VACUUM RADIATION AND SYMMETRY BREAKING IN CONFORMALLY INVARIANT QUANTUM FIELD THEORY [1]

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Abstract

The underlying reasons for the difficulty of unitarily implementing the whole conformal group $SO(4,2)$ in a massless *Quantum* Field Theory (QFT) on the Minkowski space are investigated in this paper. Firstly, we demonstrate that the singular action of the subgroup of special conformal transformations (SCT), on the standard Minkowski space M, cannot be primarily associated with the vacuum radiation problems, the reason being more profound and related to the dynamical breakdown of part of the conformal symmetry (the SCT subgroup, to be more precise) when representations of null mass are selected inside the representations of the whole conformal group. Then we show how the vacuum of the massless QFT *radiates* under the action of SCT (usually interpreted as transitions to a uniformly accelerated frame) and we calculate exactly the spectrum of the outgoing particles, which proves to be a generalization of the Planckian one, this recovered as a given limit.

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1 Introduction

The conformal group $SO(4,2)$ has ever been recognized as a symmetry of the Maxwell equations for *classical* electro-dynamics [C-B], and more recently considered as an invariance of general, non-abelian, maseless gauge field theories at the classical level. However, the *quantum* theory raises, in general, serious problems in the implementation of conformal symmetry, and much work has been devoted to the study of the physical reasons for that (see e.g. Ref. [Fr]). Basically, the main trouble associated with this quantum symmetry (at the second quantization level) lies in the difficulty of finding a vacuum, which is *stable* under special conformal transformations acting on the Minkowski space in the form:

$$x^\mu \to x'^\mu = \frac{x^\mu + c^\mu x^2}{\sigma(x,c)}, \quad \sigma(x,c) = 1 + 2cx + c^2x^2. \tag{1}$$

These transformations, which can be interpreted as transitions to systems of relativistic, uniformly accelerated observers (see e.g. Ref. [H]), cause *vacuum radiation*, a phenomenon analogous to the Fulling-Unruh effect [Fu, U] in a non-inertial reference frame. To be more precise, if $a(k), a^+(k)$ are the Fourier components of a scalar massless field $\phi(x)$, satisfying the equation

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi(x) = 0\,, \tag{2}$$

then, the Fourier components $a'(k), a'^+(k)$ of the transformed field $\phi'(x') = \sigma^{-l}(x,c)\phi(x)$ by (1) (l being the conformal dimension) are expressed in terms of both $a(k), a^+(k)$ through a Bogolyubov transformation

$$a'(\lambda) = \int dk\,[A_\lambda(k)a(k) + B_\lambda(k)a^+(k)]\,. \tag{3}$$

In the second quantized theory, the vacuum states defined by the conditions $a(k)|0\rangle = 0$ and $a'(\lambda)|0'\rangle = 0$, are not identical if the coefficients $B_\lambda(k)$ in (3) are not zero. In this case the new vacuum has a non-trivial content of untransformed particle states.

This situation is always present when quantizing field theories in curved space as well as in flat space, whenever some kind of global mutilation of the space is involved. This is the case of the natural quantization in Rindler coordinates [BD], which leads to a quantization inequivalent to the normal Minkowski quantization, or that of a quantum field in a box, where a dilatation produces a rearrangement of the vacuum [Fu]. Nevertheless, it must be stressed that the situation for SCT is more peculiar. The rearrangement of the vacuum in a massless QFT due to SCT, even though they are a symmetry of the classical system, behaves as if the conformal group were *spontaneously broken*, and this fact can be interpreted as a kind of topological *anomaly*.

Thinking of the underlying reasons for this anomaly, we are tempted to make the singular action of the transformations (1) in Minkowski space responsible for it, as has been in fact pointed out in [GU]. However, a deeper analysis of the interconnection between symmetry and quantization will reveal a more profound obstruction to the possibility of implementing unitarily STC in a generalized Minkowski space, free from singularities, when conformally invariant fields are forced to evolve in time. This way, the quantum time evolution itself destroys the conformal symmetry, leading to some sort of *dynamical symmetry breaking* which preserves the Weyl subgroup (Poincaré + dilatations).

This obstruction is traced back to the impossibility of representing the entire $SO(4,2)$ group unitarily and irreducibly on a space of functions depending arbitrarily on $\vec{x}$ (see e.g. Ref. [Fr]), so that a Cauchy surface determines the evolution in time. Natural representations, however, can be constructed by means of wave functions having support on the hole space-time and evolving in some kind of *proper time*.

From the point of view of particle quantum mechanics (or "first" quantization), the free arguments of wave functions in the configuration-space "representation" correspond to half of the canonically conjugated variables in phase space (or classical solution manifold), and this phase space is usually defined as a co-adjoint orbit of the basic symmetry group characterizing the physical system. Thus, for instance, for the Galilei or Poincaré group the phase space associated with massive spinless particles has dimension 6 and the corresponding wave functions in configuration space have the time variable factorized out. However, as mentioned above, this is not the case for the conformal group, for which we shall realize that *time is a quantum observable* subject to uncertainty relations; this fact extends covariance rules to the quantum domain.

The present study is developed in the framework of a Group Approach to Quantization (GAQ)[AA1, ANR], which proves to be specially suitable for facing those quantization problems arising from specific symmetry requirements. Furthermore, this formalism has the virtue of providing in a natural way the space on which the wave functions are defined. A very brief report on GAQ is presented in Sec. 2. In Sec. 3 we apply this quantization technique to the particular case of the group $SO(2,2)$, which is the 1+1 dimensional version of the $SO(4,2)$ symmetry. Although the conformal symmetry in 1+1 dimensions is far richer, we proceed in a way that can be straightforwardly extended to the realistic dimension. In this example we show how a (compact) configuration space, locally isomorphic to Minkowski space time, can be found inside the $SO(2,2)$ group manifold on which the whole conformal group acts without singularities. We also prove that the unitarity of the irreducible representations of $SO(2,2)$ requires the dynamical character of the time variable, or that which is similar, prevents the existence of a conformally invariant quantum evolution equation in the time variable. We examine two cases corresponding to non-compact and compact "proper time" dynamics in Subsec. 3.1 and 3.2 , respectively. Sec. 4 is devoted to the application of GAQ to a very special infinite-dimensional Lie group $\tilde{G}^{(2)}(\mathcal{H}(\tilde{G}),\tilde{G})$ directly attached to the quantum mechanical Hilbert space $\mathcal{H}(\tilde{G})$ of a "first"-quantized system characterized by the *quantizing group* $\tilde{G}$ (a central extension of $G = SO(2,2)$, for the present case). This mechanism is nothing other than a group version of the "second"-quantization algorithm. With this algorithm at hand we formulate in Sec. 4.1 a conformally invariant quantum field theory and, in Sec. 4.2, we investigate the effect of a SCT on a Weyl vacuum and the associated radiation phenomenon. We calculate exactly the spectrum of an accelerated Weyl vacuum, which proves to be a generalization of the black body spectrum, this recovered as a given limit. Final comments are presented in the last Sec. 5.

2 Quantization on a group $\tilde{G}$

The starting point of GAQ is a group $\tilde{G}$ (the quantizing group) with a principal fibre bundle structure $\tilde{G}(B,T)$, having T as the structure group and B being the base. The group T generalizes the phase invariance of Quantum Mechanics. Although the situation can be more general [ANR], we shall start with the rather general case in which $\tilde{G}$ is a central extension of a group G by T [$T = U(1)$ or even $T = C^* = \Re^+ \otimes U(1)$]. For the one-parametric group $T = U(1)$, the group law for $\tilde{G} = \{\tilde{g} = (g, \zeta)/g \in G, \zeta \in U(1)\}$ adopts the following form:

$$\tilde{g}' * \tilde{g} = (g' * g, \zeta'\zeta e^{i\xi(g',g)}) \tag{4}$$

where $g'' = g' * g$ is the group operation in G and $\xi(g', g)$ is a two-cocycle of G on $\Re$ fulfilling:

$$\xi(g_2, g_1) + \xi(g_2 * g_1, g_3) = \xi(g_2, g_1 * g_3) + \xi(g_1, g_3) \ \ , g_i \in G. \tag{5}$$

In the general theory of central extensions [B], two two-cocycles are said to be equivalent if they differ in a coboundary, i.e. a cocycle which can be written in the form $\xi(g', g) = \delta(g' * g) - \delta(g') - \delta(g)$, where $\delta(g)$ is called the generating function of the coboundary. However, although cocycles differing on a coboundary lead to equivalent central extensions as such, there are some coboundaries which provide a non-trivial connection on the fibre bundle $\tilde{G}$ and Lie-algebra structure constants different from that of the direct product $G \otimes U(1)$. These are generated by a function δ with a non-trivial gradient at the identity, and can be divided into equivalence Pseudo-cohomology subclasses: two pseudo-cocycles are equivalent if they differ in a coboundary generated by a function with trivial gradient at the identity [S, AA2, AGM]. Pseudo-cohomology plays an important role in the theory of finite-dimensional semi-simple group, as they have trivial cohomology. For them, Pseudo-cohomology classes are associated with coadjoint orbits [AGM].

The right and left finite actions of the group $\tilde{G}$ on itself provide two sets of mutually commuting (left- and right-, respectively) invariant vector fields:

$$\tilde{X}^L_{\tilde{g}^i} = \left.\frac{\partial \tilde{g}''^j}{\partial \tilde{g}^i}\right|_{\tilde{g}=e} \frac{\partial}{\partial \tilde{g}^j}, \quad \tilde{X}^R_{\tilde{g}^i} = \left.\frac{\partial \tilde{g}''^j}{\partial \tilde{g}'^i}\right|_{\tilde{g}'=e} \frac{\partial}{\partial \tilde{g}^j}, \quad \left[\tilde{X}^L_{\tilde{g}^i}, \tilde{X}^R_{\tilde{g}^j}\right] = 0, \tag{6}$$

where $\{\tilde{g}^j\}$ is a parameterization of $\tilde{G}$. The GAQ program continues finding the left-invariant 1-form Θ (the *Quantization 1-form*) associated with the central generator $\tilde{X}^L_\zeta = \tilde{X}^R_\zeta, \zeta \in T$, i.e. the T-component $\tilde{\theta}^{L(\zeta)}$ of the canonical left-invariant 1-form $\tilde{\theta}^L$ on $\tilde{G}$. This constitutes the generalization of the Poincaré-Cartan form of Classical Mechanics (see [AM]). The differential $d\Theta$ is a *presymplectic* form and its *characteristic module*, $Ker\Theta \cap Kerd\Theta$, is generated by a left subalgebra $\mathcal{G}_\Theta$ named *characteristic subalgebra*. The quotient $(\tilde{G}, \Theta)/\mathcal{G}_\Theta$ is a *quantum manifold* in the sense of Geometric Quantization [GQ]. The trajectories generated by the vector fields in $\mathcal{G}_\Theta$ constitute the generalized equations of motion of the theory (temporal evolution, rotations, etc...), and the "Noether" invariants under those equations are $F_{\tilde{g}^j} \equiv i_{\tilde{X}^R_{\tilde{g}^j}} \Theta$, i.e. the contraction of right-invariant vector fields with the Quantization 1-form.

Let $\mathcal{B}(\tilde{G})$ be the set of complex-valued *T-functions* on $\tilde{G}$ in the sense of principal bundle theory:

$$\psi(\zeta * \tilde{g}) = D_T(\zeta)\psi(\tilde{g}), \ \ \zeta \in T\,, \tag{7}$$

where D_T is the natural representation of T on the complex numbers C. The representation of $\tilde{G}$ on $\mathcal{B}(\tilde{G})$ generated by $\tilde{\mathcal{G}}^R = \{\tilde{X}^R\}$ is called *Bohr Quantization* and is *reducible*. The reduction can be achieved by means of the restrictions imposed by a *full polarization* $\mathcal{P}$:

$$\tilde{X}^L \psi_{\mathcal{P}} = 0, \ \ \forall \tilde{X}^L \in \mathcal{P}\,, \tag{8}$$

which is a maximal, horizontal (excluding $\tilde{X}^L_\zeta$) left subalgebra of $\tilde{\mathcal{G}}^L$ which contains $\mathcal{G}_\Theta$. It should be noted that the existence of a full polarization, containing the whole subalgebra $\mathcal{G}_\Theta$, is not guaranteed. In case of such a breakdown, called *anomaly*, or simply for the desire of choosing of a preferred representation space, a higher-order polarization has to be imposed [ABLN]. A higher-order polarization is a maximal, horizontal subalgebra of the left enveloping algebra $U\tilde{\mathcal{G}}^L$ which contains $\mathcal{G}_\Theta$.

The group $\tilde{G}$ is irreducibly represented on the space $\mathcal{H}(\tilde{G}) \equiv \{|\psi\rangle\}$ of polarized wave functions, and on its dual $\mathcal{H}^*(\tilde{G}) \equiv \{\langle\psi|\}$. If we denote by

$$\langle \tilde{g}_{\mathcal{P}}|\psi\rangle \equiv \psi_{\mathcal{P}}(\tilde{g})\,,\ \ \langle\psi'|\tilde{g}_{\mathcal{P}}\rangle \equiv \psi'^*_{\mathcal{P}}(\tilde{g}) \tag{9}$$

the coordinates of the "ket" $|\psi\rangle$ and the "bra" $\langle\psi'|$ in a representation defined through a polarization $\mathcal{P}$ (first- or higher-order), then, a scalar product on $\mathcal{H}(\tilde{G})$ can be naturally defined as:

$$\langle\psi'|\psi\rangle \equiv \int_{\tilde{G}} v(\tilde{g})\psi'^{\,*}_{\mathcal{P}}(\tilde{g})\psi_{\mathcal{P}}(\tilde{g}), \tag{10}$$

where

$$v(\tilde{g}) \equiv \theta^L_{\tilde{g}^i}\wedge \overset{\dim(\tilde{G})}{\cdots} \wedge\theta^L_{\tilde{g}^n} \tag{11}$$

is the left-invariant integration volume in $\tilde{G}$ and

$$1 = \int_{\tilde{G}} |\tilde{g}_{\mathcal{P}}\rangle v(\tilde{g})\langle\tilde{g}_{\mathcal{P}}| \tag{12}$$

formally represents a *closure* relation. A direct computation proves that, with this scalar product, the group $\tilde{G}$ is unitarily represented through the *left* finite action (ρ denotes the representation)

$$\langle\tilde{g}_{\mathcal{P}}|\rho(\tilde{g}')|\psi\rangle \equiv \psi_{\mathcal{P}}(\tilde{g}'^{-1} * \tilde{g})\,. \tag{13}$$

The *adjoint* action is then defined as

$$\langle\psi'|\rho^\dagger(\tilde{g}')|\psi\rangle \equiv \langle\psi|\rho(\tilde{g}')|\psi'\rangle^*, \ \text{ i.e } \ \langle\tilde{g}_{\mathcal{P}}|\rho^\dagger(\tilde{g}')|\psi\rangle = \psi_{\mathcal{P}}(\tilde{g}' * \tilde{g})\,. \tag{14}$$

We can relate the coordinates of $|\psi\rangle$ in two given representations, corresponding with two different polarizations $\mathcal{P}_1$ and $\mathcal{P}_2$, as follows:

$$\psi_{\mathcal{P}_1}(\tilde{g}) = \langle\tilde{g}_{\mathcal{P}_1}|\psi\rangle = \int_{\tilde{G}} v(\tilde{g}')\langle\tilde{g}_{\mathcal{P}_1}|\tilde{g}'_{\mathcal{P}_2}\rangle\langle\tilde{g}'_{\mathcal{P}_2}|\psi\rangle \equiv \int_{\tilde{G}} v(\tilde{g}')\Delta_{\mathcal{P}_1\mathcal{P}_2}(\tilde{g},\tilde{g}')\psi_{\mathcal{P}_2}(\tilde{g}')\,, \tag{15}$$

where $\Delta_{\mathcal{P}_1\mathcal{P}_2}(\tilde{g},\tilde{g}')$ is a "polarization changing" operator. An explicit expression of $\Delta_{\mathcal{P}_1\mathcal{P}_2}$ can be obtained by making use of a basis $\{|n\rangle\}_{n\in I}$ (I is a set of indices) of $\mathcal{H}(\tilde{G})$, as follows:

$$\Delta_{\mathcal{P}_1\mathcal{P}_2}(\tilde{g},\tilde{g}') = \langle\tilde{g}_{\mathcal{P}_1}|\tilde{g}'_{\mathcal{P}_2}\rangle = \sum_{n\in I}\psi^*_{\mathcal{P}_1,n}(\tilde{g})\psi_{\mathcal{P}_2,n}(\tilde{g}')\,, \tag{16}$$

where $\psi_{\mathcal{P}_i,n}(\tilde{g}) \equiv \langle\tilde{g}_{\mathcal{P}_i}|n\rangle$ are the coordinates of $|n\rangle$ in a polarization $\mathcal{P}_i$.

