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The dimensional character of permeability: Dimensionless groups that govern Darcy's flow in anisotropic porous media

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Abstract

The dimensional character of permeability in anisotropic porous media, that is, its dimension or dimensional equation, is an information that allows setting the dimensionless groups that govern the solution of the flow equation in terms of hydraulic potential patterns. However, employing the dimensional basis {L, M, T} (length, mass, time), the dimensionless groups containing the anisotropic permeability do not behave as independent monomials that rule the solutions. In this work, the contributions appearing in the literature on the dimensional character of permeability are discussed and a new approach based on discriminated and general dimensional analysis is presented. This approach leads to the emergence of a new and accurate dimensionless group, $\frac{k_x}{k_y}\frac{l_y^{*2}}{l_x^{*2}}$, a ratio of permeabilities corrected by the squared value of an aspect factor, being l_x^* and l_y^* two arbitrary lengths of the domain in the directions that are indicated in their subscripts. Specific values of this lengths, which we name 'hidden characteristic lengths', are also discussed in this article. To check the validity of this dimensionless group, numerical simulations of two illustrative 2-D seepage scenarios have been solved.

KEYWORDS
anisotropy, Darcy's law, dimensionless groups, permeability dimensions, porous media

1 | INTRODUCTION

The importance of the study of permeability in porous media is demonstrated by the great effort made by many researchers both in experimental and theoretical fields. In the latter area, they have tried to devise complex formulae for its estimation. Nevertheless, this effort has not been orientated towards the search for the dimensional character of this parameter, necessary information for finding reliable dimensionless groups that govern the solution of problems related to flow in porous media, especially in anisotropic scenarios. These groups, when correctly established, lead to the representation of universal solutions as well as they simplify the study of the sensitivity analysis of the physical and/or geometrical parameters involved in hydraulic processes.

Apart from Muskat[1] and, to a lesser extent, Taylor,[2] there are few contributions in the literature on the study of the dimensional aspects involved in Darcy's and Forchheimer's laws. Muskat, who was very knowledgeable regarding the Pi theorem,[3] insisted on justifying these empirical laws as limit cases of a general dependence in which the effects of pressure, inertial and viscous forces were involved. However, his conclusions are of little use when applied to the deduction of dimensionless groups in anisotropic scenarios. On the other hand, the application of discriminated dimensional analysis in its spatial version,[4,5] besides leading to the deduction of these laws, justifies two facts. First, the emergence of 'ζR_e' (the product of the friction factor and Reynolds number) as a dimensionless group, instead of ζ and R_e separately; second, the derivation of new information on the dimensional character of the permeability, even though it does not lead to a correct characterization of the anisotropic scenarios.

The absence of inertial forces in Darcy's flow allows removing mass from the dimensional basis. Instead, the energetic potential of the fluid (ξ) is introduced to the basis, justified in the application of the general discrimination concept of the dimensional theory[6] that allows defining the dimensional basis according to the physical phenomenon involved. Although the new basis, $\{L_x, L_y, L_z, T, \xi\}$, does not provide a specific dimensional equation for permeability, it does so for hydraulic conductivity (the ratio of the permeability and the viscosity), obtaining a new and precise dimensionless group in anisotropic domains. This new group (which we name discriminated) degenerates into a simpler one by removing the fluid viscosity, as the numerical value of this parameter does not depend on the spatial direction.

In order to verify these results, we numerically solve the equations for the flow in two anisotropic scenarios of seepage under dams with and without a sheet pile. We prove that the emergent group (a ratio of permeabilities corrected by a geometrical shape factor), together with the other discriminated aspect factors, behave as independent monomials that define the solution of the problem in terms of equipotential distribution. Two characteristic lengths, one for each spatial direction, must be defined to establish this new dimensionless group, although it can be a problem in some semi-infinite scenarios. However, the introduction of these hidden lengths as unknowns (whose physical meaning is defined later) solves this difficulty.

The paper is organized as follows. Sections 2 present the derivation of Darcy's and Forchheimer's laws by the Pi theorem. For the first derivation, based on a commentary of the work of Muskat, classical dimensional analysis has been applied. The second uses the spatial discriminated analysis and enables more detailed on the dimensional character of permeability. Section 2 also introduces a new dimensional basis employing general discrimination that leads to a dimensional equation for hydraulic conductivity and derives accurate dimensionless groups for characterizing anisotropic problems. Section 3 discusses the results obtained and the advantage of using this new group for a better understanding and characterization of hydrological processes in anisotropic media. In Section 4, the deduced dimensionless group are verified with an example of a flow net for two anisotropic scenarios. Finally, Section 5 collects the contributions and conclusions of the paper.

2 | THEORETICAL FRAMEWORK AND METHODS

2.1 | Darcy's law and the Pi theorem

Darcy's law is written as $Q = A\varkappa i$, with Q the water flow (m^3/s), A the cross-section area (m^2), and i the hydraulic gradient, $\frac{dh}{dx}$, where h is the hydraulic potential or head (energy per unit weight of fluid). Parameter $\varkappa$ is known as hydraulic conductivity and its unit, deduced from Darcy's law, is m/s. Muskat,[1] who employed the pressure variable (p) instead of h, expressed this law as

$$v = \frac{k}{\mu}\frac{dp}{dx} \text{ or } Q = C_o \frac{A}{L_o}(\Delta p) \tag{1}$$

He introduced the parameter permeability k (m^2) whose relation to conductivity is given by $k = \frac{\varkappa\mu}{\rho g}$, where g is the gravitational constant, ρ the fluid density and μ the fluid viscosity. Permeability can be obtained through a formulation in which the geometrical properties of the soil are involved or, more precisely, by laboratory tests.[7–11]

