a net current which has the form given in Figure 1.

Note that for half-integer values of $k$, the net current has no definite sign, as a consequence, precisely, of the double degeneracy of states, in such a way that states with opposite signs of velocity have the same energy and therefore there is no energy cost to pass from one to the other.

## 3.1 Failure of the Ehrenfest theorem

As mentioned in the introduction, the most common problem appearing in systems with topologically non-trivial configuration space is the failure of Ehrenfest theorem for certain operators ("anomalous" operators) [**E**]. Ehrenfest theorem asserts that the expectation values of quantum operators follow classical equations of motion:

$$\frac{d}{dt} < \hat{A} > = \frac{i}{\hbar} < \left[\hat{H}, \hat{A}\right] > \tag{53}$$

In the framework of AQG, this is a natural consequence of the appearance of bad operators, those which do not preserve the Hilbert space of wave functions that verify the $T$-function condition.

In Ref. [**E**] it is claimed that when the operator $\hat{A}$ does not keep invariant the domain of $\hat{H}$, then an extra term appears in the r.h.s. of (53), which is interpreted as an anomaly. In the language of AQG, we would say that $\hat{A}$ is a bad operator, so that neither the left-hand side nor the right-hand side of (53) would make sense, since the operator $\hat{A}$ takes the wave function off the Hilbert space where the Hamiltonian $\hat{H}$ is self-adjoint (of course

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$\hat{H}$ is a good operator; otherwise the temporal evolution would take the physical states off the Hilbert space, and the system would have no physical meaning). The appearance of the "anomalous" term violating the Ehrenfest theorem is a consequence of this fact.

Returning to the free Galilean particle on the circumference, the Ehrenfest theorem will fail for the position operator, which is a bad operator and therefore Eq. (53) makes no sense in this regard.

In conclusion, whenever there are bad operators in the theory, the Ehrenfest theorem will fail for each of these operators and, in general, any expectation value involving these operators will be ill-defined, giving extra terms that can eventually be interpreted as topologic anomalies.

# 4 Charged particle in a homogeneous magnetic field on the torus

Now, we shall consider the most interesting problem of a charged particle moving on a torus in the presence of a homogeneous magnetic field. This problem is related to the Schwinger model [**M**], and has important applications in the Quantum Hall effect [**K-D-P**, **La**, **T**]. The magnetic field is perpendicular to the torus surface, and the total flux is quantized (as we shall see), much in the same manner the Dirac monopole charge is quantized [**W-Y**]. The actual connection of this system with Quantum Hall Effect is based on the fact that the wave function of the complete system factorizes in a relative-coordinate dependent term (which includes interactions) and a centre of mass dependent term, which behaves essentially as a particle in a transverse homogeneous magnetic field, and on the effective topology of the experimental device in the latter system; the topology of the (semiconductor) strip along with the current and voltage leads is that of a punctured torus [**T**].

Firstly, we shall study the planar case, i.e. the charged particle on the plane, to clarify the meaning of the different magnitudes appearing in the problem, and to obtain a proper parameterization of the system.

## 4.1 Charged particle in a homogeneous magnetic field

The movement of a charged particle in a homogeneous magnetic field can be factorized into a 2-dimensional problem (on the plane normal to the magnetic field) times a free movement in the direction of the magnetic field. Thus, we restrict ourselves to a 2-dimensional system characterized by a non-zero commutator between the translation generators, $[\tilde{X}_{x^1}^L, \tilde{X}_{x^2}^L] = im\omega_c/\hbar$, where $\omega_c$ is the cyclotron frequency, $\omega_c = \frac{q\mathcal{H}}{mc}$, $\mathcal{H}$ the magnetic field strength and $q$ the particle electric charge [**L-L**, **C-D-L**].