Constraints are consistently incorporated into the theory by enlarging the structure group T (which always includes $U(1)$), i.e. through T-function conditions:

$$\rho(\tilde{t})|\psi\rangle = D^{(\epsilon)}_T(\tilde{t})|\psi\rangle\,, \ \ \tilde{t}\in T \tag{17}$$

or, for continuous transformations,

$$\tilde{X}^R_{\tilde{t}}|\psi\rangle = dD^{(\epsilon)}_T(\tilde{t})|\psi\rangle \ , \tag{18}$$

$D^{(\epsilon)}_T$ means a specific representation of T [the index ϵ parametrizes different (inequivalent) quantizations] and $dD^{(\epsilon)}_T$ is its differential.

It is obvious that, for a non-central structure group T, not all the right operators $\tilde{X}^R_{\tilde{g}}$ will preserve these constraints; a sufficient condition for a subgroup $\tilde{G}_T \subset \tilde{G}$ to preserve the constraints is (see [ACG]):

$$\left[\tilde{G}_T, T\right] \subset \mathrm{Ker} D^{(\epsilon)}_T \tag{19}$$

[note that, for the trivial representation of T, the subgroup $\tilde{G}_T$ is nothing other than the *normalizer* of T]. $\tilde{G}_T$ takes part of the set of *good* operators [ANR], of the enveloping algebra $U\tilde{\mathcal{G}}^R$ in general, for which the subgroup T behaves as a *gauge* group (see [NAC] for a thorough study of gauge symmetries and constraints from the point of view of GAQ). A more general situation can be posed when the constraints are lifted to the higher-order level, not necessarily first order as in (18), that is, they are a subalgebra of the right enveloping algebra $U\tilde{\mathcal{G}}^R$. An interesting example of this last case arises when one selects representations labelled by a value ϵ of some Casimir operator Q of a subgroup $\tilde{G}_Q$ of $\tilde{G}$. This is exactly the case that interests us: null mass representations ($\epsilon = m = 0$) of the Poincaré group ($\tilde{G}_Q = SO(3,1) \otimes_s T_4\,,\ Q = P_\nu P^\nu$) inside the conformal group ($\tilde{G} = SO(4,2)$).

In the more general case in which T is not a trivial central extension, $T \neq \check{T} \times U(1)$, where $\check{T} \equiv T/U(1)$ -i.e. T contains second-class constraints- the conditions (18) are not all compatible and we must select a subgroup $T_B = T_p \times U(1)$, where T_p is the subgroup associated with a right polarization subalgebra of the central extension T (see [ANR]).

For simplicity, we have sometimes made use of infinitesimal (geometrical) concepts, but all this language can be translated to their finite (algebraic) counterparts (see [ANR]), a desirable way of proceeding when discrete transformations are incorporated into the theory.

3 Conformally invariant Quantum Mechanics

Conformally invariant Quantum Mechanics (in 1+1D) will be developed by finding the unitary irreducible representations of the centrally extended $SO(2,2)$ group in exactly the same way the Hilbert space of the Galilean particle is obtained from the unitary irreducible representations of the centrally extended Galilei group (see e.g. Ref. [AA1]). The configuration space of the theory or, rather, an analytic continuation of Minkowski space, will arise as a homogeneous space of the group, on which the wave functions supporting the irreducible representation take arguments.

Except for discrete symmetries, which are not relevant at the Lie algebra level, $SO(2,2) \sim SU(1,1) \otimes SU(1,1)$ so that we shall look at the structure of

$$SU(1,1) = \left\{ U = \begin{pmatrix} z_1 & z_2 \\ z_2^* & z_1^* \end{pmatrix}, z_i, z_i^* \in C/\det(U) = |z_1|^2 - |z_2|^2 = 1 \right\} . \tag{20}$$

$SU(1,1)$ is a fibre bundle with fibre $U(1)$ and base the hyperboloid. A system of coordinates adapted to this fibration is the following:

$$\eta \equiv \frac{z_1}{|z_1|},\ \ \alpha \equiv \frac{z_2}{z_1},\ \ \alpha^* \equiv \frac{z_2^*}{z_1^*}, \quad \eta \in U(1),\ \ \alpha, \alpha^* \in D_1, \tag{21}$$

where D_1 is the unit disk and the coordinates α, α^* are related to the stereographical projection of the hyperboloid on the disk. The inverse transformation is:

$$z_1 = \sqrt{\frac{1}{1-\alpha\alpha^*}}\eta, \quad z_2 = \sqrt{\frac{1}{1-\alpha\alpha^*}}\alpha\eta. \tag{22}$$

The group law $U'' = U'U$ in η, α, α^* coordinates is:

$$\begin{aligned}
\eta'' &= \frac{z_1''}{|z_1''|} = \frac{\eta'\eta + \eta'\eta^*\alpha'\alpha^*}{\sqrt{(1+\eta^{*2}\alpha'\alpha^*)(1+\eta^2\alpha\alpha^{*\prime})}} \\
\alpha'' &= \frac{z_2''}{z_1''} = \frac{\alpha\eta^2 + \alpha'}{\eta^2 + \alpha'\alpha^*} \\
\alpha^{*\prime\prime} &= \frac{z_2''^*}{z_1''^*} = \frac{\alpha^*\eta^{-2} + \alpha^{*\prime}}{\eta^{-2} + \alpha^{*\prime}\alpha},
\end{aligned} \tag{23}$$

from which we can extract the left- and right-invariant vector fields:

$$\begin{aligned}
X^L_\eta &= \eta\frac{\partial}{\partial\eta} - 2\alpha\frac{\partial}{\partial\alpha} + 2\alpha^*\frac{\partial}{\partial\alpha^*} \\
X^L_\alpha &= -\frac{1}{2}\eta\alpha^*\frac{\partial}{\partial\eta} + \frac{\partial}{\partial\alpha} - \alpha^{*2}\frac{\partial}{\partial\alpha^*} \\
X^L_{\alpha^*} &= \frac{1}{2}\eta\alpha\frac{\partial}{\partial\eta} - \alpha^2\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\alpha^*} \\
X^R_\eta &= \eta\frac{\partial}{\partial\eta} \\
X^R_\alpha &= \frac{1}{2}\eta^{-1}\alpha^*\frac{\partial}{\partial\eta} + \eta^{-2}(1-\alpha\alpha^*)\frac{\partial}{\partial\alpha} \\
X^R_{\alpha^*} &= -\frac{1}{2}\eta^3\alpha\frac{\partial}{\partial\eta} + \eta^2(1-\alpha\alpha^*)\frac{\partial}{\partial\alpha^*}.
\end{aligned} \tag{24}$$

They close the Lie algebra:

$$\left[X^L_\eta, X^L_\alpha\right] = 2X^L_\alpha\,, \quad \left[X^L_\eta, X^L_{\alpha^*}\right] = -2X^L_{\alpha^*}\,, \quad \left[X^L_\alpha, X^L_{\alpha^*}\right] = X^L_\eta\,, \tag{25}$$

and the corresponding right version by changing the sign to the structure constants.

Let us parameterize $G = SO(2,2)$ as two copies of $SU(1,1)$ with parameters $\{(\eta, \alpha, \alpha^*); (\bar\eta, \bar\alpha, \bar\alpha^*)\}$. There are two possibilities of combining the generators in the Lie algebra

$$\mathcal{G}^L = \{X^L_\eta, X^L_\alpha, X^L_{\alpha^*}, X^L_{\bar\eta}, X^L_{\bar\alpha}, X^L_{\bar\alpha^*}\} \tag{26}$$

of G, in order to get the usual conformal generators

$$\mathcal{G}^L = \{D^L, M^L, {P_0}^L, {P_1}^L, {K_0}^L, {K_1}^L\} \tag{27}$$

which fulfil the ordinary commutation relations [K]:

$$\begin{array}{lll}
\left[P^L_0, D^L\right] = -P^L_0 & \left[P^L_1, D^L\right] = -P^L_1 & \left[P^L_0, M^L\right] = -P^L_1 \\
\left[P^L_1, M^L\right] = -P^L_0 & \left[P^L_0, K^L_0\right] = -2D^L & \left[P^L_1, K^L_0\right] = -2M^L \\
\left[P^L_0, K^L_1\right] = 2M^L & \left[P^L_1, K^L_1\right] = 2D^L & \left[K^L_0, D^L\right] = K^L_0 \\
\left[K^L_1, D^L\right] = K^L_1 & \left[K^L_0, M^L\right] = -K^L_1 & \left[K^L_1, M^L\right] = -K^L_0 \\
\left[D^L, M^L\right] = 0 & \left[P^L_0, P^L_1\right] = 0 & \left[K^L_0, K^L_1\right] = 0
\end{array} \tag{28}$$

where D, M, P_μ, K_μ are the generators of dilatation, boosts, space-time translations and special conformal transformations, respectively. One of the two mentioned choices lead to a non-compact dilatation subgroup, whereas the other leads to a compact one. Let us show what this combinations are in both cases:

$$
\begin{array}{ll}
\text{COMPACT D} & \text{NON COMPACT D} \\
D^L = -\frac{1}{2}\left(X^L_\eta + X^L_{\bar\eta}\right) & D^L = -\frac{i}{2}\left(X^L_\alpha - X^L_{\alpha^*} + X^L_{\bar\alpha} - X^L_{\bar\alpha^*}\right) \\
M^L = \frac{1}{2}\left(X^L_\eta - X^L_{\bar\eta}\right) & M^L = \frac{i}{2}\left(X^L_\alpha - X^L_{\alpha^*} - X^L_{\bar\alpha} + X^L_{\bar\alpha^*}\right) \\
P^L_0 = -\left(X^L_{\alpha^*} + X^L_{\bar\alpha^*}\right) & P^L_0 = \frac{1}{2}\left(X^L_\alpha + X^L_{\alpha^*} - X^L_{\bar\alpha} - X^L_{\bar\alpha^*} - i(X^L_\eta - X^L_{\bar\eta})\right) \\
P^L_1 = X^L_{\alpha^*} - X^L_{\bar\alpha^*} & P^L_1 = -\frac{1}{2}\left(X^L_\alpha + X^L_{\alpha^*} + X^L_{\bar\alpha} + X^L_{\bar\alpha^*} - i(X^L_\eta + X^L_{\bar\eta})\right) \\
K^L_0 = X^L_\alpha + X^L_{\bar\alpha} & K^L_0 = \frac{1}{2}\left(-X^L_\alpha - X^L_{\alpha^*} + X^L_{\bar\alpha} + X^L_{\bar\alpha^*} - i(X^L_\eta - X^L_{\bar\eta})\right) \\
K^L_1 = X^L_\alpha - X^L_{\bar\alpha} & K^L_1 = -\frac{1}{2}\left(X^L_\alpha + X^L_{\alpha^*} + X^L_{\bar\alpha} + X^L_{\bar\alpha^*} + i(X^L_\eta + X^L_{\bar\eta})\right)
\end{array}
\tag{29}
$$

The group $G = SU(1,1) \otimes SU(1,1)$ is non-compact and semisimple. The left-invariant integration volume can be expressed as:

$$
v(g) \equiv \theta^L_{g^i} \wedge \overset{\dim(G)}{\cdots} \wedge \theta^L_{g^n} = -\frac{1}{\eta(1-\alpha\alpha^*)^2}\frac{1}{\bar\eta(1-\bar\alpha\bar\alpha^*)^2} d\alpha \wedge d\alpha^* \wedge d\eta \wedge d\bar\alpha \wedge d\bar\alpha^* \wedge d\bar\eta, \tag{30}
$$

which becomes singular for values $|\alpha|, |\bar\alpha| \to 1$ (unit circumferences surrounding both open disks). However, resorting to a central extension $\tilde G$ of G, necessarily trivial since G is semisimple and finite-dimensional, we shall turn the extended scalar product, between wave functions on the group, finite for some range of the extension parameter.

There are several central extensions of the conformal group, but we are interested in one that afterwards leads to a generalized Minkowski space. This choice corresponds to an extension by a coboundary locally generated by the dilatation parameter, which we shall consider as a "proper time" (see Ref. [AA3]).

We shall separate the two cases: a) non-compact and b) compact dilatation subgroup, in two subsections, respectively. The essentials of the problem we are involved in are insensitive to the topological character of the dilatation subgroup; however, whereas the non compact dilatation case will be useful to connect with some standard expressions in Minkowski space, the compact dilatation case will be more manageable to construct and illustrate the second quantization program. It can be proven that a consistent quantum theory needs the group C^* as the structure group T for the first case, whereas a pseudo-extension by $U(1)$ is enough for the second one.

3.1 Non-compact dilatation subgroup

Let us look for a $T = C^* = \{z = r\zeta;\, r \in \Re^+,\, \zeta \in U(1)\}$-pseudo-extension

$$
z'' = z'ze^{\xi(g',g)}, \quad \xi(g',g) = \delta(g' * g) - \delta(g') - \delta(g), \quad z \in C^* \tag{31}
$$

where $\delta(g) = -i\beta(\alpha - \alpha^* + \bar\alpha - \bar\alpha^*)$ is the function which generates the coboundary and $\beta = \beta_1 + i\beta_2$ is a complex parameter characterizing the representation.