In order to deduce Darcyťs law by applying dimensional considerations and citing Bridgman,[12] Muskat started with a relevant list of six variables, $\{\Delta p, \rho, \mu, d, v, \Delta s\}$. In this list, d (m) is a characteristic length (average grain or pore size), v (m/s) is the Darcy-velocity, and Δs (m) is the length of the porous medium sample. With the dimensional equations of these variables, $[\Delta p] = ML^{-1}T^{-2}$, $[\rho] = ML^{-3}$, $[d] = L$, $[v] = LT^{-1}$, $[\mu] = ML^{-1}T^{-1}$ and $[\Delta s] = L$, Muskat determined the three dimensionless groups that characterize the problem. These are

$$\pi_1 = \frac{(\Delta p)\,\rho d^2}{\mu^2}, \quad \pi_2 = \frac{v\rho d}{\mu}, \quad \pi_3 = \frac{(\Delta s)}{d} \tag{2}$$

Based on the assumption (derived from experimental results) of linear dependence between $\frac{v\rho d}{\mu}$ and $\frac{(\Delta s)}{d}$, and using the Pi theorem, Muskat provided the solution

$$\frac{\Delta p}{\Delta s} = \frac{\mu^2}{\rho d} F\left(\frac{v\rho d}{\mu}\right) \tag{3}$$

where F is an unknown function. He summarized his conclusions regarding Darcy's law (exclusively referring to sandy soils) stating that it was a very reliable approach to what he called the 'law of flow'. Its validity range is, however, difficult to establish due to the lack of definition of the parameter 'd'. Properties such as porosity, connectivity, tortuosity, grain shape, compaction and cementation degrees are the cause of the unclear definition and uncertainty of this characteristic length.

2.2 | Discrimination and Darcy's and Forchheimer's laws

In the theory of dimensional analysis,[3,12] discrimination involves (through physical arguments) increasing the quantities of the dimensional basis. This is a little-known and applied concept even though in its 'spatial discrimination' version it has been known for decades. Its application has led to more accurate solutions in many engineering problems, particularly in anisotropic media.[13–16] Spatial discrimination states that each one of the spatial directions, whatever the geometry used, is represented by a different quantity (length) in the dimensional basis. Therefore, other quantities in which length is involved, such as velocity or acceleration, would have different dimensional equations, also according to their spatial direction. In rectangular geometries, for example, the spatial discriminated basis consists of five magnitudes, $\{L_x, L_y, L_z, M, T\}$. As a consequence of spatial discrimination in anisotropic media, it might be that some of the classical groups such as Reynolds, Peclet, Rayleigh, etc. neither have null dimension nor represent balances of quantities that counteract in the domain. In contrast, discriminated dimensionless groups do express the balance of quantities.

In the dimensional study of permeability, the use of the classic basis {L, M, T} attributes to the potential energy h (energy per unit weight of the fluid) the same dimension as that of width or length of the domain. This causes confusion about the physical meaning of the dimensional equations and produces imprecise results in the dimensionless groups to which this basis gives rise. Furthermore, the use of the classical basis in an anisotropic problem involving lengths associated with different spatial directions might lead to aspect ratios (quotients of these lengths) containing lengths in different spatial directions. As shown in the illustrations in Section 4, such aspect ratios must be discriminated to rule as independent groups.

To apply discriminated dimensional analysis to the deduction of Darcy's law, a discriminated intrinsic dimensional basis for any geometry is adopted, $\{M, L_{\rightarrow}, L_{vis}, L_n, T\}$. In this basis, $L_{\rightarrow}$ is the spatial dimension in the direction of the fluid velocity, L_{vis} the spatial dimension in a normal direction to the fluid velocity, so both directions define the planes of viscous surfaces, and L_n is the third spatial dimension, which is normal to the other two. The viscous or sliding surface has the dimensional equation $[S_{vis}] = L_{\rightarrow}L_{vis}$. The relevant list of variables $\{\Delta p/L, d, v, \rho, \mu\}$, as well as the friction factor and Reynolds number, have in this basis the dimensional equations shown in Table 1. Figure 1 shows the directions of $L_{\rightarrow}$, L_{vis} and L_n, as well as S_{vis}, in granular soils.

The direction normal to the viscous surface has been intentionally selected for d, so its value would be related to the average grain size. It is also important to highlight that the chosen direction for the forces due to the pressure is the one of the velocities since this is the direction in which the pressure produces its effects on flow. According to these dimensional equations, the product $\zeta\ \ R_e = (\frac{(\Delta p)d}{2L\rho v^2})\ (\frac{v\rho d}{\mu}) = \frac{(\Delta p)d^2}{2Lv\mu}$ is dimensionless.

Now, if we delete ρ from the list of relevant variables as this is the parameter essentially linked to (negligible) inertial effects, no other discriminated groups can be formed. The other variables cannot be deleted since v is related to viscous effects, and L and d, as boundaries of the domain in which the forces are balanced, are linked to inertial, viscous and pressure effects. Thus, when inertial effects are neglected in respect of those of viscosity and pressure (or sufficiently low velocities are considered to assume this hypothesis), ζR_e is the only discriminated dimensionless group that rules the solution

$$\pi_{pre-vis} = \xi R_e = \frac{(\Delta p)\, d^2}{2Lv\mu} \sim 1 \tag{4}$$

TABLE 1 Discriminated dimensional equations of the list of variables $\{\Delta p/L, d, v, \rho, \mu\}$ and of the friction factor and Reynolds number, ζ and R_e