We have to build up a group law for this system, which must be a deformation of the Galilean group law (in two dimensions) due to the non-zero commutator between the

translation generators. In fact, the Galilean group does not admit any central extension giving rise to $[\tilde{X}^L_{x^1}, \tilde{X}^L_{x^2}] = im\omega_c/\hbar$, and a deformation of the non-extended algebra is required: $\left[X^L_t, X^L_{\vec{x}}\right] = \omega_c \hat{\mathrm{J}} \cdot X^L_{\vec{x}}$. We then arrive at the following Lie algebra as the quantum symmetry for our system:

$$
\begin{aligned}
\left[\tilde{X}^L_t, \tilde{X}^L_{\vec{x}}\right] &= \omega_c \hat{\mathrm{J}} \cdot \tilde{X}^L_{\vec{x}} \\
\left[\tilde{X}^L_t, \tilde{X}^L_{\vec{p}}\right] &= -\frac{1}{m}\tilde{X}^L_{\vec{x}} \\
\left[\tilde{X}^L_{x^i}, \tilde{X}^L_{p^j}\right] &= \frac{\delta_{ij}}{\hbar}\Xi \\
\left[\tilde{X}^L_{x^i}, \tilde{X}^L_{x^j}\right] &= \frac{m\omega_c}{\hbar}\epsilon_{ij}\Xi
\end{aligned}
\qquad (54)
$$

A group law for this centrally extended Lie algebra becomes:

$$
\begin{aligned}
t'' &= t' + t \\
\vec{x}'' &= \vec{x} + \hat{\mathrm{M}}^{-1}(t) \cdot \vec{x}' + \frac{1}{m\omega_c}\left(\hat{\mathrm{N}}^{-1}(t) + \hat{\mathrm{J}}\right) \cdot \vec{p}' \\
\vec{p}'' &= \vec{p}' + \vec{p} \\
\zeta'' &= \zeta'\zeta e^{\frac{i}{\hbar}\xi(g',g)}
\end{aligned}
\qquad (55)
$$

where the cocycle is given by:

$$
\xi(g', g) = \frac{1}{2}\left\{m\omega_c \vec{x}' \cdot \hat{\mathrm{N}}(t) \cdot \vec{x} - \vec{p}' \cdot \hat{\mathrm{M}}(t) \cdot \vec{x} + \vec{x}' \cdot \hat{\mathrm{M}}(t) \cdot \vec{p} + \frac{1}{m\omega_c}\vec{p}' \cdot \left(\hat{\mathrm{N}}(t) - \hat{\mathrm{J}}\right) \cdot \vec{p}\right\}
\qquad (56)
$$

The $2 \times 2$ matrices are given by $\hat{\mathrm{M}}(t) \equiv \cos\omega_c t\,\hat{\mathrm{I}} - \sin\omega_c t\,\hat{\mathrm{J}}$, $\hat{\mathrm{N}}(t) \equiv \sin\omega_c t\,\hat{\mathrm{I}} + \cos\omega_c t\,\hat{\mathrm{J}}$, $\hat{\mathrm{M}}(t)$ and $\hat{\mathrm{N}}(t)$ being orthogonal, and $\hat{\mathrm{J}}_{ij} \equiv \epsilon_{ij}$, $\epsilon_{12} = 1$. We have not taken into account the rotations, since they do not play any dynamical rôle, although they are of interest in that, when considered on the torus, they represent a very simple example of a local (in the strict mathematical sense) symmetry of the equation of motion which cannot be realized globally.

The left and right invariants vector fields are easily deduced from the group law:

$$
\begin{aligned}
\tilde{X}^L_t &= \frac{\partial}{\partial t} + \frac{\vec{p}}{m} \cdot \frac{\partial}{\partial \vec{x}} - \omega_c \vec{x} \cdot \hat{\mathrm{J}} \cdot \frac{\partial}{\partial \vec{x}} \\
\tilde{X}^L_{\vec{x}} &= \frac{\partial}{\partial \vec{x}} - \frac{1}{2\hbar}\left[\vec{p} + m\omega_c \hat{\mathrm{J}} \cdot \vec{x}\right]\Xi \\
\tilde{X}^L_{\vec{p}} &= \frac{\partial}{\partial \vec{p}} + \frac{\vec{x}}{2\hbar}\Xi \\
\tilde{X}^L_{\zeta} &= i\zeta\frac{\partial}{\partial \zeta} \equiv \Xi
\end{aligned}
\qquad (57)
$$