The extended left- and right-invariant vector fields in $\tilde G$ are

$$
\begin{aligned}
\tilde{X}^L_r &= \tilde{X}^R_r = r\frac{\partial}{\partial r} \\
\tilde{X}^L_\zeta &= \tilde{X}^R_\zeta = \zeta\frac{\partial}{\partial \zeta} \\
\tilde{X}^L_\eta &= X^L_\eta + 2i\beta_1(\alpha+\alpha^*)\tilde{X}^L_r - 2\beta_2(\alpha+\alpha^*)\tilde{X}^L_\zeta \\
\tilde{X}^L_\alpha &= X^L_\alpha - i\beta_1\alpha^{*2}\tilde{X}^L_r + \beta_2\alpha^{*2}\tilde{X}^L_\zeta \\
\tilde{X}^L_{\alpha^*} &= X^L_{\alpha^*} + i\beta_1\alpha^2\tilde{X}^L_r - \beta_2\alpha^2\tilde{X}^L_\zeta \\
\tilde{X}^R_\eta &= X^R_\eta \\
\tilde{X}^R_\alpha &= X^R_\alpha - i\beta_1(\eta^{-1}(1-\alpha\alpha^*)-1)\tilde{X}^R_r + \beta_2(\eta^{-1}(1-\alpha\alpha^*)-1)\tilde{X}^R_\zeta \\
\tilde{X}^R_{\alpha^*} &= X^R_{\alpha^*} + i\beta_1(\eta^2(1-\alpha\alpha^*)-1)\tilde{X}^R_r - \beta_2(\eta^2(1-\alpha\alpha^*)-1)\tilde{X}^R_\zeta
\end{aligned} \tag{32}
$$

and similar expression for the $\bar{\eta}, \bar{\alpha}, \bar{\alpha}^*$ parameters. The new commutation relations for the extended conformal Lie algebra $\tilde{\mathcal{G}}$ of $\tilde{G}$ are two copies of:

$$
\begin{aligned}
\left[\tilde{X}^L_\eta, \tilde{X}^L_\alpha\right] &= 2\tilde{X}^L_\alpha - 2i\beta_1\tilde{X}^L_r + 2\beta_2\tilde{X}^L_\zeta \\
\left[\tilde{X}^L_\eta, \tilde{X}^L_{\alpha^*}\right] &= -2\tilde{X}^L_{\alpha^*} - 2i\beta_1\tilde{X}^L_r + 2\beta_2\tilde{X}^L_\zeta \\
\left[\tilde{X}^L_\alpha, \tilde{X}^L_{\alpha^*}\right] &= \tilde{X}^L_\eta \\
\left[\tilde{X}^L_r, \mathrm{all}\right] &= 0 \\
\left[\tilde{X}^L_\zeta, \mathrm{all}\right] &= 0\ ,
\end{aligned} \tag{33}
$$

the right ones changing a global sign in the structure constants. The only change induced in the Lie algebra commutators, when expressed in terms of

$$
\tilde{\mathcal{G}}^L = \{\tilde{D}^L, \tilde{M}^L, \tilde{P}^L_0, \tilde{P}^L_1, \tilde{K}^L_0, \tilde{K}^L_1, \tilde{X}^L_\zeta, \tilde{X}^L_r\} \tag{34}
$$

(having the same functional form as in the right hand side of eq. (29)), is in the following two commutators:

$$
\begin{aligned}
\left[\tilde{P}^L_0, \tilde{K}^L_0\right] &= -2\tilde{D}^L + 4\beta_1\tilde{X}^L_r + 4i\beta_2\tilde{X}^L_\zeta \\
\left[\tilde{P}^L_1, \tilde{K}^L_1\right] &= 2\tilde{D}^L - 4\beta_1\tilde{X}^L_r - 4i\beta_2\tilde{X}^L_\zeta\ ,
\end{aligned} \tag{35}
$$

the remainder keeping the same expression as in Eq. (28). These relations show that the two couples of generators $\tilde{P}^L_0, \tilde{K}^L_0$ and $\tilde{P}^L_1, \tilde{K}^L_1$ are canonically conjugate, i.e. they give rise to central terms at the right-hand side of the corresponding commutator. Central extensions of this kind were already considered in Refs. [AAB, AA3]. From (35) we conclude that, like the space operator, time is not deprived of dynamical character, that is, it is an operator subject to uncertainty relations (see [JR] for another definition of space-time position operators inside the enveloping algebra of the conformal group).

The quantization form and its characteristic module are

$$
\Theta = \left(\Theta^{(r)}, \Theta^{(\zeta)}\right)
$$

$$\begin{aligned}
\Theta^{(r)} &= \beta_1(\Gamma(\eta,\alpha,\alpha^*)+\Gamma(\bar\eta,\bar\alpha,\bar\alpha^*))+r^{-1}dr \\
\Theta^{(\zeta)} &= -\beta_2(\Gamma(\eta,\alpha,\alpha^*)+\Gamma(\bar\eta,\bar\alpha,\bar\alpha^*))+i\zeta^{-1}d\zeta \\
\Gamma(\eta,\alpha,\alpha^*) &\equiv \frac{1}{1-\alpha\alpha^*}\Big(-2i(\alpha+\alpha^*)\eta^{-1}d\eta - i\alpha\alpha^* d\alpha + i\alpha\alpha^* d\alpha^*\Big) \\
\mathcal{G}_\Theta &= <\tilde{D}^L, \tilde{M}^L> .
\end{aligned} \tag{36}$$

Let $\mathcal{B}(\tilde G)$ be the set of complex valued T-*functions* on $\tilde G$ in the sense of principal bundle theory: $\psi(z*\tilde g)=D_T(z)\psi(\tilde g)$ and let us choose the representation $D_T(z)=z^p$, where p has to be a negative integer for single-valuedness and "square integrable" condition of the wave function. In order to reduce the representation of $\tilde G$ on $\mathcal{B}(\tilde G)$, we impose the full polarization subalgebra:

$$\mathcal{P}=<\tilde D^L,\tilde M^L,\tilde K_0^L,\tilde K_1^L> \ . \tag{37}$$

The solution to the polarization conditions leads to a Hilbert space $\mathcal{H}(\tilde G)$ made of wave functions of the form

$$\begin{aligned}
\psi^{(\beta)}(\eta,\alpha,\alpha^*,\bar\eta,\bar\alpha,\bar\alpha^*,z) &= z^p W_\beta(\alpha,\alpha^*,\bar\alpha,\bar\alpha^*)\phi(\mu,\bar\mu) \\
W_\beta(\alpha,\alpha^*,\bar\alpha,\bar\alpha^*) &= w_\beta(\alpha,\alpha^*)w_\beta(\bar\alpha,\bar\alpha^*) \\
w_\beta(\alpha,\alpha^*) &= (1-\alpha\alpha^*)^{p\beta}(\alpha+i)^{-p\beta}(\alpha^*-i)^{-p\beta}e^{ip\beta(\alpha-\alpha^*)} ,
\end{aligned} \tag{38}$$

where W_β is a "generating function" and ϕ is an arbitrary power series

$$\phi(\mu,\bar\mu)=\sum_{n,\bar n=-\infty}^{\infty} a_{n,\bar n}\phi_{n,\bar n}(\mu,\bar\mu), \quad \phi_{n,\bar n}(\mu,\bar\mu)\equiv \mu^n\bar\mu^{\bar n} \tag{39}$$

in the variables

$$\mu=\frac{\alpha^*-i}{\alpha+i}\eta^{-2}=\frac{z_2^*-iz_1^*}{z_2+iz_1}, \quad \bar\mu=\frac{\bar\alpha^*-i}{\bar\alpha+i}\bar\eta^{-2}=\frac{\bar z_2^*-i\bar z_1^*}{\bar z_2+i\bar z_1} . \tag{40}$$

Note that $(\mu,\bar\mu)$ are defined in a two-dimensional torus $T^2=S^1\times S^1$ (the 1+1 dimensional version of the 3+1 dimensional projective cone $S^3\times S^1/Z_2$). Let us show how the conformal group act on T^2 *free from singularities*. For this, we have only to translate the group composition law, originally written in global variables $z_i,\bar z_i$, $i=1,2$, in Eq.(22), to the variables $\mu,\bar\mu$:

$$\begin{aligned}
\mu \;&\to\; \mu''\equiv\frac{z_2^{*\prime\prime}-iz_1^{*\prime\prime}}{z_2''+iz_1''}=\frac{z_2^{*\prime}z_2+z_1^{*\prime}z_1^*-i(z_2^{*\prime}z_1+z_1^{*\prime}z_2^*)}{z_1'z_2+z_2'z_1^*+i(z_1'z_1+z_2'z_2^*)}=\frac{\mu-i\alpha^{*\prime}}{\eta'^2(1+i\mu\alpha')} \\
\bar\mu \;&\to\; \bar\mu''\equiv\frac{\bar z_2^{*\prime\prime}-i\bar z_1^{*\prime\prime}}{\bar z_2''+i\bar z_1''}=\frac{\bar z_2^{*\prime}\bar z_2+\bar z_1^{*\prime}\bar z_1^*-i(\bar z_2^{*\prime}\bar z_1+\bar z_1^{*\prime}\bar z_2^*)}{\bar z_1'\bar z_2+\bar z_2'\bar z_1^*+i(\bar z_1'\bar z_1+\bar z_2'\bar z_2^*)}=\frac{\bar\mu-i\bar\alpha^{*\prime}}{\bar\eta'^2(1+i\bar\mu\bar\alpha')} .
\end{aligned} \tag{41}$$

This action is always well defined and transitive on T^2 (see Ref. [LM] for a more detailed study of the global properties of a similar space in 3+1 dimensions), in contrast to the action on the Minkowski space, which can be seen as a local chart of T^2 obtained by stereographical projection ($\mu\equiv e^{i\theta}$, $\bar\mu\equiv e^{i\bar\theta}$):

$$\begin{aligned}
t &= \frac{1}{2}(\cot\frac{\theta}{2}+\cot\frac{\bar\theta}{2}) \\
x &= \frac{1}{2}(\cot\frac{\theta}{2}-\cot\frac{\bar\theta}{2}),
\end{aligned} \tag{42}$$

as can be checked by realizing that the expression of the generators of the conformal group in T^2 (see Eq. (44)) acquire the standard form in Minkowski space –except for (quantum) inhomogeneous terms proportional to the extension parameter β– (see [K] for instance) when expressed in terms of t, x. The manifold T^2 is thus a natural space-time on which a globally-defined 1+1 conformaly invariant QFT can live.

The invariant integration volume is $v(\tilde{g}) = v(g) \wedge (r^{-1}dr) \wedge (i\zeta^{-1}d\zeta)$ (see Eq.(30)). The scalar product of two wave functions (38) will be finite when the factor $((1-\alpha\alpha^*)(1-\bar{\alpha}\bar{\alpha}^*))^{2p\beta}$, coming from W_β (see Eq.(38)), cancels out the singularity of $v(\tilde{g})$ at the boundary of the unit disk due to the factor $((1-\alpha\alpha^*)(1-\bar{\alpha}\bar{\alpha}^*))^{-2}$. This is possible when

$$p\beta_1 > 1/2\,, \tag{43}$$

with no restriction in the parameter β_2 (this is the reason why the pseudo-extension by the real positive line, with parameter $\beta_1 \neq 0$, is fundamental for this case).

The action of the right-invariant vector fields (operators in the theory) on polarized wave functions (see Eq. (38)) has the explicit form:

$$\begin{aligned}
\tilde{D}^R\psi^{(\beta)} &= z^p W_\beta \cdot \left(-\frac{1}{2}(\mu^2-1)\frac{\partial}{\partial\mu} - \frac{1}{2}(\bar{\mu}^2-1)\frac{\partial}{\partial\bar{\mu}} - p\beta(\mu+\mu^{-1}+\bar{\mu}+\bar{\mu}^{-1}-2)\right)\phi(\mu,\bar{\mu}) \\
\tilde{M}^R\psi^{(\beta)} &= z^p W_\beta \cdot \left(\frac{1}{2}(\mu^2-1)\frac{\partial}{\partial\mu} - \frac{1}{2}(\bar{\mu}^2-1)\frac{\partial}{\partial\bar{\mu}} - p\beta(-\mu-\mu^{-1}+\bar{\mu}+\bar{\mu}^{-1})\right)\phi(\mu,\bar{\mu}) \\
\tilde{P}_0^R\psi^{(\beta)} &= z^p W_\beta \cdot \left(-\frac{i}{2}(\mu-1)^2\frac{\partial}{\partial\mu} + \frac{i}{2}(\bar{\mu}-1)^2\frac{\partial}{\partial\bar{\mu}} - p\beta(\mu-\mu^{-1}-\bar{\mu}+\bar{\mu}^{-1})\right)\phi(\mu,\bar{\mu}) \\
\tilde{P}_1^R\psi^{(\beta)} &= z^p W_\beta \cdot \left(\frac{i}{2}(\mu-1)^2\frac{\partial}{\partial\mu} + \frac{i}{2}(\bar{\mu}-1)^2\frac{\partial}{\partial\bar{\mu}} - p\beta(-\mu+\mu^{-1}-\bar{\mu}+\bar{\mu}^{-1})\right)\phi(\mu,\bar{\mu}) \\
\tilde{K}_0^R\psi^{(\beta)} &= z^p W_\beta \cdot \left(\frac{i}{2}(\mu+1)^2\frac{\partial}{\partial\mu} - \frac{i}{2}(\bar{\mu}+1)^2\frac{\partial}{\partial\bar{\mu}} + p\beta(\mu-\mu^{-1}-\bar{\mu}+\bar{\mu}^{-1})\right)\phi(\mu,\bar{\mu}) \\
\tilde{K}_1^R\psi^{(\beta)} &= z^p W_\beta \cdot \left(\frac{i}{2}(\mu-1)^2\frac{\partial}{\partial\mu} + \frac{i}{2}(\bar{\mu}-1)^2\frac{\partial}{\partial\bar{\mu}} - p\beta(-\mu+\mu^{-1}-\bar{\mu}+\bar{\mu}^{-1})\right)\phi(\mu,\bar{\mu}) \\
\tilde{X}_r^R\psi^{(\beta)} &= p\psi^{(\beta)},\ \tilde{X}_\zeta^R\psi^{(\beta)} = p\psi^{(\beta)}.
\end{aligned} \tag{44}$$

This representation is irreducible for the extended conformal group $\tilde{G}$ and this is a consequence, according to the general formalism, of the maximality of the full polarization subalgebra $\mathcal{P}$ in Eq. (37), i.e. $\mathcal{P}$ cannot be further enlarged nor the representation further reduced. The process of obtaining the unitary irreducible representations ends here. Any restriction desired on our wave functions should then be imposed as constraints.