Variable	Discriminated dimensional equation
$[\Delta p/L] = \left[\frac{f}{S\cdot L}\right]$	$\frac{ML_{\rightarrow}T^{-2}}{L_{vis}L_nL_{\rightarrow}} = ML_{vis}^{-1}L_n^{-1}T^{-2}$
$[d]$	L_n
$[v]$	$L_{\rightarrow}T^{-1}$
$[\mu] = \left[\frac{f_{vis}}{S_{vis}\frac{\partial v}{\partial n}}\right]$	$\frac{M}{L_{vis}L_{\rightarrow}L_n} = ML_{\rightarrow}^{-1}L_{vis}^{-1}\ L_n^{-1} = MS_{vis}^{-1}L_n^{-1}$
$[\rho] = \left[\frac{mass}{volume}\right]$	$\frac{M}{L_{vis}L_{\rightarrow}L_n} = ML_{\rightarrow}^{-1}L_{vis}^{-1}\ L_n^{-1} = MS_{vis}^{-1}L_n^{-1}$
$[\zeta] = \left[\frac{(\Delta p)d}{2L\rho v^2}\right] = \frac{ML_{\rightarrow}L_{vis}^{-1}L_n^{-1}T^{-2}L_n}{L_{\rightarrow}ML_{\rightarrow}^{-1}L_{vis}^{-1}L_n^{-1}L_{\rightarrow}^2T^{-2}}$	$\frac{ML_{\rightarrow}L_{vis}^{-1}L_n^{-1}T^{-2}L_n}{L_{\rightarrow}ML_{\rightarrow}^{-1}L_{vis}^{-1}L_n^{-1}L_{\rightarrow}^2T^{-2}} = \frac{L_n}{L_{\rightarrow}}$
$[R_e] = \left[\frac{v\rho d}{\mu}\right]$	$\frac{L_{\rightarrow}T^{-1}ML_{\rightarrow}^{-1}L_{vis}^{-1}L_n^{-1}L_n}{ML_{\rightarrow}^{-1}L_{vis}^{-1}L_nT^{-1}} = \frac{L_{\rightarrow}}{L_n}$

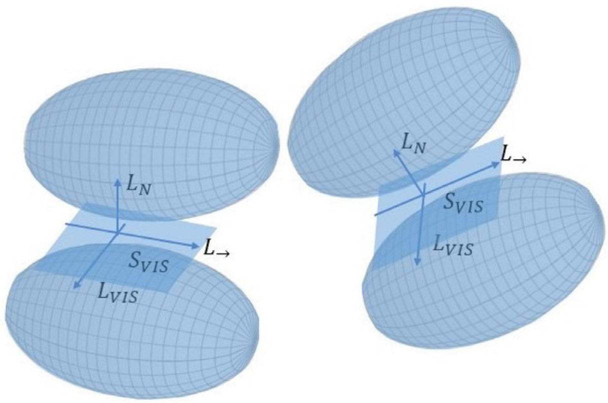

FIGURE 1 Directions of $L_{\rightarrow}$, L_{vis}, L_n and S_{vis}, in granular 3-D soils

TABLE 2 Possible pairs of discriminated dimensionless groups for the list of variables $\{\Delta p/L, d, v, \rho, \mu\}$

First solution	$\pi_{pre-vis} = \frac{(\Delta p)d^2}{2Lv\mu}$,	$\pi_{pre-ine} = \frac{\Delta p}{\rho v^2}$
Second solution	$\pi_{pre-vis} = \frac{(\Delta p)d^2}{2Lv\mu}$,	$\pi_{ine-vis} = \frac{\pi_{pre-ine}}{\pi_{pre-vis}} = \frac{\rho vd^2}{2L\mu}$
Third solution	$\pi_{ine-vis} = \frac{\pi_{pre-ine}}{\pi_{pre-vis}} = \frac{\rho vd^2}{2L\mu}$,	$\pi_{pre-ine} = \frac{\Delta p}{\rho v^2}$

The subscript 'pre-vis' expresses that pressure and viscous forces per unit of volume are balanced because the monomial has an order of magnitude of unity. The fluid loses pressure (or pressure energy) as the energy is dissipated due to the viscous friction. From the former equation, we can write $v \sim \frac{(\Delta p)d^2}{L\mu} = (\frac{d^2}{\mu})\ \frac{\Delta p}{L}$, a result that is coherent with Darcy's law.

For a general study that assumes pressure, inertial and viscous forces of the same order of magnitude, the independent dimensionless groups that emerge from the list of variables $\{\Delta p/L, d, v, \rho, \mu\}$ are two, which can be chosen in three ways (Table 2).

Adopting any of these pairs, for example the first one, the Pi theorem provides the solution $\pi_{pre-vis} = F(\pi_{pre-ine})$, or, $\frac{(\Delta p)d^2}{2Lv\mu} = F(\frac{\Delta p}{\rho v^2})$, with F an unknown function. From this, Darcy's law can be written in its most general form as

$$v = \frac{d^2}{\mu}\left(\frac{\Delta p}{L}\right)F\left(\frac{\Delta p}{\rho v^2}\right) \tag{5}$$

TABLE 3 Discriminated dimensional analysis solutions for Darcyťs and Forchheimerťs laws

Negligible inertial effects (Darcy's law)	$\pi_{pre-vis} = \frac{(\Delta p)d^2}{2Lv\mu} \sim 1$	$\Rightarrow$	$v \propto \frac{d^2}{\mu}(\frac{\Delta p}{L})$
Negligible viscous effects (Forchheimer's law)	$\pi_{pre-ine} = \frac{\Delta p}{\rho v^2} \sim 1$	$\Rightarrow$	$v^2 \propto \frac{L}{\rho}(\frac{\Delta p}{L})$

The limit cases of this result are: i) negligible inertial effects, ii) negligible viscous effects and, iii) negligible pressure forces. Case i) removes ρ from the list of variables providing the solution $\pi_{pre-vis} \sim 1$. Case ii) removes μ, providing the solution $\pi_{pre-ine} \sim 1$. Case iii) represents a kind of problem of a different nature related to the existence of a boundary layer that does not have to be discussed here, as pressure forces are essential in the problems of interest in this study. Table 3 summarizes the solutions of the two first cases.

2.3 | The dimensional character of permeability and the emergence of the dimensionless group $\frac{k_x}{k_y}\frac{l_y^{*2}}{l_x^{*2}}$ in anisotropic media

In its original definition of Darcyťs law, $v = \varkappa\frac{\Delta h}{L}$ or $v = C_o\frac{\Delta p}{\Delta x}$, constants $\varkappa$ and C_o collect influences of the fluid and the porous media, which is the reason why its dimensions cannot be assigned to a specific property. Muskat went back to dimensional arguments and argued that the equation $v = C(\frac{d^2}{\mu})(\frac{dp}{dx})$ should be the starting point since it separates the influence of viscosity from the influence of the characteristic length, with C being a constant in which the rest of the properties of the porous medium are collected.