$$
\begin{aligned}
\tilde{X}_t^R &= \frac{\partial}{\partial t} \\
\tilde{X}_{\vec{x}}^R &= \hat{\mathrm{M}}(t) \cdot \frac{\partial}{\partial \vec{x}} + \frac{1}{2\hbar}\left[\hat{\mathrm{M}}(t)\cdot\vec{p} + m\omega_c\hat{\mathrm{N}}(t)\cdot\vec{x}\right]\Xi \\
\tilde{X}_{\vec{p}}^R &= \frac{\partial}{\partial\vec{p}} + \frac{1}{m\omega_c}\left(\hat{\mathrm{N}}(t) - \hat{\mathrm{J}}\right)\cdot\frac{\partial}{\partial\vec{x}} - \frac{1}{2\hbar}\left[\hat{\mathrm{M}}(t)\cdot\vec{x} - \frac{1}{m\omega_c}\left(\hat{\mathrm{N}}(t) - \hat{\mathrm{J}}\right)\cdot\vec{p}\right]\Xi \\
\tilde{X}_\zeta^R &= i\zeta\frac{\partial}{\partial\zeta} \equiv \Xi
\end{aligned}
\tag{58}
$$

and from (57) the quantization 1-form is computed:

$$
\Theta = \frac{1}{2}\left[\vec{p}\cdot d\vec{x} - \vec{x}\cdot d\vec{p} - m\omega_c\vec{x}\cdot\hat{\mathrm{J}}\cdot d\vec{x}\right] - \left[\frac{p^2}{2m} + \omega_c\vec{p}\cdot\hat{\mathrm{J}}\cdot\vec{x} + \frac{m\omega_c^2}{2}\vec{x}^2\right]dt + \hbar\frac{d\zeta}{i\zeta}
\tag{59}
$$

the characteristic module of which is $\mathcal{G}_\Theta =< \tilde{X}_t^L >$. From this, the classical equations of motion are written:

$$
\begin{aligned}
\vec{p} &= \vec{P} \\
\vec{x} &= \hat{\mathrm{M}}^{-1}(t)\cdot\vec{r}_0 + \frac{1}{m\omega_c}\hat{\mathrm{J}}\cdot\vec{P}
\end{aligned}
\tag{60}
$$

where $\vec{P}$ and $\vec{r}_0$ are arbitrary constant vectors, parameterizing the (classical) solution manifold. With the aid of the constant $\omega_c$, we may introduce $\vec{R} \equiv \frac{1}{m\omega_c}\hat{\mathrm{J}}\cdot\vec{P}$, so that the second line of the equation above reads $\vec{x} = \hat{\mathrm{M}}^{-1}(t)\cdot\vec{r}_0 + \vec{R}$, i.e. the classical trajectories are circumferences centred at $\vec{R}$, with radius $|\vec{r}_0|$.

The Noether invariants, in terms of the constants $\vec{r}_0$ and $\vec{R}$, are:

$$
\begin{aligned}
i_{\tilde{X}_t^R}\Theta &= \frac{m\omega_c^2}{2}\vec{r}_0{}^2 \equiv H \\
i_{\tilde{X}_{\vec{x}}^R}\Theta &= m\omega_c\hat{\mathrm{J}}\cdot\vec{r}_0 \\
i_{\tilde{X}_{\vec{p}}^R}\Theta &= -(\vec{r}_0 + \vec{R}) \equiv \vec{x}_0
\end{aligned}
\tag{61}
$$

where $H$ is the classical energy of the system. It should be noted that the energy depends only on the radius $|\vec{r}_0|$ of the circumference, and not on the position $\vec{R}$ of its centre, as corresponds to a system with translational invariance [The system possesses translational invariance in the more conventional sense (the magnetic field is homogeneous) although the translation generator $\tilde{X}_{\vec{x}}^R$ does not commute with the Hamiltonian $\tilde{X}_t^R$. In fact, as we shall see later, there exists a translation generator in the Lie algebra (the magnetic translations) which commutes with the Hamiltonian generator].