We are interested, however, in null mass representations, and these can be achieved by selecting those wave functions $\psi_c^{(\beta)}$ in $\mathcal{H}(\tilde{G})$ which are nullified by the Casimir $\tilde{Q}^R \equiv (\tilde{P}_0^R)^2 - (\tilde{P}_1^R)^2$ of the Poincaré subgroup. More explicitly, wave functions which fulfil:

$$\begin{aligned}
\tilde{Q}^R\psi_c^{(\beta)} = 0 \;\; &\Rightarrow \;\; \left(\frac{(\bar{\mu}-1)^2}{(\bar{\mu}-\bar{\mu}^{-1})}\frac{\partial}{\partial\bar{\mu}} + p\beta\right)\left(\frac{(\mu-1)^2}{(\mu-\mu^{-1})}\frac{\partial}{\partial\mu} + p\beta\right)\phi(\mu,\bar{\mu}) = 0 \\
&\Rightarrow \;\; \frac{\partial\varphi(\mu,\bar{\mu})}{\partial\mu\partial\bar{\mu}} = 0\,,
\end{aligned} \tag{45}$$

where

$$\phi(\mu,\bar{\mu}) \equiv \left(\frac{(\mu-1)^2}{\mu}\frac{(\bar{\mu}-1)^2}{\bar{\mu}}\right)^{-p\beta}\varphi(\mu,\bar{\mu}). \tag{46}$$

This Klein-Gordon-like evolution equation (in a light-cone-like coordinates) is then interpreted as a constraint in the theory and leads to a new Hilbert space $\mathcal{H}_c(\tilde{G})$ made of constrained wave functions of the form:

$$\psi_c^{(\beta)} = z^p W_\beta \left(\frac{(\mu-1)^2}{\mu} \frac{(\bar{\mu}-1)^2}{\bar{\mu}} \right)^{-p\beta} (\varphi(\mu) + \bar{\varphi}(\bar{\mu})), \tag{47}$$

that is, wave functions for which the arbitrary part splits up into functions which depend on μ and $\bar{\mu}$ separately (they resemble the standard left- and right-hand moving modes). So long as this constraint is imposed by means of generators of the left translation on the group, not all the operators $\tilde{X}^R_{g^i}$ will preserve this constraint; only the ones called *good* in the general approach of Refs. [ANR, ACG] will do. One can obtain the good operators for the condition (45) by looking at the (right) commutators:

$$\begin{aligned}
\left[\tilde{D}^R, \tilde{Q}^R\right] &= -2\tilde{Q}^R \\
\left[\tilde{M}^R, \tilde{Q}^R\right] &= 0 \\
\left[\tilde{P}^R_0, \tilde{Q}^R\right] &= 0 \\
\left[\tilde{P}^R_1, \tilde{Q}^R\right] &= 0 \\
\left[\tilde{K}^R_0, \tilde{Q}^R\right] &= -4\tilde{P}^R_0\tilde{D}^R + 4\tilde{P}^R_1\tilde{M}^R - 8ip\beta\tilde{P}^R_0 \\
&= f_0(\mu,\bar{\mu})\tilde{Q}^R - 8ip\beta\tilde{P}^R_0 \\
\left[\tilde{K}^R_1, \tilde{Q}^R\right] &= -4\tilde{P}^R_1\tilde{D}^R + 4\tilde{P}^R_0\tilde{M}^R - 8ip\beta\tilde{P}^R_1 \\
&= f_1(\mu,\bar{\mu})\tilde{Q}^R - 8ip\beta\tilde{P}^R_1\,,
\end{aligned} \tag{48}$$

[$f_\nu(\mu,\bar{\mu})$ are some functions on the torus], from which we can conclude that the set of (first-order) good operators is

$$\mathcal{G}_{good} =< \tilde{D}^R, \tilde{M}^R, \tilde{P}^R_0, \tilde{P}^R_1, \tilde{X}^R_r, \tilde{X}^R_\zeta >, \tag{49}$$

and close a subalgebra (Poincare+dilatation≡Weyl) of the extended conformal Lie algebra in $1+1$ dimensions.

The fact that $\tilde{K}^R_0$ and $\tilde{K}^R_1$ are *bad* operators, i.e. they do not preserve the $\mathcal{H}_c(\tilde{G})$ Hilbert space, will be relevant in the "second quantization" of the constrained theory. The new (Weyl) vacuum will no longer be annihilated by the second quantized version of $\tilde{K}^R_0$ and $\tilde{K}^R_1$ but, rather, it will appear to be "polarized" from an accelerated frame (see Subsec. 4.2). This way, the profound reason for the rearrangement of the vacuum (under special conformal transformations) in (massless) Quantum Conformal Field Theories is not a singular action of this subgroup on the space-time but, rather, the impossibility of properly implementing these transformations in the constrained Hilbert space $\mathcal{H}_c(\tilde{G})$. Note that the combinations $A_+ \equiv \frac{1}{2}(\tilde{K}^R_0 + \tilde{K}^R_1)$ and $A_- \equiv \frac{1}{2}(\tilde{K}^R_0 - \tilde{K}^R_1)$ are "partially good", in the sense that they preserve the left- and right-hand moving modes subspaces, respectively; we shall see (Subsec. 4.2) how its finite action on a Weyl vacuum (in the second quantized theory) give rise to a thermal bath of left- and right-hand moving scalar photons, respectively.

As far as the classical field theories is concerned, the existence of a well defined scalar product does not really matter; the condition (43) can be violated by putting $\beta = 0$, thus leading to a reducible representation where the operators $\tilde{K}_0^R$ and $\tilde{K}_1^R$ leave the equation $\tilde{Q}^R\psi_c^{(\beta)} = 0$ invariant, as it can be easily checked from the two last commutators in (48). However, for this particular case, the loss of unitarity can give rise to some problems in the quantization procedure, especially concerning the definition of the field propagators in the quantum field theory (see Sec. 4). Thus, for the null mass case, the conformal symmetry is "spontaneously broken" in the sense that it is a symmetry of the classical massless field theory, whereas the corresponding quantum field theory is only invariant under the Weyl subgroup. The appearance of terms proportional to β at the right hand side of some commutators, as in (48), can be seen as an "anomaly" ; however, this time, anomaly does not means obstruction to quantization but, on the contrary, it is intrinsic to the quantization procedure and necessary for the good behaviour of the theory.

Note that for *massive* field theories the situation is slightly different. The only symmetry which survives (both for classical and quantum theories), after the constraint

$$\tilde{Q}^R\psi_c^{(\beta)} = D^{(m)}(\tilde{Q}^R)\psi_c^{(\beta)} = m^2\psi_c^{(\beta)} \tag{50}$$

is imposed, is the Poincaré subgroup. Indeed, the dilatation generator is now a bad operator (it does not preserve the constraint (50), as can be seen from the first line of (48)). Its finite action, of course being bad, is not "so bad" in the sense that it changes from one representation $D^{(m)}(\tilde{Q}^R)$ to another $D^{(m')}(\tilde{Q}^R)$ with $m' = e^{2\lambda}m$, where λ is the parameter of the transformation. That is, it plays the role of a "quantization-changing operator" (see Ref. [ACG] for other relevant examples), its domain being the union $\bigcup_{m\in\Re^+}\mathcal{H}_c^{(m)}(\tilde{G})$ of all the constrained Hilbert spaces corresponding to different masses (i.e. a theory with continuum mass spectrum).

One can look for a physical interpretation of those facts and say that "quantum conformal fields do not evolve in time". The representation (44) is irreducible for the whole conformal group, but reducible under Poincare+dilatation (Weyl) subgroup. Some external perturbation breaks the conformal symmetry and forces the fields to evolve in time and acquire a fixed value for the mass (we are interested in the massless case), so that these fields carry an irreducible representation of the Poincare(+dilatation) subgroup. In this way, the dynamical symmetry breaking and the fixing of the mass, even null, come together.

3.2 Compact dilatation subgroup

It can be proved that, for this case, a $T = U(1)$-pseudo-extension is enough to have a well defined quantum theory. It has the form:

$$\zeta'' = \zeta'\zeta e^{i\xi(g',g)} = \zeta'\zeta\left(\eta''\eta'^{-1}\eta^{-1}\bar{\eta}''\bar{\eta}'^{-1}\bar{\eta}^{-1}\right)^{-2N}, \tag{51}$$

where $\xi(g',g)$ is the two-cocycle (in fact, coboundary) generated by a multiple of $i(\log\eta+\log\bar{\eta})$, and the parameter N labels the irreducible representations and it must be quantized, taking the values

$$N = \frac{j}{2}, \quad j \in Z, \tag{52}$$

for globality conditions.

The extended left- and right-invariant vector fields on $\tilde{G}$ are:

$$
\begin{array}{ll}
\tilde{X}^L_\eta = X^L_\eta & \tilde{X}^R_\eta = X^R_\eta \\
\tilde{X}^L_\alpha = X^L_\alpha + N\alpha^*\tilde{X}^R_\zeta & \tilde{X}^R_\alpha = X^R_\alpha - N\eta^{-2}\alpha^*\tilde{X}^R_\zeta \\
\tilde{X}^L_{\alpha^*} = X^L_{\alpha^*} - N\alpha\tilde{X}^L_\zeta & \tilde{X}^R_{\alpha^*} = X^R_{\alpha^*} + N\eta^2\alpha\tilde{X}^R_\zeta
\end{array}
\tag{53}
$$

and similar expressions for the $\bar{\eta}, \bar{\alpha}, \bar{\alpha}^*$ parameters. The new commutation relations for the extended conformal Lie algebra $\tilde{\mathcal{G}}$ of $\tilde{G}$ are two copies of:

$$
\begin{aligned}
\left[\tilde{X}^L_\eta, \tilde{X}^L_\alpha\right] &= 2\tilde{X}^L_\alpha \\
\left[\tilde{X}^L_\eta, \tilde{X}^L_{\alpha^*}\right] &= -2\tilde{X}^L_{\alpha^*} \\
\left[\tilde{X}^L_\alpha, \tilde{X}^L_{\alpha^*}\right] &= \tilde{X}^L_\eta - 2N\tilde{X}^L_\zeta \\
\left[\tilde{X}^L_\zeta, \mathrm{all}\right] &= 0.
\end{aligned}
\tag{54}
$$

which, expressed in terms of the basis $\{\tilde{D}^L, \tilde{M}^L, \tilde{P}^L_0, \tilde{P}^L_1, \tilde{K}^L_0, \tilde{K}^L_1, \tilde{X}^L_\zeta\}$, lead now to

$$
\begin{aligned}
\left[\tilde{P}^L_0, \tilde{K}^L_0\right] &= -2\tilde{D}^L - 4N\tilde{X}^L_\zeta \\
\left[\tilde{P}^L_1, \tilde{K}^L_1\right] &= 2\tilde{D}^L + 4N\tilde{X}^L_\zeta
\end{aligned}
\tag{55}
$$

and the same expression as in (28) for the remainder.

The left-invariant 1-form Θ has now the form:

$$
\begin{aligned}
\Theta &= \frac{iN}{1-\alpha\alpha^*}\left(4\alpha\alpha^*\eta^{-1}d\eta + \alpha^* dc - \alpha d\alpha^*\right) \\
&+ \frac{iN}{1-\bar{\alpha}\bar{\alpha}^*}\left(4\bar{\alpha}\bar{\alpha}^*\bar{\eta}^{-1}d\bar{\eta} + \bar{\alpha}^* d\bar{\alpha} - \bar{\alpha} d\bar{\alpha}^*\right) - i\zeta^{-1}d\zeta,
\end{aligned}
\tag{56}
$$

the characteristic module $\mathcal{G}_\Theta$ and the polarization subalgebra having the same content in fields as in the previous section. The polarized $U(1)$-functions (we choose the faithful representation for $U(1)$) have now the form

$$
\begin{aligned}
\psi^{(N)}(\eta,\alpha,\alpha^*,\bar{\eta},\bar{\alpha},\bar{\alpha}^*,\zeta) &= \zeta W_N(\alpha,\alpha^*,\bar{\alpha},\bar{\alpha}^*)\phi(s,\bar{s}) \\
W_N &= w_N(\alpha,\alpha^*)w_N(\bar{\alpha},\bar{\alpha}^*) \\
w_N(\alpha,\alpha^*) &= (1-\alpha\alpha^*)^N
\end{aligned}
\tag{57}
$$

where W_N is a "generating function" and ϕ is an arbitrary power series

$$
\phi(s,\bar{s}) = \sum_{n,\bar{n}=0}^{\infty} a_{n,\bar{n}} s^n \bar{s}^{\bar{n}}
\tag{58}
$$

in the variables

$$
s = \eta^{-2}\alpha^* = \frac{z_2^*}{z_1}, \quad \bar{s} = \bar{\eta}^{-2}\bar{\alpha}^* = \frac{\bar{z}_2^*}{\bar{z}_1}.
\tag{59}
$$

Let us show how the conformal group act on $s, \bar{s}$ free from singularities. For this, let us proceed as in Eq. (41):

$$
\begin{aligned}
s &\to s'' \equiv \frac{z_2^{*\prime\prime}}{z_1{}''} = \frac{z_2^{*\prime} z_2 + z_1^{*\prime} z_1^*}{z_1{}' z_1 + z_2{}' z_2^*} = \frac{s+\alpha^{*\prime}}{\eta'^2(1+s\alpha')} \\
\bar{s} &\to \bar{s}'' \equiv \frac{\bar{z}_2^{*\prime\prime}}{\bar{z}_1{}''} = \frac{\bar{z}_2^{*\prime}\bar{z}_2 + \bar{z}_1^{*\prime}\bar{z}_1^*}{\bar{z}_1\bar{z}_1 + \bar{z}_2{}'\bar{z}_2^*} = \frac{\bar{s}+\bar{\alpha}^{*\prime}}{\bar{\eta}'^2(1+\bar{s}\bar{\alpha}')}
\end{aligned}
\tag{60}
$$

This action is always well defined and transitive on this space.

The invariant integration volume can be now chosen as $v(\tilde{g}) = -(2\pi)^{-5} v(g) \wedge (i\zeta^{-1} d\zeta)$ and the scalar product of two basic functions $\check{\psi}^{(N)}_{n,\bar{n}} \equiv \zeta W_N s^n \bar{s}^{\bar{n}}$ and $\check{\psi}^{(N)}_{m,\bar{m}} \equiv \zeta W_N s^m \bar{s}^{\bar{m}}$ is:

$$\begin{aligned}
\langle \check{\psi}^{(N)}_{n,\bar{n}} | \check{\psi}^{(N)}_{m,\bar{m}} \rangle &= \frac{n!(2N-2)!}{(2N+n-1)!} \frac{\bar{n}!(2N-2)!}{(2N+\bar{n}-1)!} \delta_{nm}\delta_{\bar{n}\bar{m}} = C^{(N)}_n C^{(N)}_{\bar{n}} \delta_{nm}\delta_{\bar{n}\bar{m}} \\
C^{(N)}_n &\equiv \frac{n!(2N-2)!}{(2N+n-1)!},
\end{aligned} \tag{61}$$

where we are assuming that $N > \frac{1}{2}$, a necessary condition for having a well defined (finite) scalar product [this condition can be relaxed to $N > 0$ by going to the universal covering group of G]. The set

$$B(\mathcal{H}_N(\tilde{G})) = \left\{ \psi^{(N)}_{n,\bar{n}} \equiv \frac{1}{\sqrt{C^{(N)}_n C^{(N)}_{\bar{n}}}} \check{\psi}^{(N)}_{n,\bar{n}} \right\} \tag{62}$$

is then orthonormal and complete, i.e. an orthonormal base of $\mathcal{H}_N(\tilde{G})$.