The use of either the classical dimensional analysis (with $[\frac{\Delta p}{\Delta x}] = ML^{-2}T^{-2}$, $[d] = L$, $[v] = LT^{-1}$, and $[\mu] = ML^{-1}T^{-1}$) or discrimination (Table 1), assigns C a dimensionless character. This means that properties such as tortuosity or angularity compensate each other dimensionally. The dimension of these physical characteristics of the porous media, not included in this research, are expected to be associated with the aspect ratio (of dimensions L_n/L_vis or its inverse), what we assume as a reasonable hypothesis. If not, the deduction of the dimensionless character of this constant might be supported by erroneous arguments

Discriminated dimensional analysis can also be applied to the studies of Taylor[2] based on the flow through circular paths of different section. Doing this, the following dimensional equations are obtained:

$$\left[\frac{\Delta p}{L}\right] = ML_{vi}^{-1}L_n^{-1}T^{-2}, [d_o] = L_n \text{ (duct diameter)},\ [v] = L_{\rightarrow}T^{-1},\ \text{and } [\mu] = MS_{vis}^{-1}L_nT^{-1}$$

Taylor set a similitude between Poiseuille's and Darcy's laws, $\mathbf{v} \propto \frac{d_o^2}{\mu}(\frac{\Delta p}{L})$ and $\mathbf{v} = (\frac{k}{\mu})(\frac{\Delta p}{L})$ respectively, arguing that $[k] = L_n^2$, a result that agrees with Muskat's. Yet, to conclude, Taylor admitted that '*probably there are other factors related to the shape effects of the pore section and other constants which should be considered to extrapolate the solution of his model to what occurs in the real model.*'

The paradox in this historical approach to the dimensional character of permeability is that the conclusions reached are not coherent when trying to apply them to the search of the dimensionless groups that govern the anisotropic domains. Indeed, simplifying the problem to a 2-D rectangular domain, the dimensions of k_x and k_y should be $[k_x] = L_y^2$ and $[k_y] = L_x^2$, thus the dimension of the permeability ratio is $[\frac{k_x}{k_y}] = \frac{L_y^2}{L_x^2}$. According to this, the dimensionless monomial of anisotropic scenarios would present the form $\frac{k_x}{k_y}\frac{l_x^{*2}}{l_y^{*2}}$, where l_x^* and l_y^* are whatever two lengths that define the scenario in the directions indicated by the subscript. As presented later, the dimensionless group that governs the problem is not $\frac{k_x}{k_y}\frac{l_x^{*2}}{l_y^{*2}}$, but $\frac{k_x}{k_y}\frac{l_y^{*2}}{l_x^{*2}}$.

Based on general discrimination arguments, a new way is proposed for obtaining the dimensionless groups that involves the permeability. Although this way, which prevents the energy potential from having the dimensions of a length, does not give the exact dimension for permeability, it leads to interesting results in the search for the dimensionless groups that characterize the flow in anisotropic media. In the most general conception,[17] discrimination allows introducing the fluid energetic potential variable, h, in the dimensional basis instead of mass, which can be removed since the inertial effects

TABLE 4 Dimensional equation and physical meaning of the constants involved in Darcy's law

Parameter	Energetic potential	Discriminated basis	Dimensional equation	Physical meaning
$\kappa = \frac{k\rho g}{\mu} = \frac{v}{\frac{dh}{dx}}$	h	$\{L_{\rightarrow}, L_{vis}, L_n, T, \xi\}$	$[\kappa] = T^{-1}\xi^{-1}$	velocity that causes a unity gradient of h
$\frac{k\rho}{\mu} = \frac{\kappa}{g} = \frac{v}{\frac{d(gh)}{dx}}$	gh	$\{L_{\rightarrow}, L_{vis}, L_n, T, \Theta\}$	$[\frac{\kappa}{g}] = T^{-1}\Theta^{-1}$	velocity that causes a unity gradient of gh
$\frac{k}{\mu} = \frac{\kappa}{\rho g} = \frac{v}{\frac{d(\rho gh)}{dx}}$	ρgh	$\{L_{\rightarrow}, L_{vis}, L_n, T, \Phi\}$	$[\frac{\kappa}{\rho g}] = T^{-1}\Phi^{-1}$	velocity that causes a unity gradient of ρgh

are negligible. Let us name ξ to the dimension of the potential quantity (either p or h) and define a new dimensional basis in the form $\{\xi, L_x, L_y, L_z, T\}$. With this, Darcy's law, $v = (\frac{k}{\mu})\ (\frac{\Delta p}{L})$ or $v = C_o\ (\frac{\Delta p}{L})$, permits assigning to $(\frac{k}{\mu\rho g})$ and C_o the same dimensional equation, $[\frac{k}{\mu\rho g}] = [C_o] = L^2\ T^{-1}\xi^{-1}$. Thus, the dimensions of these parameters in x and y spatial directions are

$$\left[\frac{k}{\mu\rho g}\right]_x = \left[\frac{k_x}{\rho g(\mu)_x}\right] = [C_o]_x = \frac{L_x^2 T^{-1}}{\xi}$$

$$\left[\frac{k}{\mu\rho g}\right]_y = \left[\frac{k_y}{\rho g(\mu)_y}\right] = [C_o]_y = \frac{L_y^2 T^{-1}}{\xi}$$

This allows writing $[\frac{k_x}{\rho g(\mu)_x}]/[\frac{k_y}{\rho g(\mu)_y}] = \frac{L_x^2}{L_y^2}$, an essential result, as it implies that dimensionless groups can be found by multiplying $[\frac{k_x}{\rho g(\mu)_x}]/[\frac{k_y}{\rho g(\mu)_y}]$ by the square of an appropriate aspect ratio $(\frac{l_y^*}{l_x^*})$ between the characteristic lengths involved in the problem. Thus, this new discriminated group has the form $\pi = \frac{k_x\mu_y}{k_y\mu_x}\ \frac{l_y^{*2}}{l_x^{*2}}$ and, as numerically $\mu_x = \mu_y$, such group simplifies to

$$\pi = \frac{k_x}{l_x^{*2}}\ \frac{l_y^{*2}}{k_y} \tag{6}$$

This result, different from that obtained by the previous historical approach, might as well have been obtained with the use of the energy potential per unit mass or the energy potential per unit volume, gh and ρgh, respectively. The bases and the dimensional equations of the proportionality coefficients, as well as their physical meaning for this discrimination, are shown in Table 4.