To obtain the representation in configuration space, we need to impose polarization conditions similar to those of the Galilean case [**A-B-G-N**]:

$$\mathcal{P}^{HO} = <\tilde{X}^L_{\vec{p}}, \tilde{X}^L_t - \frac{i\hbar}{2m}\left(\tilde{X}^L_{\vec{x}}\right)^2> \tag{62}$$

Solving the polarization equations we obtain for the wave functions the general form:

$$\Psi = \zeta e^{-\frac{i}{2\hbar}\vec{x}\cdot\vec{p}}\Phi(\vec{x},t) \tag{63}$$

where $\Phi(\vec{x},t)$ satisfies the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Phi = \left\{-\frac{\hbar^2}{2m}\vec{\nabla}^2 + i\hbar\frac{\omega_c}{2}\vec{x}\cdot\hat{J}\cdot\vec{\nabla} + \frac{m\omega_c^2}{8}\vec{x}^2\right\}\Phi \tag{64}$$

The quantum operators are:

$$\begin{aligned}
\hat{E}\Psi &= i\hbar\frac{\partial}{\partial t}\Psi = \zeta e^{-\frac{i}{2\hbar}\vec{p}\cdot\vec{x}}\left[-\frac{\hbar^2}{2m}\vec{\nabla}^2 + i\hbar\frac{\omega_c}{2}\vec{x}\cdot\hat{J}\cdot\vec{\nabla} + \frac{m\omega_c^2}{8}\vec{x}^2\right]\Phi(\vec{x},t) \\
\hat{\vec{P}}\Psi &= \zeta e^{-\frac{i}{2\hbar}\vec{p}\cdot\vec{x}}\left[-i\hbar\hat{M}(t)\cdot\vec{\nabla} + \frac{m\omega_c}{2}\hat{N}(t)\cdot\vec{x}\right]\Phi(\vec{x},t) \\
\hat{\vec{X}}\Psi &= \zeta e^{-\frac{i}{2\hbar}\vec{p}\cdot\vec{x}}\left[\frac{1}{2}\left(\hat{M}(t)+\hat{I}\right)\cdot\vec{x} + \frac{i\hbar}{m\omega_c}\left(\hat{N}(t)-\hat{J}\right)\cdot\vec{\nabla}\right]\Phi(\vec{x},t)
\end{aligned} \tag{65}$$

Instead of proceeding further and solving the Schrödinger equation explicitly, we shall perform a change of variables which will clarify the meaning of the different magnitudes entering the theory and which will make facilitate the accomplishment of AQG in the next subsection. If we define $\vec{r} \equiv \hat{M}^{-1}(t)\cdot\vec{r}_0$, we can easily rewrite the group law (55) and (56) in terms of $\vec{r}$ and $\vec{R}$:

$$\begin{aligned}
t'' &= t' + t \\
\vec{r}'' &= \vec{r} + \hat{M}^{-1}(t)\cdot\vec{r}' \\
\vec{R}'' &= \vec{R}' + \vec{R} \\
\zeta'' &= \zeta'\zeta e^{\frac{i}{\hbar}m\omega_c\left[\frac{1}{2}\vec{r}'\cdot\hat{N}(t)\cdot\vec{r} - \left((1+\lambda)R_1'R_2 - \lambda R_2'R_1\right)\right]}
\end{aligned} \tag{66}$$

where we have added the coboundary generated by $-m\omega_c(\frac{1}{2}+\lambda)R_1R_2$ to accommodate the cocycle, in its $\vec{R}$-dependent term, to the expression of Sec. 2.2 (except for a global minus sign).