The actions of the right-invariant vector fields (operators in the theory) on polarized wave functions (see Eq. (57)) have the explicit form:

$$\begin{aligned}
\tilde{D}^R \psi^{(N)} &= \zeta W_N \cdot (s\frac{\partial}{\partial s} + \bar{s}\frac{\partial}{\partial \bar{s}})\phi(s,\bar{s}) \\
\tilde{M}^R \psi^{(N)} &= \zeta W_N \cdot (-s\frac{\partial}{\partial s} + \bar{s}\frac{\partial}{\partial \bar{s}})\phi(s,\bar{s}) \\
\tilde{P}^R_0 \psi^{(N)} &= \zeta W_N \cdot (-\frac{\partial}{\partial s} - \frac{\partial}{\partial \bar{s}})\phi(s,\bar{s}) \\
\tilde{P}^R_1 \psi^{(N)} &= \zeta W_N \cdot (\frac{\partial}{\partial s} - \frac{\partial}{\partial \bar{s}})\phi(s,\bar{s}) \\
\tilde{K}^R_0 \psi^{(N)} &= \zeta W_N \cdot (-s^2\frac{\partial}{\partial s} - \bar{s}^2\frac{\partial}{\partial \bar{s}} - 2N(s+\bar{s}))\phi(s,\bar{s}) \\
\tilde{K}^R_1 \psi^{(N)} &= \zeta W_N \cdot (-s^2\frac{\partial}{\partial s} + \bar{s}^2\frac{\partial}{\partial \bar{s}} - 2N(s-\bar{s}))\phi(s,\bar{s}) \\
\tilde{X}^L_\zeta \psi^{(N)} &= \psi^{(N)}.
\end{aligned} \tag{63}$$

The finite (left) action (13) of an arbitrary element $\tilde{g}' = (\eta', \alpha', \alpha^{*\prime}, \bar{\eta}', \bar{\alpha}', \bar{\alpha}^{*\prime}, \zeta') \in \tilde{G}$ on an arbitrary wave function

$$\psi^{(N)}(\tilde{g}) = \sum_{n,\bar{n}=0}^{\infty} a_{n,\bar{n}} \psi^{(N)}_{n,\bar{n}}(\tilde{g}), \tag{64}$$

can be given through the matrix elements $\rho^{(N)}_{mn;\bar{m}\bar{n}}(\tilde{g}') \equiv \langle \psi^{(N)}_{m,\bar{m}} | \rho(\tilde{g}') | \psi^{(N)}_{n,\bar{n}} \rangle$ of ρ in the base $B(\mathcal{H}_N(\tilde{G}))$. They have the following expression:

$$\begin{aligned}
\rho^{(N)}_{mn;\bar{m}\bar{n}}(\tilde{g}) &= \zeta^{-1} \rho^{(N)}_{mn}(\eta,\alpha,\alpha^*) \rho^{(N)}_{\bar{m}\bar{n}}(\bar{\eta},\bar{\alpha},\bar{\alpha}^*) \\
\rho^{(N)}_{mn}(\eta,\alpha,\alpha^*) &= \sqrt{\frac{C^{(N)}_m}{C^{(N)}_n}} \sum_{l=\theta_{nm}}^{n} \begin{pmatrix} n \\ l \end{pmatrix} \begin{pmatrix} 2N+m+l-1 \\ m-n+l \end{pmatrix} \times \\
& \quad (-1)^l \eta^{2m} \alpha^{*l} \alpha^{m-n+l} (1-\alpha\alpha^*)^N
\end{aligned} \tag{65}$$

where the function θ_{nm} in the lower limit of the last summatory is defined by $\theta_{nm} \equiv (n - m)\frac{\mathrm{sign}(n-m)+1}{2}$, the function sign$(n)$ being the standard sign function (sign$(0) = 1$); it guarantees the following inequality $m - n + l \geq 0$. These expressions will be very useful for the construction of the corresponding quantum field theory in the next section.

The constrained wave functions $\psi_c^{(N)}$ obeying

$$\tilde{Q}^R \psi_c^{(N)} = ((\tilde{P}_0^R)^2 - (\tilde{P}_1^R)^2)\psi_c^{(N)} = 0 \Rightarrow \frac{\partial^2 \phi}{\partial s \partial \bar{s}} = 0 \tag{66}$$

have now the form

$$\psi_c^{(N)} = \zeta W_N \cdot (\varphi(s) + \bar{\varphi}(\bar{s})). \tag{67}$$

We arrive at the same conclusions as in the non-compact dilatation case, concerning good and bad operators. For this case, N plays the same role as β did in the former.

Let us investigate the conformal quantum field theory associated with this "first quantized" theory and how to interpret the dynamical symmetry breaking of the conformal group in the context of the corresponding "second quantized" theory. To this end, let us show how this second quantization approach can be discussed within the GAQ framework.

4 "Second Quantization" on a group $\tilde{G}$: a model for a conformally invariant QFT

In this subsection we shall develop a general approach to the quantization of linear, complex quantum fields defined on a group manifold $\tilde{G}$ (more precisely, on the quotient $\tilde{G}/(T \cup \mathcal{P})$). This formalism can be seen as a "second quantization" of a "first quantized" theory defined by a group $\tilde{G}$ and a Hilbert space $\mathcal{H}(\tilde{G})$ of polarized wave functions.

The construction of the quantizing group $\tilde{G}^{(2)}$ for this complex quantum field is as follows. Given a Hilbert space $\mathcal{H}(\tilde{G})$ and its dual $\mathcal{H}^*(\tilde{G})$, we define the direct sum

$$\begin{aligned} \mathcal{F}(\tilde{G}) &\equiv \mathcal{H}(\tilde{G}) \oplus \mathcal{H}^*(\tilde{G}) \\ &= \left\{ |f\rangle = |A\rangle + |B^*\rangle;\ |A\rangle \in \mathcal{H}(\tilde{G}),\, |B^*\rangle \in \mathcal{H}^*(\tilde{G}) \right\}, \end{aligned} \tag{68}$$

where we have denoted $|B^*\rangle$ according to $\langle \tilde{g}^*_{\mathcal{P}} | B^* \rangle \equiv \langle B | \tilde{g}_{\mathcal{P}} \rangle = B^*_{\mathcal{P}}(\tilde{g})$. The group $\tilde{G}$ acts on this vectorial space as follows:

$$\rho(\tilde{g}')|f\rangle = \rho(\tilde{g}')|A\rangle + \rho(\tilde{g}')|B^*\rangle\,, \tag{69}$$

where

$$\langle \tilde{g}^*_{\mathcal{P}} | \rho(\tilde{g}') | B^* \rangle \equiv \langle B | \rho^\dagger(\tilde{g}') | \tilde{g}_{\mathcal{P}} \rangle = B^*_{\mathcal{P}}(\tilde{g}'^{-1} * \tilde{g}). \tag{70}$$

We can also define the dual space

$$\begin{aligned} \mathcal{F}^*(\tilde{G}) &\equiv \mathcal{H}^*(\tilde{G}) \oplus \mathcal{H}^{**}(\tilde{G}) \\ &= \left\{ \langle f| = \langle A| + \langle B^*|\,;\ \ \langle A| \in \mathcal{H}^*(\tilde{G}),\, \langle B^*| \in \mathcal{H}^{**}(\tilde{G}) \sim \mathcal{H}(\tilde{G}) \right\}, \end{aligned} \tag{71}$$

where $\tilde{G}$ acts according to the adjoint action

$$\langle f|\rho^\dagger(\tilde{g}') = \langle A|\rho^\dagger(\tilde{g}') + \langle B^*|\rho^\dagger(\tilde{g}') \tag{72}$$

and now

$$\langle B^*|\rho^\dagger(\tilde{g}')|\tilde{g}^*_{\mathcal{P}}\rangle \equiv \langle \tilde{g}_{\mathcal{P}}|\rho(\tilde{g}')|B\rangle. \tag{73}$$

Using the closure relation (12), the product of two arbitrary elements of $\mathcal{F}(\tilde{G})$ is

$$\langle f'|f\rangle = \langle A'|A\rangle + \overbrace{\langle A'|B^*\rangle}^{0} + \overbrace{\langle B'^*|A\rangle}^{0} + \langle B'^*|B^*\rangle\,, \tag{74}$$

indeed, the second and third integrals

$$\int_{\tilde{G}} v(\tilde{g}) A'_{\mathcal{P}}{}^*(\tilde{g}) B^*_{\mathcal{P}}(\tilde{g}) = 0 = \int_{\tilde{G}} v(\tilde{g}) B'_{\mathcal{P}}(\tilde{g}) A_{\mathcal{P}}(\tilde{g}) \tag{75}$$

are zero because of the integration on the central parameter $\zeta \in U(1)$. Thus, the subspaces $\mathcal{H}(\tilde{G})$ and $\mathcal{H}^*(\tilde{G})$ are orthogonal with respect to this scalar product in $\mathcal{F}(\tilde{G})$. A basis for $\mathcal{F}(\tilde{G})$ is provided by the set $\{|n\rangle + |m^*\rangle\}_{n,m\in I}$.

The space $\mathcal{M}(\tilde{G}) \equiv \mathcal{F}(\tilde{G}) \otimes \mathcal{F}^*(\tilde{G})$ can be endowed with a simplectic structure

$$S(f', f) \equiv \frac{-i}{2}(\langle f'|f\rangle - \langle f|f'\rangle)\,, \tag{76}$$

thus defining $\mathcal{M}(\tilde{G})$ as a phase space. This phase space can be naturally embedded into a quantizing group

$$\tilde{G}^{(2)} \equiv \left\{ \tilde{g}^{(2)} = (g^{(2)}; \varsigma) \equiv (\tilde{g}, |f\rangle, \langle f|; \varsigma) \right\}\,, \tag{77}$$

which is a (true) central extension by $U(1)$, with parameter ς, of the semidirect product $G^{(2)} \equiv \tilde{G} \otimes_\rho \mathcal{M}(\tilde{G})$ of the basic group $\tilde{G}$ and the phase space $\mathcal{M}(\tilde{G})$. The group law of $\tilde{G}^{(2)}$ is formally:

$$\begin{aligned} \tilde{g}'' &= \tilde{g}' * \tilde{g} \\ |f''\rangle &= |f'\rangle + \rho(\tilde{g}')|f\rangle \\ \langle f''| &= \langle f'| + \langle f|\rho^\dagger(\tilde{g}') \\ \varsigma'' &= \varsigma'\varsigma e^{i\xi^{(2)}(g^{(2)\prime}, g^{(2)})}\,, \end{aligned} \tag{78}$$

where $\xi^{(2)}(g^{(2)\prime}, g^{(2)})$ is a two-cocycle defined as

$$\xi^{(2)}(g^{(2)\prime}, g^{(2)}) \equiv \kappa S(f', \rho(\tilde{g}')f) \tag{79}$$

and κ is intended to kill any possible dimension of S.

A system of coordinates for $\tilde{G}^{(2)}$ corresponds to a choice of representation associated with a given polarization $\mathcal{P}$

$$\begin{array}{ll} f^{(+)}_{\mathcal{P}}(\tilde{g}) \equiv \langle \tilde{g}_{\mathcal{P}}|f\rangle\,, & f^{(-)}_{\mathcal{P}}(\tilde{g}) \equiv \langle \tilde{g}^*_{\mathcal{P}}|f\rangle\,, \\ f^{*(+)}_{\mathcal{P}}(\tilde{g}) \equiv \langle f|\tilde{g}^*_{\mathcal{P}}\rangle\,, & f^{*(-)}_{\mathcal{P}}(\tilde{g}) \equiv \langle f|\tilde{g}_{\mathcal{P}}\rangle\,. \end{array} \tag{80}$$

This splitting of f is the group generalization of the more standard decomposition of a field in positive and negative frequency parts. If we make use of the closure relation $1 = \int_{\tilde{G}} v(\tilde{g})\{|\tilde{g}_{\mathcal{P}}\rangle\langle\tilde{g}_{\mathcal{P}}| + |\tilde{g}^*_{\mathcal{P}}\rangle\langle\tilde{g}^*_{\mathcal{P}}|\}$ for $\mathcal{F}(\tilde{G})$, the explicit expression of the cocycle (79) in this coordinate system (for simplicity, we discard the semidirect action of $\tilde{G}$),

$$\begin{aligned} \xi^{(2)}(g^{(2)\prime}, g^{(2)}) &= \frac{-i\kappa}{2} \iint_{\tilde{G}} v(\tilde{g}')v(\tilde{g}) \Big\{ f'^{*(-)}_{\mathcal{P}}(\tilde{g}')\Delta^{(+)}_{\mathcal{P}}(\tilde{g}', \tilde{g}) f^{(+)}_{\mathcal{P}}(\tilde{g}) \\ &- f^{*(-)}_{\mathcal{P}}(\tilde{g}')\Delta^{(+)}_{\mathcal{P}}(\tilde{g}', \tilde{g}) f'^{(+)}_{\mathcal{P}}(\tilde{g}) + f'^{*(+)}_{\mathcal{P}}(\tilde{g}')\Delta^{(-)}_{\mathcal{P}}(\tilde{g}', \tilde{g}) f^{(-)}_{\mathcal{P}}(\tilde{g}) \\ &- f^{*(+)}_{\mathcal{P}}(\tilde{g}')\Delta^{(-)}_{\mathcal{P}}(\tilde{g}', \tilde{g}) f'^{(-)}_{\mathcal{P}}(\tilde{g}) \Big\}\,, \end{aligned} \tag{81}$$

where

$$\begin{aligned}
\Delta^{(+)}_{\mathcal{P}}(\tilde g',\tilde g) &\equiv \langle \tilde g'_{\mathcal{P}}|\tilde g_{\mathcal{P}}\rangle = \sum_{n\in I}\psi_{\mathcal{P},n}(\tilde g')\psi^*_{\mathcal{P},n}(\tilde g)\,,\\
\Delta^{(-)}_{\mathcal{P}}(\tilde g',\tilde g) &\equiv \langle \tilde g'^*_{\mathcal{P}}|\tilde g^*_{\mathcal{P}}\rangle = \Delta^{(+)}_{\mathcal{P}}(\tilde g,\tilde g')\,,
\end{aligned} \tag{82}$$

shows that the vector fields associated with the co-ordinates in (80) are canonically conjugated

$$\begin{aligned}
\left[\tilde X^L_{f^{*(-)}_{\mathcal{P}}(\tilde g')},\tilde X^L_{f^{(+)}_{\mathcal{P}}(\tilde g)}\right] &= \kappa\Delta^{(+)}_{\mathcal{P}}(\tilde g',\tilde g)\tilde X^L_{\varsigma}\,,\\
\left[\tilde X^L_{f^{*(+)}_{\mathcal{P}}(\tilde g')},\tilde X^L_{f^{(-)}_{\mathcal{P}}(\tilde g)}\right] &= \kappa\Delta^{(-)}_{\mathcal{P}}(\tilde g',\tilde g)\tilde X^L_{\varsigma}\,.
\end{aligned} \tag{83}$$

Here, the functions $\Delta^{(\pm)}_{\mathcal{P}}(\tilde g',\tilde g)$ play the role of *propagators* (central matrices of the cocycle). At this point, we must stress the importance of a well defined scalar product in $\mathcal{H}(\tilde G)$ as regards the good behaviour of the two-cocycle (81), an essential ingredient in the corresponding QFT. The non-zero value of the central extension parameter of $\tilde G$ –see Eq. (43,52) and comments after Eq. (61)– which prevents the whole conformal group from being an exact symmetry of the massless *quantum* field theory (remember the comments after Eq. (48)) proves to be an essential prerequisite for a proper definition of the conformal *quantum* field theory through the group $\tilde G^{(2)}$.