Some authors use an alternative way to investigate the dimensionless groups that govern a given problem. They introduce the dimensionless variables in the governing equation to make it dimensionless and deduce the dimensionless groups from the resulting equation (see references in the next section). Introducing discrimination into this procedure,[6] the previously obtained result is confirmed. Indeed, steady-state flow in porous media is governed by $\frac{k_x}{\mu_x}\frac{\partial^2 p}{\partial x^2} + \frac{k_y}{\mu_y}\ \frac{\partial^2 p}{\partial y^2} = 0$, which come from Darcy's law and mass conservation. Introducing the dimensionless variables $p' = \frac{p}{\Delta p_o}$, $x' = \frac{x}{l_x^*}$ and $y' = \frac{y}{l_y^*}$, with Δp_o, l_x^* and l_y^* suitable references, this equation is written in its dimensionless form

$$\left[\frac{k_x\mu_y}{k_y\mu_x}\frac{l_y^{*2}}{l_x^{*2}}\right]\frac{\partial^2 p'}{\partial x'^2} + \frac{\partial^2 p'}{\partial y'^2} = 0 \tag{7}$$

The only group governing the equation is the coefficient of $\frac{\partial^2 p'}{\partial x'^2}$, that is $\pi = \frac{k_x \mu_y}{k_y \mu_x} \frac{l_y^{*2}}{l_x^{*2}}$ or, deleting the viscosity, $\pi = \frac{k_x}{k_y} \frac{l_y^{*2}}{l_x^{*2}}$. An identical result to that obtained using the energy potential in the discriminated dimensional basis, equation (6).

3 | DISCUSSION

Historically, in literature referring to Darcy's law and its applications to different hydrogeological problems, permeability dimensions have always been accepted as a squared length (L^2), either for isotropic or anisotropic soils. Therefore, its unit in the International System is m^2. Consequently, the ratio k_x/k_y is commonly considered as dimensionless. The same occurs for the hydraulic conductivity, whose dimension and unit is LT^{-1} and m/s, respectively. Without further considerations to its dimensional character, permeability values have been obtained either through formulations in which geometrical properties of the soil are involved or carrying out laboratory tests. Developed formulae for permeability calculation are given in many former works, for example those of Kozeny[8] and Wyllie and Rose.[18] The latter, which applied to porous media with grains of constant diameter, uses dimensional analysis arguments and includes a modification for soils with different grain sizes by introducing a shape factor. However, many authors have demonstrated that the different theoretical and even semi-empirical formulations proposed for the calculation of the permeability fail to a good degree Loudon.[19] In, A general theory has not been found to date despite the countless efforts of some authors, such as Åberg[20,21] and Scheidegger.[22] Åberg used a simple stochastic model to perform precise calculations of porosity, bulk density and permeability of granular soils depending on the size of grain, its shape, degree of densification and other physical characteristics. Later, Odong[23] provided empirical formulae for permeability based on grain-size analysis. Finally, Shin[9] introduced the tortuosity in the determination of Kozeny hydraulic diameter, providing formulations that have been used to date as the most accurate when applied to porous media with regular grain size. According to Shin, the 'tortuous hydraulic diameter' is the most important characteristic parameter governing flow aspects. Introducing this variable, the difference between permeability values calculated by numerical computation and those obtained by experimentation decreases up to 1.67%.

As far as we have investigated the dimensional aspects of permeability, the new emerging group brought by general discrimination arguments that add the dimension of the energetic potential to the base, $\pi = \frac{k_x}{k_y} \frac{l_y^{*2}}{l_x^{*2}}$, constitutes an advance for the understanding and dimensionless characterization of hydrological processes in anisotropic media. On the one hand, two of the classic monomials of many problems, namely the quotient of hydraulic conductivities ($\frac{k_x}{k_y}$) and the ratio of two suitably chosen lengths ($\frac{l_y^*}{l_x^*}$), appear together in a single group. This significantly reduces the set of universal curves or abaci that represent the universal solution of a flow problem in porous media. On the other hand, discrimination forces to select the geometric shape factors, or quotients between pairs of lengths that define the geometry of the scenario, so that the two lengths involved in the relationship have different spatial direction. Undoubtedly, these results set a better and more consistent way to represent the solution of an anisotropic problem in porous media. As seen in the previous section, the group $\frac{k_x}{k_y} \frac{l_y^{*2}}{l_x^{*2}}$ can be also derived from the governing equation $\frac{k_x}{\mu_x} \frac{\partial^2 p}{\partial x^2} + \frac{k_y}{\mu_y} \frac{\partial^2 p}{\partial y^2} = 0$ and has the meaning of a balance between the two addends of the equation.

Assuming that the dimensionless variables are chosen in such a way that they and their spatial derivatives can be averaged to the unit in the global scenario (a reasonable hypothesis), we can consider that the order of magnitude of the value of the monomials, which are defined by the ratio of coefficients of the simplified equation that only contains physical and geometrical parameters, is unity. This is an inherent property of the groups deduced under the protocol of nondimensionalization.[6] Higher (or zero value) orders of magnitude would make one of the terms of the equation predominant over another, allowing neglecting the influence of the monomial in the solution of the problem, which is the same as deleting the associated non-influential term in the governing equation. These arguments also justify that, for correctly deduced dimensionless groups, a range of values that covers the order of magnitude of unity is representative of all practical cases.