From this group law we can compute again the left- and right-invariant vector fields:

$$\begin{aligned}
\tilde{X}^L_t &= \frac{\partial}{\partial t} - \omega_c\vec{r}\cdot\hat{J}\cdot\frac{\partial}{\partial\vec{r}} & \tilde{X}^R_t &= \frac{\partial}{\partial t} \\
\tilde{X}^L_{\vec{r}} &= \frac{\partial}{\partial\vec{r}} - \frac{m\omega_c}{2\hbar}\hat{J}\cdot\vec{r}\,\Xi & \tilde{X}^R_{\vec{r}} &= \hat{M}(t)\cdot\frac{\partial}{\partial\vec{r}} + \frac{m\omega_c}{2\hbar}\hat{N}(t)\cdot\vec{r}\,\Xi \\
\tilde{X}^L_{R_1} &= \frac{\partial}{\partial R_1} - \frac{\lambda}{\hbar}m\omega_c R_2\,\Xi & \tilde{X}^R_{R_1} &= \frac{\partial}{\partial R_1} - \frac{1+\lambda}{\hbar}m\omega_c R_2\,\Xi \\
\tilde{X}^L_{R_2} &= \frac{\partial}{\partial R_2} - \frac{1+\lambda}{\hbar}m\omega_c R_1\,\Xi & \tilde{X}^R_{R_2} &= \frac{\partial}{\partial R_2} - \frac{\lambda}{\hbar}m\omega_c R_1\,\Xi \\
\tilde{X}^L_{\zeta} &= i\zeta\frac{\partial}{\partial\zeta} \equiv \Xi & \tilde{X}^R_{\zeta} &= i\zeta\frac{\partial}{\partial\zeta} \equiv \Xi
\end{aligned} \tag{67}$$

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and the commutation relations are now:

$$
\begin{aligned}
\left[\tilde{X}_t^L, \tilde{X}_{\vec{r}}^L\right] &= \omega_c \hat{\mathbf{J}} \cdot \tilde{X}_{\vec{r}}^L \\
\left[\tilde{X}_t^L, \tilde{X}_{\vec{R}}^L\right] &= 0 \\
\left[\tilde{X}_{r^i}^L, \tilde{X}_{r^j}^L\right] &= \frac{m\omega_c}{\hbar}\epsilon_{ij}\Xi \\
\left[\tilde{X}_{R^i}^L, \tilde{X}_{R^j}^L\right] &= -\frac{m\omega_c}{\hbar}\epsilon_{ij}\Xi \\
\left[\tilde{X}_{\vec{r}}^L, \tilde{X}_{\vec{R}}^L\right] &= 0
\end{aligned}
\tag{68}
$$

A glance at the algebra (68) reveals that it is the central extension of the direct sum of the harmonic oscillator algebra and the Heisenberg algebra. Consequently, the wave function factorizes into a harmonic oscillator wave function (depending on $t$ and $\vec{r}$) times a function of $\vec{R}$, and the energy spectrum coincides with that of the harmonic oscillator, the degeneracy being infinite due to the Heisenberg-Weyl symmetry, which in the plane has only infinite-dimensional unitary irreducible representations.

We are interested in a configuration-space representation, so that a second-order polarization is needed. This is found to be:

$$
\mathcal{P}^{HO} = < \tilde{X}_p^L, \tilde{X}_t^L - \frac{i\hbar}{2m}\left(\tilde{X}_{\vec{r}}^L\right)^2, \vec{n}\cdot\tilde{X}_{\vec{R}}^L, \vec{n}'\cdot\tilde{X}_{\vec{r}}^L >
\tag{69}
$$

where $\vec{n}$ and $\vec{n}'$ are arbitrary unit vectors. They can be chosen to be $(1,0)$ or $(0,1)$, for instance.