The propagators in two different parametrizations of $\tilde G^{(2)}$, corresponding to two different polarization subalgebras $\mathcal{P}_1$ and $\mathcal{P}_2$ of $\tilde{\mathcal{G}}^L$ (or $U\tilde{\mathcal{G}}^L$), are related through polarization-changing operators (16) as follows:

$$\begin{aligned}
\Delta^{(\pm)}_{\mathcal{P}_2}(\tilde h',\tilde h) &= \iint_{\tilde G} v(\tilde g')v(\tilde g)\Delta^{(\pm)}_{\mathcal{P}_2\mathcal{P}_1}(\tilde h',\tilde g')\Delta^{(\pm)}_{\mathcal{P}_1}(\tilde g',\tilde g)\Delta^{(\pm)}_{\mathcal{P}_1\mathcal{P}_2}(\tilde g,\tilde h)\\
\Delta^{(+)}_{\mathcal{P}_i\mathcal{P}_j}(\tilde h,\tilde g) &\equiv \Delta_{\mathcal{P}_i\mathcal{P}_j}(\tilde h,\tilde g)\,,\quad \Delta^{(-)}_{\mathcal{P}_i\mathcal{P}_j}(\tilde h,\tilde g)\equiv\Delta_{\mathcal{P}_i\mathcal{P}_j}(\tilde g,\tilde h)\,.
\end{aligned} \tag{84}$$

To apply the GAQ formalism to $\tilde G^{(2)}$, it is appropriate to use a "Fourier-like" parametrization, alternative to the field-like parametrization above [see (80)]. If we denote by

$$\begin{aligned}
a_n \equiv \langle n|f\rangle\,,\quad b_n\equiv\langle n^*|f\rangle\,,\\
a^*_n \equiv \langle f|n\rangle\,,\quad b^*_n\equiv\langle f|n^*\rangle\,,
\end{aligned} \tag{85}$$

the Fourier coefficients of the "particle" and the "antiparticle", a polarization subalgebra $\mathcal{P}^{(2)}$ for $\tilde G^{(2)}$ is always given by the corresponding left-invariant vector fields $\tilde X^L_{a_n}, \tilde X^L_{b_n}$ and the whole Lie algebra $\tilde{\mathcal{G}}^L$ of $\tilde G$, which is the characteristic subalgebra $\mathcal{G}_{\Theta^{(2)}}$ of the second-quantized theory (see next subsection). The operators of the theory are the right-invariant vector fields of $\tilde G^{(2)}$; in particular, the basic operators are: the annihilation operators of particles and anti-particles, $\hat a_n \equiv \tilde X^R_{a^*_n}$, $\hat b_n \equiv \tilde X^R_{b^*_n}$, and the corresponding creation operators $\hat a^\dagger_n \equiv -\frac{1}{\kappa}\tilde X^R_{a_n}$, $\hat b^\dagger_n \equiv -\frac{1}{\kappa}\tilde X^R_{b_n}$. The operators corresponding to the subgroup $\tilde G$ [the second-quantized version $\tilde X^{R(2)}_{\tilde g^j}$ of the first-quantized operators $\tilde X^R_{\tilde g^j}$ in (6)] are written in terms of the basic ones, since they are in the characteristic subalgebra $\mathcal{G}_{\Theta^{(2)}}$ of the second-quantized theory.

The group $\tilde G$ plays a key role in picking out a preferred vacuum state and defining the notion of a "particle", in the same way as the Poincaré group plays a central role in relativistic quantum theories defined on Minkowski space. In general, standard QFT in curved space suffers from the lack of a preferred definition of particles. The infinite-dimensional character of the symplectic solution manifold of a field system is responsible for the existence of an infinite number of unitarily inequivalent irreducible representations of the Heisenberg-Weyl (H-W) relations and there is no criterion to select a preferred vacuum of the corresponding quantum field (see, for example, [W, BD]). This situation is not present in the finite-dimensional case, according to the Stone-von Newman theorem [St, N]. In our language, the origen of this fact is related to the lack of a characteristic module for the H-W subgroup $\tilde{G}^{(2)}/\tilde{G}$ of $\tilde{G}^{(2)}$; i.e., for the infinite-dimensional H-W group, we can polarize the wave functions in arbitrary, non-equivalent directions. Thus, so long as we can embed the (curved) space M into a given group $\tilde{G}$, the existence of a characteristic module -generated by $\tilde{\mathcal{G}}^L$- in the polarization subalgebra helps us in picking out a preferred vacuum state. This vacuum state will be characterized by being annihilated by the right version of the polarization subalgabra dual to $\mathcal{P}^{(2)}$, i.e, it will be invariant under the action of $\tilde{G} \subset \tilde{G}^{(2)}$ and annihilated by the right-invariant vector fields $\tilde{X}^R_{a^*_n}, \tilde{X}^R_{b^*_n}$.

Other vacuum states might be selected as those states being invariant under a subgroup $\tilde{G}_Q \subset \tilde{G}$ only, for example, the uniparametric subgroup of time evolution (see e.g. [A] for a discussion of vacuum states in de Sitter space). From our point of view, this situation would correspond to a breakdown of the symmetry and could be understood as a constrained theory of the original one. Indeed, let us comment on the influence of the constraints in the first quantized theory at the second quantization level. Associated with a constrained wave function satisfying (18), there is a corresponding constrained quantum field subjected to the condition:

$$\mathrm{ad}_{\tilde{X}^{R(2)}_{\tilde{t}}}\left(\tilde{X}^R_{|f\rangle}\right) \equiv \left[\tilde{X}^{R(2)}_{\tilde{t}}, \tilde{X}^R_{|f\rangle}\right] = dD^{(\epsilon)}_T(\tilde{t})\tilde{X}^R_{|f\rangle}\,, \tag{86}$$

where $\tilde{X}^{R(2)}_{\tilde{t}}$ stands for the “second-quantized version” of $\tilde{X}^R_{\tilde{t}}$. It is straightforward to generalize the last condition to higher-order constraints:

$$\begin{aligned} \tilde{X}^R_1\tilde{X}^R_2...\tilde{X}^R_j|\psi\rangle &= \epsilon|\psi\rangle \rightarrow \\ \mathrm{ad}_{\tilde{X}^{R(2)}_1}\left(\mathrm{ad}_{\tilde{X}^{R(2)}_2}\left(...\mathrm{ad}_{\tilde{X}^{R(2)}_j}\left(\tilde{X}^R_{|f\rangle}\right)...\right)\right) &= \epsilon\tilde{X}^R_{|f\rangle}\,. \end{aligned} \tag{87}$$

The selection of a given Hilbert subspace $\mathcal{H}^{(\epsilon)}(\tilde{G}) \subset \mathcal{H}(\tilde{G})$ made of wave functions ψ_c obeing a higher-order constraint $Q\psi_c = \epsilon\psi_c$, where $Q = \tilde{X}^R_1\tilde{X}^R_2...\tilde{X}^R_j$ is some Casimir operator of $\tilde{G}_Q \subset \tilde{G}$, manifests, at second quantization level, as a new (broken) QFT. The vacuum for the new observables of this broken theory (the good operators in (87)) does not have to coincide with the vacuum of the original theory, and the action of the rest of the operators (the bad operators) could make this new vacuum radiate. This is precisely the problem we are involved, where $Q \equiv \tilde{Q}^R$ is the Casimir of the Poincaré subgroup inside the conformal group (see later in Sec. 4.2).

In general, constraints lead to gauge symmetries in the constrained theory and, also, the property for a subgroup $N \subset \tilde{G}$ of being gauge is heritable at the second-quantization level.

To conclude this subsection, it is important to note that the representation of $\tilde{G}$ on $\mathcal{M}(\tilde{G})$ is reducible, but it is irreducible under $\tilde{G}$ together with the *charge conjugation* operation $a_n \leftrightarrow b_n$, which could be implemented on $\tilde{G}^{(2)}$. For simplicity, we have preferred to discard this transformation; however, a treatment including it, would be relevant as a revision of the CPT symmetry in quantum field theory. The Noether invariant associated with $\tilde{X}^{R(2)}_\zeta$ is nothing other than the *total electric charge* (the total number of particles in the case of a real field $b_n \equiv a_n$)

and its central character, inside the "dynamical" group $\tilde{G}$ of the first-quantized theory, now ensures its conservation under the action of the subgroup $\tilde{G} \subset \tilde{G}^{(2)}$. To account for non-abelian charges (iso-spin, color, etc), a non-abelian structure group $T \subset \tilde{G}$ is required.

4.1 The case of the conformal group

Let us now apply the GAQ formalism to the centrally extended group $\tilde{G}^{(2)}$ given through the group law in (78) for the case of $\tilde{G} = SO(2,2)$ and compact dilatation. We shall consider the case of a real field and we shall use a "Fourier" parametrization in terms of the coefficients $a_{n,\bar{n}}$ rather than a "field" parametrization in terms of $f_{\mathcal{P}}(\tilde{g})$. The explicit group law is:

$$
\begin{aligned}
\tilde{g}'' &= \tilde{g}' * \tilde{g} \\
{a_{m,\bar{m}}}'' &= {a_{m,\bar{m}}}' + \sum_{n,\bar{n}=0}^{\infty} \rho^{(N)}_{mn;\bar{m}\bar{n}}(\tilde{g}') a_{n,\bar{n}} \\
{a^*_{m,\bar{m}}}'' &= {a^*_{m,\bar{m}}}' + \sum_{n,\bar{n}=0}^{\infty} \rho^{(N)*}_{mn;\bar{m}\bar{n}}(\tilde{g}') a^*_{n,\bar{n}} \\
\varsigma'' &= \varsigma'\varsigma \exp \frac{\kappa}{2} \sum_{m,\bar{m}=0}^{\infty} \sum_{n,\bar{n}=0}^{\infty} ({a^*_{m,\bar{m}}}' \rho^{(N)}_{mn;\bar{m}\bar{n}}(\tilde{g}') a_{n,\bar{n}} - {a_{m,\bar{m}}}' \rho^{(N)*}_{mn;\bar{m}\bar{n}}(\tilde{g}') a^*_{n,\bar{n}})\,.
\end{aligned}
\tag{88}
$$

The left- and right-invariant vector fields (we denote $\partial_{m,\bar{m}} \equiv \frac{\partial}{\partial a_{m,\bar{m}}}$, $\partial^*_{m,\bar{m}} \equiv \frac{\partial}{\partial a^*_{m,\bar{m}}}$) are:

$$
\begin{aligned}
\tilde{X}^L_{\varsigma} &= \tilde{X}^R_{\varsigma} = \varsigma \frac{\partial}{\partial \varsigma} \\
\tilde{X}^L_{a_{n,\bar{n}}} &= \sum_{m,\bar{m}=0}^{\infty} \rho^{(N)}_{mn;\bar{m}\bar{n}}(\tilde{g}) \partial_{m,\bar{m}} + \frac{\kappa}{2} \sum_{m,\bar{m}=0}^{\infty} \rho^{(N)}_{mn;\bar{m}\bar{n}}(\tilde{g}) a^*_{m,\bar{m}} \tilde{X}^L_{\varsigma} \\
\tilde{X}^L_{a^*_{n,\bar{n}}} &= \sum_{m,\bar{m}=0}^{\infty} \rho^{(N)*}_{mn;\bar{m}\bar{n}}(\tilde{g}) \partial^*_{m,\bar{m}} - \frac{\kappa}{2} \sum_{m,\bar{m}=0}^{\infty} \rho^{(N)*}_{mn;\bar{m}\bar{n}}(\tilde{g}) a_{m,\bar{m}} \tilde{X}^L_{\varsigma} \\
\tilde{D}^{L(2)} &= \tilde{D}^L,\ \tilde{M}^{L(2)} = \tilde{M}^L,\ \tilde{P}_0^{L(2)} = \tilde{P}_0^L, \\
\tilde{P}_1^{L(2)} &= \tilde{P}_1^L,\ \tilde{K}_0^{L(2)} = \tilde{K}_0^L,\ \tilde{K}_1^{L(2)} = \tilde{K}_1^L,\ \tilde{X}_{\zeta}^{L(2)} = \tilde{X}^L_{\zeta} \\
\tilde{X}^R_{a_{n,\bar{n}}} &= \partial_{n,\bar{n}} - \frac{\kappa}{2} a^*_{n,\bar{n}} \tilde{X}^L_{\varsigma} \\
\tilde{X}^R_{a^*_{n,\bar{n}}} &= \partial^*_{n,\bar{n}} + \frac{\kappa}{2} a_{n,\bar{n}} \tilde{X}^L_{\varsigma} \\
\tilde{D}^{R(2)} &= \tilde{D}^R - \sum_{m,\bar{m}=0}^{\infty} (m+\bar{m})(a_{m,\bar{m}}\partial_{m,\bar{m}} - a^*_{m,\bar{m}}\partial^*_{m,\bar{m}}) \\
\tilde{M}^{R(2)} &= \tilde{M}^{R(2)} + \sum_{m,\bar{m}=0}^{\infty} (m-\bar{m})(a_{m,\bar{m}}\partial_{m,\bar{m}} - a^*_{m,\bar{m}}\partial^*_{m,\bar{m}}) \\
\tilde{P}_0^{R(2)} &= \tilde{P}_0^R + \sum_{m,\bar{m}=0}^{\infty} \Big(\sqrt{(m+1)(2N+m)}(a_{m+1,\bar{m}}\partial_{m,\bar{m}} - a^*_{m,\bar{m}}\partial^*_{m+1,\bar{m}}) \\
&+ \sqrt{(\bar{m}+1)(2N+\bar{m})}(a_{m,\bar{m}+1}\partial_{m,\bar{m}} - a^*_{m,\bar{m}}\partial^*_{m,\bar{m}+1}) \Big)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{P}_1^{R(2)} &= \tilde{P}_1^R - \sum_{m,\bar{m}=0}^{\infty} \Big(\sqrt{(m+1)(2N+m)}(a_{m+1,\bar{m}}\partial_{m,\bar{m}} - a^*_{m,\bar{m}}\partial^*_{m+1,\bar{m}}) \\
&- \sqrt{(\bar{m}+1)(2N+\bar{m})}(a_{m,\bar{m}+1}\partial_{m,\bar{m}} - a^*_{m,\bar{m}}\partial^*_{m,\bar{m}+1}) \Big) \\
\tilde{K}_0^{R(2)} &= \tilde{K}_0^R + \sum_{m,\bar{m}=0}^{\infty} \Big(\sqrt{(m+1)(2N+m)}(a_{m,\bar{m}}\partial_{m+1,\bar{m}} - a^*_{m+1,\bar{m}}\partial^*_{m,\bar{m}}) \\
&+ \sqrt{(\bar{m}+1)(2N+\bar{m})}(a_{m,\bar{m}}\partial_{m,\bar{m}+1} - a^*_{m,\bar{m}+1}\partial^*_{m,\bar{m}}) \Big) \\
\tilde{K}_1^{R(2)} &= \tilde{K}_1^R + \sum_{m,\bar{m}=0}^{\infty} \Big(\sqrt{(m+1)(2N+m)}(a_{m,\bar{m}}\partial_{m+1,\bar{m}} - a^*_{m+1,\bar{m}}\partial^*_{m,\bar{m}}) \\
&- \sqrt{(\bar{m}+1)(2N+\bar{m})}(a_{m,\bar{m}}\partial_{m,\bar{m}+1} - a^*_{m,\bar{m}+1}\partial^*_{m,\bar{m}}) \Big) \\
\tilde{X}_\zeta^{R(2)} &= \tilde{X}_\zeta^R - \sum_{m,\bar{m}=0}^{\infty} (a_{m,\bar{m}}\partial_{m,\bar{m}} - a^*_{m,\bar{m}}\partial^*_{m,\bar{m}}).
\end{aligned}
\tag{89}
$$