The studies that make direct use of the group obtained in the previous section from the general discrimination are scarce. It is worth mentioning the work of Madanayaka and Sivakugan[24] for the study of two-dimensional (confined flow) seepage axisymmetric cofferdam problems using the method of fragments.[25] Using classic arguments based on merely dimensional analysis and from the list of relevant variables of the problem, these authors choose two dimensionless groups consistent with those obtained in the previous section. This non-arbitrary choice allowed them to represent the solutions of these problems in a universal way. In a very similar work, Madanayaka and Sivakugan[26] incorporated anisotropy in the hydraulic conductivity to provide efficient solutions for the flow rate and exit gradient of confined seepage problems.

Other works that use dimensionless groups of type $\frac{k_x}{k_y}\frac{l_y^{*2}}{l_x^{*2}}$ in complex anisotropic (flow and transport) problems deduce them from the discriminated non-dimensionalization of the governing equations, which is the same protocol as that described at the end of Section 2. Many of these scenarios refer to benchmark flow and transport problems in hydrology. For example, Manteca et al.[27] studied an intrusion scenario with a salt flat in which four complex dimensionless groups emerged, one of which had the form $\frac{k_x}{k_y}\frac{l_y^{*2}}{l_x^{*2}}$. They verified the correct work of these groups by numerical simulations. Moreover, they also demonstrated that, although dimensionless groups can also be obtained by the classical non-dimensionalization technique, these lead to wrong results. Later, Manteca et al.,[28] following the same protocol, deduced the dimensionless groups of salt intrusion in Henry (anisotropic) problem. In this work, they suggested better references for benchmarking in order to obtain patterns that cover the whole domain, and not just a small region. Alhama et al.[29] and Cánovas et al. repeated the procedure for considering the length of the characteristic cell in the 2D anisotropic Bènard problem. In these two problems, it was also verified that the group $\frac{k_x}{k_y}\frac{l_y^{*2}}{l_x^{*2}}$ behaves as an independent dimensionless monomial. In addition, Alhama et al.[29] explained that the groups deduced in their work cannot be obtained with the classical non-dimensionalization technique, while Cánovas et al.[30] approached the problem introducing a subdomain instead of the whole scenario, showing a deep understanding of the phenomenon. Finally, the anisotropic Yusa problem, in which, again, $\frac{k_x}{k_y}\frac{l_y^{*2}}{l_x^{*2}}$ is one of the dimensionless groups that rule the solutions, was studied by Cánovas et al.[13]

It is interesting to note, first, that the characteristic lengths involved in the monomial $\frac{k_x}{k_y}\frac{l_y^{*2}}{l_x^{*2}}$ have no other requirement for their choice than that of spatial orientation. If there are several lengths with the same spatial orientation within the scenario,[27] the one that best represents the curves or abaci of the universal solution of the problem can be selected. Even if there are no finite lengths in any of the directions, the introduction of an unknown length in the nondimensionalization protocol makes the group $\frac{k_x}{k_y}\frac{l_y^{*2}}{l_x^{*2}}$ emerge. Thus, in Bènard problem,[30] since the only length is the thickness of the porous medium confined between two plates of sufficient large extension, it is not possible to construct monomials of the form $\frac{k_x}{k_y}\frac{l_y^{*2}}{l_x^{*2}}$, for which two characteristic lengths in perpendicular directions are required. However, as the horizontal extension of the plates is irrelevant for the physical phenomena involved in a typical cell, the introduction of an unknown length corresponding to the width of the cell makes it possible to deduce such monomials and represent the solution of the unknown length using the Pi theorem.

The most important drawback that limits the dimensional characterization and, particularly, the use of the proposed dimensionless group, is the lack of regularity in the geometry of the domain. Scenarios with sloping or curved borders make it impossible to employ the group $\frac{k_x}{k_y}\frac{l_y^{*2}}{l_x^{*2}}$. However, since most of the standard problems choose scenarios of simple geometry (even if they introduce a large number of parameters or physical properties of the medium), the utility of this monomial is essential to obtain the least number of dimensionless groups that characterize and simplify the study of these scenarios.

4 | VERIFICATION OF THE DISCRIMINATED DIMENSIONAL GROUPS INVOLVING THE PERMEABILITY

The aim of this section is to verify the influence of the group $\pi = \frac{k_x}{l_x^{*2}}\frac{l_y^{*2}}{k_y}$ (or alternatively the group $\frac{\kappa_x}{l_x^{*2}}\frac{l_y^{*2}}{\kappa_y}$) in the patterns of flow in anisotropic porous media. We will check that for the same value of the group, whatever the values of the parameters involved, the solution patterns (flow net of iso-potential and streamlines) do not change. Two illustrative applications for water flow under gravity dams, with and without a sheet pile at the end, are presented. Figure 2 shows the drawing of the problems. To enhance the effect of the monomial $\frac{k_x}{k_y}\frac{l_y^{*2}}{l_x^{*2}}$, some simplifications are assumed: horizontal lengths upstream and downstream the dam are large enough not to influence the results, and the dam foundation and sheet pilet's thickness have negligible values. The value of parameters e, h_2 and Δh is $e = h_2 = 0$ and $\Delta h = 10$.

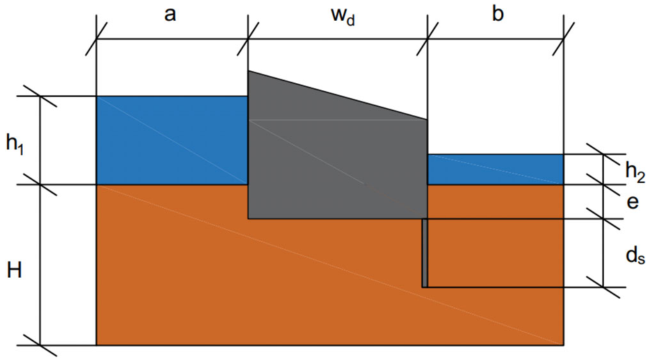

FIGURE 2 Physical scheme and nomenclature of the problem

TABLE 5 Scenario data of application 1

	κ_x	κ_y	w_d	H	a	b
Case I	0.0001	0.00008	10	20	200	200
Case II	0.0001	0.00005	10	15.81	200	200
Case III	0.0001	0.0001	8.94	20	178.89	178.89