Imposing these polarization conditions to the wave functions, we obtain the general form:

$$
\Psi = \zeta e^{-\frac{i}{\hbar}m\omega[(\lambda n_1^2 - (1+\lambda)n_2^2)y_1 y_2 + (\lambda+\frac{1}{2})n_1 n_2 y_1^2]}\Phi(y_2)e^{\frac{im\omega_c}{2\hbar}\kappa_2\kappa_1}\Omega(\kappa_2, t)
\tag{70}
$$

where $y_1 \equiv \vec{n}\cdot\vec{R}$, $y_2 \equiv \vec{n}\cdot\hat{\mathbf{J}}\cdot\vec{R}$, $\kappa_1 \equiv \vec{n}'\cdot\vec{r}$, $\kappa_2 \equiv \vec{n}'\cdot\hat{\mathbf{J}}\cdot\vec{r}$, $\Phi(y_2)$ is an arbitrary function and $\Theta(\kappa_2, t)$ satisfies the Schrödinger equation:

$$
i\hbar\frac{\partial}{\partial t}\Omega(\kappa_2, t) = \left[-\frac{\hbar^2}{2m}\vec{\nabla}_{\kappa_2}^2 + \frac{m\omega_c^2}{2}\kappa_2^2\right]\Omega(\kappa_2, t)
\tag{71}
$$

This is nothing other than the Schrödinger equation for the harmonic oscillator, so that the solutions are

$$
\Omega(\kappa_2, t) = \sum_n A_n e^{-i(n+\frac{1}{2})\omega_c t}e^{-\frac{m\omega_c}{2\hbar}\kappa_2^2}H_n\left(\sqrt{\frac{m\omega_c}{\hbar}}\kappa_2\right)
\tag{72}
$$

where $H_n$ are the Hermite polynomials.

Since the wave functions factorize, the operators $\tilde{X}_{\vec{R}}^R$ will act only on the $\vec{R}$-dependent part of it, having the same expressions as in (18) (changing there $\vec{x}$ to $\vec{R}$), and the

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operators $\tilde{X}^R_{\tilde{x}}$ will act only on the $(\vec{r}, t)$-dependent part, with the expressions:

$$\tilde{X}^R_{\kappa_1} \equiv \vec{n} \cdot \tilde{X}^R_{\vec{r}} \Omega(\kappa_2, t) = \left[ -\sin\omega_c t \frac{\partial}{\partial \kappa_2} + \frac{i}{\hbar} m\omega_c \kappa_2 \cos\omega_c t \right] \Omega(\kappa_2, t)$$

$$\tilde{X}^R_{\kappa_2} \equiv \vec{n} \cdot \hat{J} \cdot \tilde{X}^R_{\vec{r}} \Omega(\kappa_2, t) = \left[ \cos\omega_c t \frac{\partial}{\partial \kappa_2} + \frac{i}{\hbar} m\omega_c \kappa_2 \sin\omega_c t \right] \Omega(\kappa_2, t) \qquad (73)$$

once the (irrelevant) phase factors have been factorized out.

Using the dual transformation to the one taking $(\vec{x}, \vec{p})$ to $(\vec{r}, \vec{R})$, we obtain the expression of the operators $\hat{\vec{X}}$ and $\hat{\vec{P}}$ in terms of $\tilde{X}^R_{\vec{r}}$ and $\tilde{X}^R_{\vec{R}}$:

$$\frac{i}{\hbar}\hat{\vec{P}} \equiv \tilde{X}^R_{\vec{x}} = \tilde{X}^R_{\vec{r}}$$

$$-\frac{i}{\hbar}\hat{\vec{X}} \equiv \tilde{X}^R_{\vec{p}} = \frac{1}{m\omega_c}\hat{J} \cdot \left( \tilde{X}^R_{\vec{r}} - \tilde{X}^R_{\vec{R}} \right) \qquad (74)$$