The non-trivial commutators between those vector fields are:

$$
\begin{aligned}
\left[\tilde{X}^L_{a_{n,\bar{n}}}, \tilde{X}^L_{a^*_{m,\bar{m}}}\right] &= -\kappa\delta_{nm}\delta_{\bar{n}\bar{m}}\tilde{X}^L_\varsigma \\
\left[\tilde{D}^{L(2)}, \tilde{X}^L_{a_{n,\bar{n}}}\right] &= -(n+\bar{n})\tilde{X}^L_{a_{n,\bar{n}}} \\
\left[\tilde{M}^{L(2)}, \tilde{X}^L_{a_{n,\bar{n}}}\right] &= (n-\bar{n})\tilde{X}^L_{a_{n,\bar{n}}} \\
\left[\tilde{P}_0^{L(2)}, \tilde{X}^L_{a_{n,\bar{n}}}\right] &= \sqrt{n(2N+n-1)}\tilde{X}^L_{a_{n-1,\bar{n}}} + \sqrt{\bar{n}(2N+\bar{n}-1)}\tilde{X}^L_{a_{n,\bar{n}-1}} \\
\left[\tilde{P}_1^{L(2)}, \tilde{X}^L_{a_{n,\bar{n}}}\right] &= -\sqrt{n(2N+n-1)}\tilde{X}^L_{a_{n-1,\bar{n}}} + \sqrt{\bar{n}(2N+\bar{n}-1)}\tilde{X}^L_{a_{n,\bar{n}-1}} \\
\left[\tilde{K}_0^{L(2)}, \tilde{X}^L_{a_{n,\bar{n}}}\right] &= \sqrt{(n+1)(2N+n)}\tilde{X}^L_{a_{n+1,\bar{n}}} + \sqrt{(\bar{n}+1)(2N+\bar{n})}\tilde{X}^L_{a_{n,\bar{n}+1}} \\
\left[\tilde{K}_1^{L(2)}, \tilde{X}^L_{a_{n,\bar{n}}}\right] &= \sqrt{(n+1)(2N+n)}\tilde{X}^L_{a_{n+1,\bar{n}}} - \sqrt{(\bar{n}+1)(2N+\bar{n})}\tilde{X}^L_{a_{n,\bar{n}+1}} \\
\left[\tilde{X}_\zeta^{L(2)}, \tilde{X}^L_{a_{n,\bar{n}}}\right] &= \tilde{X}^L_{a_{n,\bar{n}}} \\
\left[\tilde{D}^{L(2)}, \tilde{X}^L_{a^*_{n,\bar{n}}}\right] &= (n+\bar{n})\tilde{X}^L_{a^*_{n,\bar{n}}} \\
\left[\tilde{M}^{L(2)}, \tilde{X}^L_{a^*_{n,\bar{n}}}\right] &= -(n-\bar{n})\tilde{X}^L_{a^*_{n,\bar{n}}} \\
\left[\tilde{P}_0^{L(2)}, \tilde{X}^L_{a^*_{n,\bar{n}}}\right] &= -\sqrt{(n+1)(2N+n)}\tilde{X}^L_{a^*_{n+1,\bar{n}}} - \sqrt{(\bar{n}+1)(2N+\bar{n})}\tilde{X}^L_{a^*_{n,\bar{n}+1}} \\
\left[\tilde{P}_1^{L(2)}, \tilde{X}^L_{a^*_{n,\bar{n}}}\right] &= \sqrt{(n+1)(2N+n)}\tilde{X}^L_{a^*_{n+1,\bar{n}}} - \sqrt{(\bar{n}+1)(2N+\bar{n})}\tilde{X}^L_{a^*_{n,\bar{n}+1}} \\
\left[\tilde{K}_0^{L(2)}, \tilde{X}^L_{a^*_{n,\bar{n}}}\right] &= -\sqrt{n(2N+n-1)}\tilde{X}^L_{a^*_{n-1,\bar{n}}} - \sqrt{\bar{n}(2N+\bar{n}-1)}\tilde{X}^L_{a^*_{n,\bar{n}-1}} \\
\left[\tilde{K}_1^{L(2)}, \tilde{X}^L_{a^*_{n,\bar{n}}}\right] &= -\sqrt{n(2N+n-1)}\tilde{X}^L_{a^*_{n-1,\bar{n}}} + \sqrt{\bar{n}(2N+\bar{n}-1)}\tilde{X}^L_{a^*_{n,\bar{n}-1}} \\
\left[\tilde{X}_\zeta^{L(2)}, \tilde{X}^L_{a^*_{n,\bar{n}}}\right] &= -\tilde{X}^L_{a^*_{n,\bar{n}}}\,,
\end{aligned}
\tag{90}
$$

where we have omitted the commutators corresponding to the extended conformal subgroup, which have the same form as in (28), except for the two commutators in (55).

The quantization 1-form and the characteristic module are:

$$\begin{aligned}
\Theta^{(2)} &= \frac{i\kappa}{2}\sum_{n,\bar{n}=0}^{\infty}(a_{n,\bar{n}}da^*_{n,\bar{n}} - a^*_{n,\bar{n}}da_{n,\bar{n}}) - i\varsigma^{-1}d\varsigma \\
\mathcal{G}_{\Theta^{(2)}} &= <\tilde{D}^{L(2)}, \tilde{M}^{L(2)}, \tilde{P}_0^{L(2)}, \tilde{P}_1^{L(2)}, \tilde{K}_0^{L(2)}, \tilde{K}_1^{L(2)}, \tilde{X}_\zeta^{L(2)} > \;. \qquad (91)
\end{aligned}$$

A full polarization subalgebra is:

$$\mathcal{P}^{(2)} =< \mathcal{G}_{\Theta^{(2)}}, \tilde{X}^L_{a_{n,\bar{n}}} >, \;\; \forall n, \bar{n} \geq 0 \qquad (92)$$

and the polarized $U(1)$-functions have the form:

$$\Psi[a, a^*, \tilde{g}, \varsigma] = \varsigma \exp\left\{-\frac{\kappa}{2}\sum_{n,\bar{n}=0}^{\infty} a^*_{n,\bar{n}}a_{n,\bar{n}}\right\}\Phi[a^*] \equiv \varsigma\Omega\Phi[a^*]\,, \qquad (93)$$

where Ω is the vacuum of the second quantized theory and Φ is an arbitrary power series in its argument.

The actions of the right-invariant vector fields (operators in the second-quantized theory) on polarized wave functions in (93) have the explicit form:

$$\begin{aligned}
\tilde{X}^R_{a_{n,\bar{n}}}\Psi &= \varsigma\Omega\cdot(-\kappa a^*_{n,\bar{n}})\Phi \equiv \varsigma\Omega\cdot(-\kappa\hat{a}^\dagger_{n,\bar{n}})\Phi \\
\tilde{X}^R_{a^*_{n,\bar{n}}}\Psi &= \varsigma\Omega\cdot(\partial^*_{n,\bar{n}})\Phi \equiv \varsigma\Omega\cdot(\hat{a}_{n,\bar{n}})\Phi \\
\tilde{D}^{R(2)}\Psi &= \varsigma\Omega\cdot\left(\sum_{n,\bar{n}=0}^{\infty}(n+\bar{n})\hat{a}^\dagger_{n,\bar{n}}\hat{a}_{n,\bar{n}}\right)\Phi \\
\tilde{M}^{R(2)}\Psi &= \varsigma\Omega\cdot\left(-\sum_{n,\bar{n}=0}^{\infty}(n-\bar{n})\hat{a}^\dagger_{n,\bar{n}}\hat{a}_{n,\bar{n}}\right)\Phi \\
\tilde{P}_0^{R(2)}\Psi &= \varsigma\Omega\cdot\left(-\sum_{n,\bar{n}=0}^{\infty}\sqrt{(n+1)(2N+n)}\hat{a}^\dagger_{n,\bar{n}}\hat{a}_{n+1,\bar{n}} + \sqrt{(\bar{n}+1)(2N+\bar{n})}\hat{a}^\dagger_{n,\bar{n}}\hat{a}_{n,\bar{n}+1}\right)\Phi \\
\tilde{P}_1^{R(2)}\Psi &= \varsigma\Omega\cdot\left(\sum_{n,\bar{n}=0}^{\infty}\sqrt{(n+1)(2N+n)}\hat{a}^\dagger_{n,\bar{n}}\hat{a}_{n+1,\bar{n}} - \sqrt{(\bar{n}+1)(2N+\bar{n})}\hat{a}^\dagger_{n,\bar{n}}\hat{a}_{n,\bar{n}+1}\right)\Phi \\
\tilde{K}_0^{R(2)}\Psi &= \varsigma\Omega\cdot\left(-\sum_{n,\bar{n}=0}^{\infty}\sqrt{(n+1)(2N+n)}\hat{a}^\dagger_{n+1,\bar{n}}\hat{a}_{n,\bar{n}} + \sqrt{(\bar{n}+1)(2N+\bar{n})}\hat{a}^\dagger_{n,\bar{n}+1}\hat{a}_{n,\bar{n}}\right)\Phi \\
\tilde{K}_1^{R(2)}\Psi &= \varsigma\Omega\cdot\left(-\sum_{n,\bar{n}=0}^{\infty}\sqrt{(n+1)(2N+n)}\hat{a}^\dagger_{n+1,\bar{n}}\hat{a}_{n,\bar{n}} - \sqrt{(\bar{n}+1)(2N+\bar{n})}\hat{a}^\dagger_{n,\bar{n}+1}\hat{a}_{n,\bar{n}}\right)\Phi \\
\tilde{X}_\zeta^{R(2)}\Psi &= \varsigma\Omega\cdot\left(\sum_{n,\bar{n}=0}^{\infty}\hat{a}^\dagger_{n,\bar{n}}\hat{a}_{n,\bar{n}}\right)\Phi \qquad (94)
\end{aligned}$$

where $\hat{a}_{n,\bar{n}}$ and $\hat{a}^\dagger_{n,\bar{n}}$ are interpreted as annihilation and creation operators of *modes* $|n;\bar{n}\rangle$, $\tilde{D}^{R(2)}$ is attached to the total energy (remember that the dilatation parameter plays the role of a proper time), and $\tilde{X}_\zeta^{L(2)}$ corresponds with the number operator. It should be mentioned that all those quantities appear, in a natural way, *normally ordered*; this is one of the advantages of this method of quantization: normal order does not have to be imposed by hand but, rather, it is implicitly inside the formalism itself.

We can think of the Hilbert space as composed of modes:

1. pure non-bar $|n_1, n_2, ...; 0\rangle$,
2. pure bar $|0; \bar{n}_1, \bar{n}_2, ...\rangle$,
3. mixed $|n_1, n_2, ...; \bar{n}_1, \bar{n}_2, ...\rangle$.

4.2 Breaking down to the Weyl subgroup. Vacuum radiation

In this subsection, we investigate the effect of SCT on a Weyl vacuum, i.e. a vacuum of the massless QFT obtained after constraining the conformal quantum field theory developed in the last subsection.

The field degrees of freedom of the massless field are obtained by translating the condition (66) to the second quantization level, according to the general procedure (87) that is, by imposing

$$\begin{aligned} \left[\tilde{P}_0^{R(2)} + \tilde{P}_1^{R(2)}, \left[\tilde{P}_0^{R(2)} - \tilde{P}_1^{R(2)}, \tilde{X}^R_{a_{n,\bar{n}}}\right]\right] &= \\ -4\sqrt{n(2N+n-1)}\sqrt{\bar{n}(2N+\bar{n}-1)}\tilde{X}^L_{a_{n-1,\bar{n}-1}} &= 0\,, \end{aligned} \tag{95}$$

which selects the pure non-bar and pure bar operators, i.e, $\hat{a}^\dagger_{n,0} = -\frac{1}{\kappa}\tilde{X}^R_{a_{n,0}}$ and $\hat{a}^\dagger_{0,\bar{n}} = -\frac{1}{\kappa}\tilde{X}^R_{a_{0,\bar{n}}}$. These operators, together with the Weyl generators (good operators of the first-quantized theory) close a Lie subalgebra

$$\mathcal{G}_c^{(2)} =< \tilde{D}^{R(2)}, \tilde{M}^{R(2)}, \tilde{P}_0^{R(2)}, \tilde{P}_1^{R(2)}, \tilde{X}_\zeta^{L(2)}, \hat{a}^\dagger_{n,0}, \hat{a}^\dagger_{0,\bar{n}} > \tag{96}$$

of the original Lie algebra of the conformal quantum field. The vacuum of this constrained theory does not have to coincide with the conformal vacuum $|0\rangle = |n=0; \bar{n}=0\rangle$. In fact, any conformal state made up of an arbitrary content of zero modes

$$|W\{\sigma\}\rangle \equiv \sum_{q=0}^{\infty} \sigma_q (\hat{a}^\dagger_{0,0})^q |0\rangle \tag{97}$$

behaves as a vacuum from the point of view of a *Weyl observer*, that is, it is annihilated by the Weyl generators and the destruction operators $\hat{a}_{n,0}$ and $\hat{a}_{0,\bar{n}}$, for all $n, \bar{n} \in N - \{0\}$. Note that, since the operator $\hat{a}^\dagger_{0,0}$ is central in $\mathcal{G}_c^{(2)}$ (it commutes with all the others), it would be too restrictive to require (97) being nullified by $\hat{a}_{0,0}$; the only solution would be the conformal vacuum $|0\rangle$. It is then natural to demand that $\hat{a}_{0,0}$ behave as a multiple ϑ of the identity that is, it has to leave the Weyl vacuum stable

$$\hat{a}_{0,0}|W\{\sigma\}\rangle = \vartheta |W\{\sigma\}\rangle \Rightarrow \sigma_q^{(0)} = \frac{\vartheta^q}{q!}\sigma_0\,, \tag{98}$$

condition which, after normalizing, determines the Weyl vacuum up to a complex parameter ϑ

$$\langle W\{\sigma^{(0)}\}|W\{\sigma^{(0)}\}\rangle = 1 \Rightarrow |\sigma_0| = e^{-\frac{1}{2}|\vartheta|^2}\,. \tag{99}$$

Thus, we have find a set of Weyl vacua (coherent states of the conformal quantum field, made of zero modes)

$$|0\rangle_\vartheta \equiv e^{-\frac{1}{2}|\vartheta|^2} e^{\vartheta \hat{a}^\dagger_{0,0}}|0\rangle\,, \tag{100}$$

labeled by ϑ [the existence of a degenerate ground state resembles the "θ-vacuum" phenomenon in Yang-Mills field theories [JcRb-CDG] and, in general, it is present whenever we deal with non-simply connected phase spaces and constrained theories [ACG]]. As the final result is independent of ϑ, from now on we shall implicitly choose $\vartheta = 1$, for the sake of simplicity. An orthonormal basis for the Hilbert space of the constrained theory can be obtained by taking the orbit through the vacuum (100) of the creation operators as follows:

$$|m(n_1),...,m(n_q),m(\bar{n}_1),...,m(\bar{n}_j)\rangle_\vartheta \equiv \frac{(\hat{a}^\dagger_{n_1,0})^{m(n_1)}...(\hat{a}^\dagger_{n_q,0})^{m(n_q)}(\hat{a}^\dagger_{0,\bar{n}_1})^{m(\bar{n}_1)}...(\hat{a}^\dagger_{0,\bar{n}_j})^{m(\bar{n}_j)}}{(m(n_1)!...m(n_k)!m(\bar{n}_1)!...m(\bar{n}_j)!)^{1/2}}|0\rangle_\vartheta\,. \tag{101}$$

We can make a comparison with the standard case of a massless field in 1+1 dimensional Minkowski space-time and relate the non-bar and bar modes to the left-hand and right-hand moving scalar photons, respectively. Let us introduce dimensions through the Planck constant h and the frequency mode ν, so that the total energy is given by

$$\hat{E} \equiv h\nu\left(\tilde{D}^{R(2)} + 2N\tilde{X}^{L(2)}_\zeta\right) \equiv h\nu D^{R(2)}\,; \tag{102}$$

the last redefinition of the dilatation generator is intended to render the commutation relations (55) to the usual ones (28) by destroying the pseudo-extension (51). The expected value of the energy in the general state (101) is

$$\langle\hat{E}\rangle = h\nu(\sum_{l=1}^{q} m(n_l)n_l + \sum_{l=1}^{j} m(\bar{n}_l)\bar{n}_l + 2N)\,, \tag{103}$$

where $E_0 \equiv 2Nh\nu$ represents the *zero point energy*, i.e. the expected value of the energy in the Weyl vacuum. Zero modes represent *virtual* particles (they have no energy and cannot be detected by a Weyl observer) and can be spontaneously created from the Weyl vacuum, as can be deduced from the condition (98).