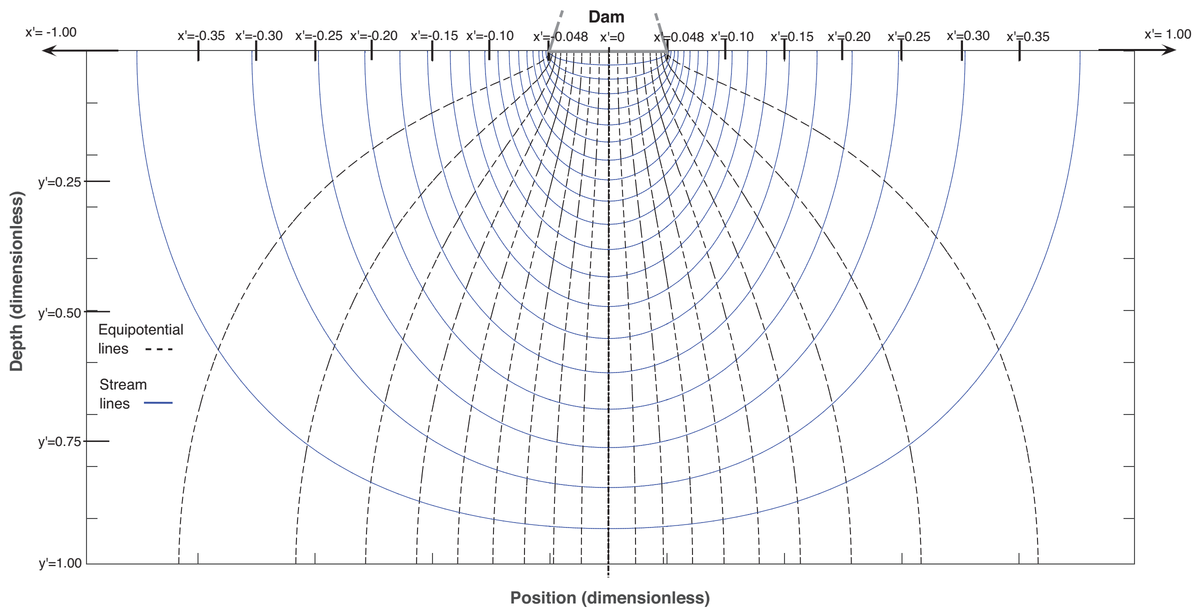

FIGURE 3 Flow net (solution pattern) for application 1. Cases I, II and III

The discriminated groups are chosen in the form

$$\pi_k = \frac{k_x}{k_y}\frac{H^2}{{w_d}^2}, \quad \pi_a = \frac{a}{w_d}, \quad \pi_b = \frac{a}{b}, \quad \pi_e = \frac{e}{H}, \quad \pi_{d_s} = \frac{d_s}{H}$$

For both applications, π_a, π_b and π_e have the same values: 20, 1 and 0, respectively. The values of π_{ds} are 0 and 0.25 for the first and second scenario, respectively. The numerical solution is carried out by the network method[31–33] and the free software Ngspice.[34]

For the first application ($\pi_k = 5$, $\pi_a = 20$, $\pi_b = 1$ and $\pi_e = 0$), three cases with different physical and geometrical parameters are chosen (Table 5). Figure 3 shows the solution of the flow net for the three cases. Iso-potential lines go from the upper side of the scenario to the lower side, while streamlines (that displays the fluid particle movement) go from

TABLE 6 Scenario data of application 2

	κ_x	κ_y	w_d	H	a	b	e	d_s	Δh	h_2
Case I	0.0001	0.00008	10	20	200	200	0	5	10	0
Case II	0.0001	0.00005	10	15.81	200	200	0	3.95	10	0
Case III	0.0001	0.0001	8.94	20	178.89	178.89	0	5	10	0

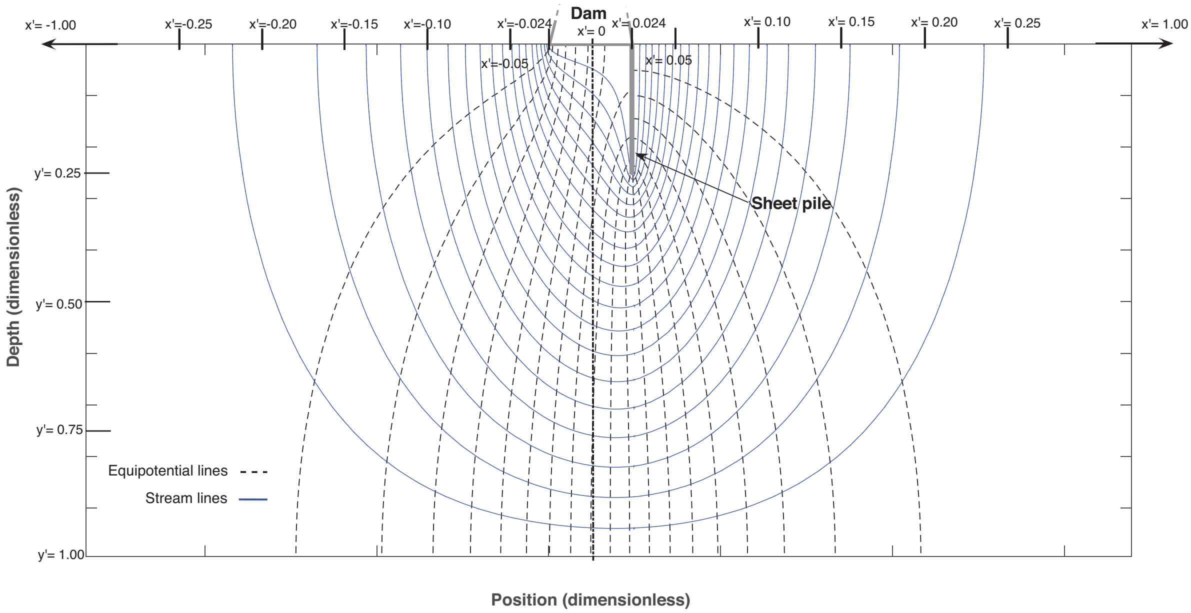

FIGURE 4 Flow net (solution pattern) for application 2. Cases I, II and III

upstream to downstream. As expected, all cases have the same pattern when represented in a normalized geometrical scale given by dimensionless coordinates $x' = \frac{x}{a+\frac{w_d}{2}}$ and $y' = \frac{y}{H}$.