In addition, by $\hat{\vec{T}}$ we denote the operator $-i\hbar \tilde{X}^R_{\vec{R}} = \hat{\vec{P}} - m\omega_c \hat{J} \cdot \hat{\vec{X}}$. It can be easily deduced that $\hat{\vec{P}}$ has the physical meaning of a linear momentum (mass times velocity), which we shall simply call momentum, while $\hat{\vec{T}}$ is a momentum commuting with the Hamiltonian, generally called magnetic translations, and this is associated with the coordinate $\vec{R}$ of the centre of the circumferences. We can still define another momentum in the theory, the canonical momentum, as $\hat{\vec{\Pi}} \equiv -\frac{i\hbar}{2}\left( \tilde{X}^R_{\vec{r}} + \tilde{X}^R_{\vec{R}} \right)$, which has the particularity that its components mutually commute, and, as can be easily checked, is a proper translation generator: it is written (for $t = 0$) as $\vec{\nabla}_{\vec{x}}$ when acting on $\Phi(x, t)$ in (63) at $t = 0$. Its explicit expression and that of $\hat{\vec{T}}$ on $\Phi(x, t)$ are:

$$\hat{\vec{\Pi}}\Psi = \zeta e^{-\frac{i}{2\hbar}\vec{p}\cdot\vec{x}} \left[ -\frac{i\hbar}{2}\left( \hat{M}(t) + \hat{I} \right) \cdot \vec{\nabla} + \frac{m\omega_c}{4}\left( \hat{N}(t) - \hat{J} \right) \cdot \vec{x} \right] \Phi(\vec{x}, t)$$

$$\hat{\vec{T}}\Psi = \zeta e^{-\frac{i}{2\hbar}\vec{p}\cdot\vec{x}} \left[ -i\hbar\vec{\nabla} - \frac{m\omega_c}{2}\hat{J}\cdot\vec{x} \right] \Phi(\vec{x}, t) \qquad (75)$$

The rôle of the different momenta can be clarified by introducing the vector potential operator in the usual form, $\hat{\vec{A}} \equiv -\frac{m\omega_c}{2}\hat{J} \cdot \hat{\vec{X}} = \frac{i\hbar}{2}\left( \tilde{X}^R_{\vec{r}} - \tilde{X}^R_{\vec{R}} \right)$. Then, the canonical momentum is rewritten $\hat{\vec{\Pi}} = \hat{\vec{P}} + \hat{\vec{A}}$, and $\hat{\vec{T}} = \hat{\vec{P}} + 2\hat{\vec{A}} = \hat{\vec{\Pi}} + \hat{\vec{A}}$. Then it is easy to verify that $\hat{E} = \frac{1}{2m}\hat{\vec{P}}^2 = \frac{1}{2m}\left( \hat{\vec{\Pi}} - \hat{\vec{A}} \right)^2 = \frac{1}{2m}\left( \hat{\vec{T}} - 2\hat{\vec{A}} \right)^2$.

## 4.2 Charged particle in a homogeneous magnetic field in the plane with periodic boundary conditions

Before imposing the periodic boundary conditions which define the torus, as in Sec. 2.2, we must determine how these boundary conditions affect each of the coordinates. Clearly,

$\vec{x}$ will be affected by the boundary conditions, but it is not clear what happens to $\vec{p}$. Let us return to $\vec{r}$ and $\vec{R}$ coordinates, where $\vec{R}$ is the (absolute) position of the centre of the circumference (the classical trajectory) and $\vec{r}$ is the (relative) position of the particle with respect to the centre of the circumference, i.e. $\vec{r} = \vec{x} - \vec{R}$. Therefore, $\vec{R}$ will be subject to periodic boundary conditions (the same that $\vec{x}$) while $\vec{r}$ will not, being a relative coordinate (since the classical energy $H$ is a function of $\vec{r_0}^2 = \vec{r}^2$, periodic boundary conditions for $\vec{r}$ would imply an upper bound to the energy spectrum, and even more, a periodic energy spectrum). This makes $\vec{r}$ and $\vec{R}$ coordinates more appropriate to describe the system with periodic boundary conditions. Now we are ready to apply the results of Sec. 2.2, having reduced the problem, roughly speaking, to the study of an harmonic oscillator times a Heisenberg-Weyl group on the torus, the latter being parameterized by $\vec{R}$.

Regarding the H-W subgroup, we can apply the results of Sec. 2.2. We also consider the two cases *i)* and *ii)*, corresponding to $T$ being a trivial or non-trivial principal fibre bundle, respectively.