It is natural to think that zero modes will play an important role in the radiation of a Weyl vacuum, as they will be made *real* by acceleration. In fact, let us show how a finite special conformal transformation, generated by $A^{(2)}_+ \equiv \frac{1}{2}(\tilde{K}^{L(2)}_0 + \tilde{K}^{L(2)}_1)$, applied to a Weyl vacuum gives rise to a "thermal bath" of no-bar modes (left-hand moving scalar photons), whereas the combination $A^{(2)}_- \equiv \frac{1}{2}(\tilde{K}^{L(2)}_0 - \tilde{K}^{L(2)}_1)$ gives rise to a "thermal bath" of bar modes (right-hand moving scalar photons). The finite action of $A^{(2)}_+$, with parameter α (the corresponding acceleration is $a \equiv -(2\pi)^2\frac{c\nu}{\log|\alpha|^2}$, where c is the speed of light), on the Fourier parameter,

$$\begin{aligned} a^*_{0,0} \to a^{*\prime}_{0,0} &= \sum_{n=0}^{\infty}(-1)^n\sqrt{\frac{C^{(N)}_0}{C^{(N)}_n}}a^*_{n,0}\alpha^n = \sum_{n=0}^{\infty} r_n a^*_{n,0}\alpha^n \\ r_n &\equiv (-1)^n\sqrt{\frac{C^{(N)}_0}{C^{(N)}_n}} \end{aligned} \tag{104}$$

(according to the general expression in the third line of (88) and the last equality in Eq. (65)), leads to the following transformation on the Weyl vacuum:

$$|0\rangle_\vartheta \to |\Psi(\alpha)\rangle_\vartheta \equiv e^{-\frac{1}{2}} e^{\hat{a}^{\dagger}_{0,0}{}'}|0\rangle = \sum_{q=0}^{\infty} \alpha^q \sum_{\substack{m_1,\ldots,m_q: \\ \sum_{n=0}^{q} n m_n = q}} \prod_{n=0}^{q} \frac{r_n^{m_n}}{m_n!} \prod_{n=0}^{q} (\hat{a}^{\dagger}_{n,0})^{m_n} |0\rangle_\vartheta \,, \quad (105)$$

where $m_0 = 0$ and we have used the general identity

$$\begin{aligned} \left(\sum_{n=0}^{\infty} \gamma_n \alpha^n\right)^l &= \sum_{q=0}^{\infty} \delta_q \alpha^q \\ \delta_0 &= \gamma_0^l \\ \delta_q &= \frac{1}{q\gamma_0} \sum_{s=1}^{q} (sm - q + s)\gamma_s \delta_{q-s} \,. \end{aligned} \quad (106)$$

The relative probability of observing a state with total energy $E_q = h\nu q + E_0$ in a Weyl vacuum from an accelerated frame (i.e. in $|\Psi(\alpha)\rangle_\vartheta$) is

$$\begin{aligned} P_q &= \Lambda(E_q)(|\alpha|^2)^q \\ \Lambda(E_q) &\equiv \sum_{\substack{m_1,\ldots,m_q: \\ \sum_{n=0}^{q} n m_n = q}} \prod_{n=0}^{q} \frac{r_n^{2m_n}}{m_n!} \,. \end{aligned} \quad (107)$$

We can associate a thermal bath with this distribution function by noticing that $\Lambda(E_q)$ represents a relative weight proportional to the number of states with energy E_q, and the factor $(|\alpha|^2)^q$ fits this weight properly to a temperature as

$$(|\alpha|^2)^q = e^{q \log|\alpha|^2} = e^{-\frac{E_q - E_0}{kT}} \,, \quad \text{where} \quad T \equiv -\frac{h\nu}{k \log|\alpha|^2} = \frac{\hbar a}{2\pi c k} \quad (108)$$

is the temperature associated with a given acceleration a, and k is the Boltzmann's constant. This simple, but profound, relation between temperature and acceleration was first considered by Unruh [U]. The balance between the "multiplicity factor" $\Lambda(E_q)$ (an increasing function of the energy) and the temperature factor (108) (a decreasing function of the energy) is favorable (maximum) for a given system of this canonical ensemble, the energy of which is a repersentative value of the mean energy. In fact, this mean energy can be calculated exactly as the expected value of the energy operator $\hat{E}$ in the state $|\Psi(\alpha)\rangle_\vartheta$. To this end, let us perform some intermediate calculations. The norm of this accelerated vacuum is

$$\mathrm{Nor}[\Psi(\alpha)] \equiv {}_\vartheta\langle\Psi(\alpha)|\Psi(\alpha)\rangle_\vartheta = \exp\left(-1 + \sum_{n=0}^{\infty} r_n^2 |\alpha|^{2n}\right) = \exp\left((1 - |\alpha|^2)^{-2N} - 1\right) \,. \quad (109)$$

The probability $P_n(m)$ of observing m particles with energy E_n coincides with the expected value of the projector $\hat{P}_n(m)$ on the state $|m(n)\rangle_\vartheta$, i.e.:

$$P_n(m) \equiv \frac{{}_\vartheta\langle\Psi(\alpha)|\hat{P}_n(m)|\Psi(\alpha)\rangle_\vartheta}{\mathrm{Nor}[\Psi(\alpha)]} = \frac{1}{\mathrm{Nor}[\Psi(\alpha)]} \frac{(r_n^2 |\alpha|^{2n})^m}{m!} \,; \quad (110)$$

it can be seen that the closure relation $\sum_{n,m=0}^{\infty} P_n(m) = 1$ is in fact verified. The mean number $\mathcal{N}_n$ of left-hand moving scalar photons with energy E_n corresponds with the expected value of the number operator $\hat{\mathcal{N}}_n \equiv \hat{a}^\dagger_{n,0}\hat{a}_{n,0}$, i.e. :

$$\mathcal{N}_n = \sum_{m=0}^{\infty} m P_n(m) = \frac{1}{\mathrm{Nor}[\Psi(\alpha)]}\; r_n^2 |\alpha|^{2n} \exp\left(r_n^2|\alpha|^{2n}\right) . \tag{111}$$

With this information at hand, we can calculate the expected value of the total energy as:

$$E[\Psi(\alpha)] \equiv E_0 + h\nu \sum_{n=1}^{\infty} n\mathcal{N}_n = E_0 + \frac{2Nh\nu|\alpha|^2}{(1-|\alpha|^2)^{2N+1}} . \tag{112}$$

If we subtract the zero-point energy, normalize by a $2N$ factor (this normalization can be seen as a reparametrization of the proper time) and make use of the relation $|\alpha|^2 = e^{-\frac{h\nu}{kT}}$ in (108), we obtain a more "familiar" expression for the *mean energy per mode*

$$\epsilon_N(\nu, T) = \frac{h\nu e^{-\frac{h\nu}{kT}}}{(1-e^{-\frac{h\nu}{kT}})^{2N+1}} , \tag{113}$$

the value $N = 0$ corresponding with the well known case of the *Bose-Einstein statistic.* Note that this particular value of N can be reached only as a limiting process formulated on the universal covering of $SU(1,1) \otimes SU(1,1)$ or, equivalently, by uncompactifying the proper time $(U(1) \to \Re)$.

Let us compare the spectral distribution of the radiation of the Weyl vacuum for different values of N, with the well known case of the *black body radiation* (Planck's spectrum). For this purpose, we have to multiply the mean energy per mode by the number of states with frequency ν which, in d dimensions, would be proportional to ν^{d-1}. If we denote this product by $u(x, N) \equiv u_0\, x^{d-1}\epsilon_N(x, T_0)$ with $x \equiv \frac{h\nu}{kT_0}$ (u_0 is a constant, for a fixed temperature T_0, with dimensions of energy per unit of volume), Figure 1 shows the departure from the Planckian spectrum ($N = 0$) for four diferent values $N = \frac{1}{2}, \frac{3}{4}, 1, 1.1$ in the realistic dimension $d = 3$. Note that the value of $N \equiv N_c = \frac{d-1}{2}$ corresponds to a *critical* situation: over this value the theory exhibits an "infrared catastrophe".

5 Other representations: some comments

We have shown that it is impossible to establish conformally invariant evolution equations in a (even compactified) Minkowski space, not only for massive fields but, also for massless *quantum* fields.

If we wish the whole conformal group to be an exact symmetry of physical laws (at least, at very high energies), then we should reconsider the convenience of the Minkowski space as the frame for describing quantum physical phenomena. In fact, there exists a wider consistent quantum dynamics in which the conformal invariance is exact. The price to be paid is the introduction of an extra dimension, thus increasing by one the number of space-time parameters. The physical interpretation of this new parameter remains obscure but, interpretations in terms of a "unit of measurement" à la Weyl [BtH] and/or a "variable mass" interpretation have already been treated in the literature, even at the non-relativistic (Galilean) level [NiBt].

In the present scheme, this kind of representation can be obtained through a higher-order polarization which, as we are going to see, carries an index m_0 eventually interpreted as a conformally invariant mass (see [BtH]). In fact, let us use the following couples of generators:

$$\tilde{\Pi}_0^L \equiv \frac{\sqrt{2}}{2}(\tilde{P}_0^L + \tilde{K}_0^L) \quad ; \quad \tilde{X}_0^L \equiv \frac{\sqrt{2}}{2}(\tilde{P}_0^L - \tilde{K}_0^L)$$
$$\text{and} \tag{114}$$
$$\tilde{\Pi}_1^L \equiv \frac{\sqrt{2}}{2}(\tilde{P}_1^L + \tilde{K}_1^L) \quad ; \quad \tilde{X}_1^L \equiv \frac{\sqrt{2}}{2}(\tilde{P}_1^L - \tilde{K}_1^L)$$

as conjugate variables. If one tries to introduce the set $\{\tilde{X}_0^L, \tilde{X}_1^L\}$ in the polarization, then we face the problem that the characteristic module generated by the operators $\tilde{D}^L, \tilde{M}^L$ is too large, since $\left[\tilde{X}_\nu^L, \tilde{D}^L\right] = -\tilde{\Pi}_\nu^L$; in fact, only $\tilde{M}^L$ possesses a compatible set of commutation relations. As we have already pointed out, the dilatation could be introduced at the price of being a higher-order operator (something similar occurs with the time operator in the free particle case as long as we stay in position representation). More precisely, with this higher-order polarization, one can reduce the representation as follows:

$$\mathcal{P} =< \tilde{M}^L, \tilde{X}_0^L, \tilde{X}_1^L, \tilde{C}^L > , \tag{115}$$

where $\tilde{C}^L$ is just a Casimir operator of the extended conformal group [we can always add an arbitrary central term to $\tilde{C}^L$]. For example, in the compact dilatation case

$$\tilde{C}^L = (\tilde{M}^L)^2 + (\tilde{D}^L + 2N\tilde{X}_\zeta^L)^2 - (\tilde{\Pi}^L)^2 + (\tilde{X}^L)^2 \, . \tag{116}$$

Polarized wave functions evolve according to a Klein-Gordon-like equation

$$(\tilde{\Pi}^L)^2\psi = (\tilde{D}^L + 2N)^2\psi \, , \tag{117}$$

which can be interpreted as the motion equation of a scalar field with variable square mass $m^2 = (\tilde{D}^L + 2N)^2 = (D^L)^2$. The value of the Casimir on polarized wave functions is $\tilde{C}^R\psi = N(N-1)\psi \equiv m_0\psi$, which justify the denomination of m_0 as a conformally invariant mass [it proves to be quantized for this case, the reason being related to the compact character of the proper time (dilatation)]. The allowed value of N, $N = 1$ thus corresponds with null conformal mass. The precise connection between N and the curvature of some homogeneous subspaces (let us say, the Anti-de Sitter universe in $2+1$ dimensions) inside the conformal group is being investigated [ACC].

Note that the Cauchy hyphersurfaces of Eq. (117) have dimension 2, and the physical interpretation of the extra dimension remains unclear. Two different approaches can be taken which could be consistent with the physical meaning of the conformal group. One is related to the Weyl idea of different lengths in different points of space time [We]. The "rule" for measuring distances changes at different positions. In Quantum Mechanics, this implies that wave functions measuring probability densities do have different integration measures as functions of space-time. This change in the measure of integration needs to be related to the extra parameter appearing in our Group Approach to Quantization of the full conformal group. The other interpretation –not necessarily unrelated to the previous one– could be attached to the variable character of mass. Even at the level of one particle ordinary conformal quantum mechanics, the inescapable consequence of a variable mass appearing in the formalism was already observed several years ago by Niederer [NiBt]. It indeed would not be a surprise should this fact also have some consequences in the full quantization. Neither interpretation, however, is without controversy, as emphasized previously by Rohrlich [R]. At any rate, we have considered here a more satisfactory point of view by examining the dynamical breaking of the conformal group down to the Weyl subgroup in the framework of the Group Approach to Quantization.

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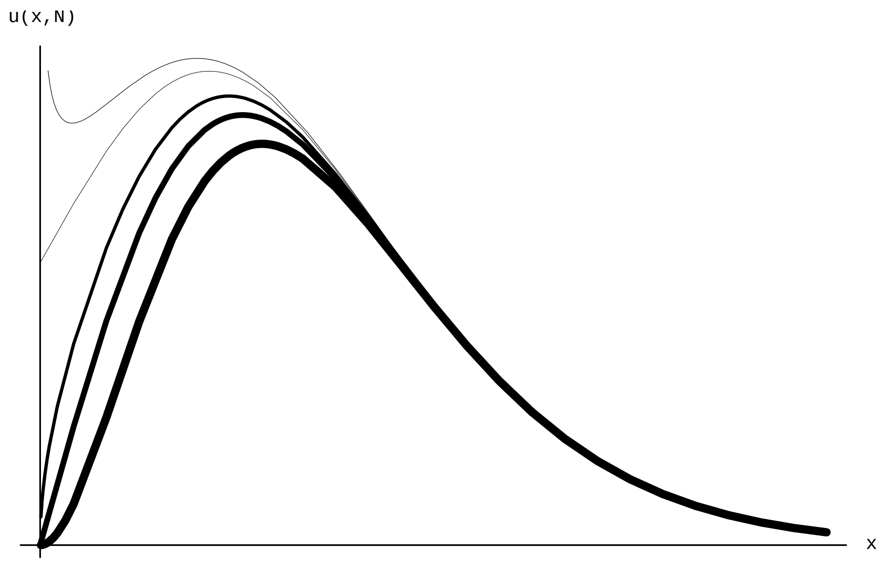

Figure 1: Departure from the Planck's spectrum (tickest line) for increasing values of N (decreasing tickness).