The second application introduces a fourth monomial, $\pi_{d_s} = \frac{d_s}{H}$, whose value is 0.25. Again, three cases are simulated, the values of the other monomials (π_k, π_a, π_b and π_e) are the same as in the first application. Table 6 shows the data of these cases. The solution pattern for the three cases is shown in Figure 4, again a coherent solution when providing the same normalized geometrical scale.

5 | FINAL COMMENTS AND CONCLUSION

There are two overall contributions of this work, where the initial objective is the search for a dimensional equation for permeability (k) that allows obtaining the accurate dimensionless groups that govern the solution of problems of flow through porous media (particularly in anisotropic scenarios).

First, an accurate dimensional equation for permeability is obtained by the application of the spatially discriminated dimensional analysis to Darcy's and Forchheimer's laws expressed in terms of pressure. However, the emerging discriminated dimensionless group in anisotropic porous media, a ratio of permeabilities corrected by the squared value of a domain aspect factor, does not behave as the independent group that governs the solutions for flow. During this process, it is evidently difficult to obtain the correct dimensional equation due to the complexity of the physical mechanism involved in this parameter (grain size, porosity, connectivity, tortuosity, etc.). Working with the energetic potential instead of pressure, which has the same dimension as any length in the problem, also fails to provide accurate results.

Second, the problem is approached with a discriminated dimensional general basis, which specifically contains the dimension of the quantity h 'energetic potential of the fluid'. In this basis, although the dimensional equation for permeability is not found, the equation for the permeability/viscosity ratio is obtained. This result allows deducing an accurate and new dimensionless group: a permeability ratio corrected by the squared value of the domain aspect ratio, but this time the aspect ratio is the inverse of that of the first approach. The new group, also derived from the dimensionless form of the governing equation, does behave like a monomial that rule the problem.

The proposed group has been used and verified by different authors in recent works. To reinforce the verification, a numerical simulation in 2-D seepage anisotropic scenarios is carried out, checking that the stream function and potential isoline patterns of flow are the same when keeping constant the values of the groups. The emergence of the new group ($\pi = \frac{k_x}{l_x^{*2}} \frac{l_y^{*2}}{k_y}$) allows obtaining universal curves which characterize anisotropic scenarios of fluid flow in porous media in future work. This research is open to future research on the dimensional character of other physical characteristics of the macroscopic porous medium such as tortuosity, angularity, connectivity or grain shape.

NOMENCLATURE

A	cross section (m^2)
a	upstream horizontal length (m)
b	downstream horizontal length (m)
C	dimensionless constant
C_o	dimensional constant in Darcy's law, equation (1), ($m \cdot kg^1 \cdot s^1$)
d	mean size of the pore or grain (m), also derivative symbol
d_o	duct diameter (m)
d_s	sheet pile length (m)
e	foundation length (m)
f	force (N)
F	arbitrary functions involving the monomials
g	gravitational acceleration ($m \cdot s^{-2}$)
H	layer thickness (m)
h	hydraulic potential or head ($J \cdot N^{-1}$ or m)
h_1	upstream hydraulic potential (m)
h_2	downstream hydraulic potential (m)
i	hydraulic gradient (dimensionless)
k	hydraulic permeability or, simple, permeability (m^2)
L	dimension of the quantity length, also the length of the domain (m)
$L_{\rightarrow}$	spatial dimension in the direction of the fluid velocity (m)
L_n	spatial dimension normal to the $L_{\rightarrow}$ and L_{vis} (m)
L_o	length of the Darcy's domain (m)
L_{vis}	spatial dimension in a normal direction to $L_{\rightarrow}$ and parallel to viscous surface (m)
l*	arbitrary length (m)
M	dimension of the quantity mass (kg)
p	pressure (Pa)
p'	dimensionless pressure
Q	water flow ($m^3 \cdot s^{-1}$)
R_e	Reynolds number (dimensionless)
S	surface (m^2)
T	dimension of the quantity time (s)
v	Darcy-velocity ($m \cdot s^{-1}$)
v	Darcy-velocity vector ($m \cdot s^{-1}$)
w_d	dam width (m)
x, y	spatial rectangular coordinates (m)
x',y'	dimensionless spatial rectangular coordinates
Δh	difference of hydraulic potential (m), $\Delta h = h_1 - h_2$
Δp_o	reference pressure (Pa)
Δs	length of the sample (m)
∂	partial derivative
ζ	friction factor (dimensionless)
Θ	dimension of the quantity gh, the energetic hydraulic potential referred to mass, (J by mass unit)

κ	hydraulic conductivity or effective permeability ($m \cdot s^{-1}$)
μ	dynamic viscosity ($kg \cdot m^{-1} \cdot s^{-1}$)
ξ	dimension of the quantity h, the energetic potential of the fluid, (J by weight unit)
$\pi_1, \pi_2 \ldots \pi_a, \pi_b \ldots$	$\pi_a, \pi_b \ldots$ dimensionless groups
ρ	density ($kg \cdot m^{-3}$)
Φ	dimension of the quantity, ρgh the energetic hydraulic potential referred to volume, (J by volume unit)
Δ	finite increment
$\sim$	denotes order of magnitude
$\propto$	proportional to
[]	used to express the dimension of a quantity
{}	used to list the relevant variables of a problem
{}	used to set the dimensional basis

Subscripts

ine	refers to inertial effects
pre	refers to pressure effects
vis	refers to viscous effects
x, y, z	refer to spatial rectangular direction

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DATA AVAILABILITY STATEMENT

The author has provided the required Data Availability Statement, and if applicable, included functional and accurate links to said data therein.

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