Let us consider first the case *i)* (Sec. 2.2.1), which is the more conventional one. The actual condition to be satisfied is

$$\frac{m\omega_c L_1 L_2}{2\pi\hbar} = n \in Z \tag{76}$$

which implies, as already anticipated, a quantization of the magnetic flux through the torus surface, in the same manner as in the Dirac monopole case. If this flux were produced by a monopole charge, the quantization of the magnetic charge would follow. This kind of quantization condition guarantees, for instance, that the Wilson loop variables in gauge theories are single-valued [**M**].

The wave functions turn out to be (70), where $\Phi(y_2)$ is subject to exactly the same restrictions as in Sec. 2.2.1, thus leading to the expression (for $\vec{n} = (1,0)$):

$$\Phi^{\vec{\alpha}}(R_2) = \sum_{k=0}^{n-1} a_k \Lambda_k^{\vec{\alpha}}(R_2) \tag{77}$$

where $\vec{\alpha}$ is defined as before. The wave function is therefore peaked at $R_2 = \alpha_2 + \frac{k}{n} L_2$, $k \in Z$ ($R_1 = \alpha_2 + \frac{k}{n} L_1$, $k \in Z$ for $\vec{n} = (0,1)$).

The subgroup $G_{\mathcal{H}}$ of good transformations (the ones that preserve the structure of the wave function) is the subgroup of $\widetilde{G}$ with the parameters $\vec{R}$ restricted to be $\vec{R} = \frac{1}{n}\vec{L}_{\vec{k}}$. The quantum operators $\hat{E}$ and $\tilde{X}_{\vec{r}}^R$ are good operators (since the harmonic oscillator part is not subject to constraints), while the operator $\tilde{X}_{\vec{R}}^R$ is a bad operator. If we analyse these results in terms of the operators $\hat{\vec{X}}$, $\hat{\vec{P}}$, $\hat{\vec{T}}$ and $\hat{\vec{\Pi}}$ by means of the expressions given in Sec. 4.1, we conclude that the operator $\hat{\vec{P}}$ is a good operator, while $\hat{\vec{X}}$, $\hat{\vec{T}}$ and $\hat{\vec{\Pi}}$ are not. Consequently, the momentum (or velocity) of the particle is a measurable quantity, but the position, the canonical momentum and the magnetic translations are not observables.

The vector potential operator is of course also a bad operator. For all the bad operators, their finite expressions (counterparts of $\hat{\eta}$ of Sec 2.1.1) can be nevertheless constructed, since all these expressions are good operators.

For the case $ii$), only the fractional case $a$) is physically meaningful. Now the wave function is defined on a torus $r$ times greater in one direction [**N-T-W**], or, what is the same, it is a vector-valued (with $r$ components) function; or, in FQHE terminology, the centre of mass function corresponding to the vacuum is degenerate, $r$ being the degeneration. In this case the Hall conductivity is associated with the quotient $\frac{n}{r}$ of the Chern class of the associated determinant bundle by the rank of the vector bundle [**V**]. This result lends support to the idea that Fractional Quantum Hall Effect is always associated with multiple-valued wave functions, i.e. degenerate vacua.

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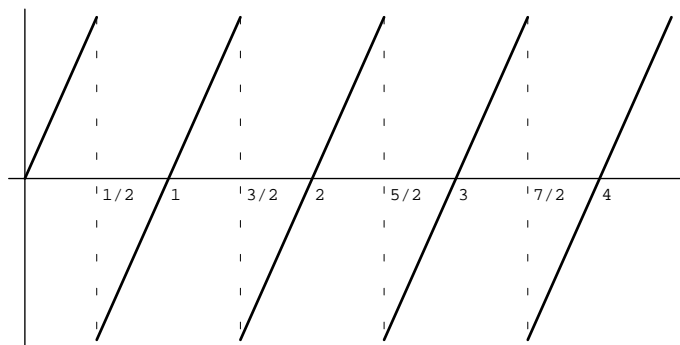
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Figure 1: Net current in the superconducting ring against $\Phi/\Phi_0$.