

## Group Quantization on Configuration Space: Gauge Symmetries and Linear Fields\*

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### **Abstract**

A new, configuration-space picture of a formalism of group quantization, the GAQ formalism, is presented in the context of a previous, algebraic generalization. This presentation serves to make a comprehensive discussion in which other extensions of the formalism, principally to incorporate gauge symmetries, are developed as well. Both images are combined in order to analyse, in a systematic manner and with complete generality, the case of linear fields (abelian current groups). To illustrate these developments we particularize them for several fields and, in particular, we carry out the quantization of the abelian Chern-Simons models over an arbitrary closed surface in detail.

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# I Introduction

At present the main goal of Theoretical Physics is to unify Quantum Theory and General Relativity. Symmetry is increasingly important in both theories and, because of that, it is expected to play a principal role in the future fundamental theory whatever it might be. Therefore it is desirable to understand as much as possible about Physics without using information other than that provided by the symmetries of the systems. The formalisms of quantization on a group, such as the Group Approach to Quantization (GAQ) formalism, are intended to perform this task as far as the process of quantization is concerned.

The GAQ formalism was introduced several years ago [1] as an improved version, in some respects, of Geometric Quantization and the Kirillov coadjoint-orbit methods of quantization [2, 3]. It is conceived basically as an algorithm for associating quantum systems with already given groups. However, most classical systems are commonly specified by a set of different equations or by a classical Lagrangian. Therefore in order to quantize these system with the GAQ formalism, it would be important to be able to derive, from the equations of motion or the Lagrangian, a group naturally associated with the system and large enough so as to reproduce, in some way, the classical theory. In so doing, solving the classical equations of motion has been required up to now. Nevertheless, in ref. [4] indications have been presented that there must be a way of circumventing this difficulty so that the basic steps, at least, of the GAQ formalism – such as finding out the quantizing group – may be carried out without previously solving the equations of motion. This procedure constitutes the **configuration-space picture** of the GAQ formalism and its further development is the main purpose of the present paper. As our first step, we shall consider linear fields only while non-abelian fields will be analysed in future studies.

An improvement of the GAQ formalism which is specially relevant to our purposes is its algebraic reformulation, which, instead of the infinitesimal calculus, uses the finite (algebraic) properties of the group [5]. This reformulation, therefore, enables us to incorporate discrete symmetries and to deal with non-Lie groups, that is, groups with no differential structure. The basic aspects of this reformulation were previously presented in ref. [5]. Here this picture of the formalism is presented in a unified manner so as to clarify several previous, heterogeneous developments. To make the discussion as self-contained as possible, the algebraic formulation is also further developed, in particular the characterization of gauge symmetries (gauge subgroup) is presented, and the way in which the GAQ formalism incorporates them at the quantum level is also shown.

When working in configuration space, with no explicit expression for the group in terms of the phase-space coordinates of the fields, to use the differential calculus over this group is clearly not feasible. It is necessary, therefore, to use algebraic group transformations. This fact provides additional support for using the algebraic picture of the GAQ formalism.

The quantization of linear fields, unlike non-linear ones whose quantization is considered to be a completely different and a much more difficult problem, is generally assumed to be well understood. There is in fact one good reason for such a different behaviour

between one case and the other: the huge (abelian) symmetry which underlies abelian fields. However, in spite of this fact, the usual way of presenting the quantization of linear fields does not make it explicit whether or not this underlying symmetry is involved. This fact does not help to identify the real difficulties in quantizing non-linear fields. Also, if the difference lies in the great symmetry which underlies linear fields, we should examine whether or not it is possible to construct non-linear fields, related to non-abelian current groups, which could be quantized with procedures similar to those applied in the linear case.

In addition to all this, and in spite of the (almost) general assumption, the quantization of linear fields is not always so trivial. There are many important cases, such as the one of fields in curved space (see for instance [6]), or when topological issues arise, in which the quantization presents difficulties and ambiguities with no simple solution.

The motivation to study linear fields is therefore twofold: on the one hand, they are important on their own, and, on the other hand this analysis may provide the key to generalize to non-linear fields.

In the present paper linear fields are thoroughly studied, relying as much as possible on their underlying symmetry and trying to be as general as possible. The structure of this paper is as follows: In Part 1, after a brief review of the Geometric Quantization and the GAQ formalism over a connected Lie group, the algebraic and configuration-space pictures of the GAQ formalism are considered. The results of this part are valid for arbitrary groups and fields. In Part 2, the theory of linear fields is thoroughly analysed by applying to it the (algebraic) GAQ formalism on configuration space. As an illustration of how to apply the formalism, several aspects of the electromagnetic field are briefly considered in section V – the interested reader may also consult ref. [7] and, above all, refs. [8, 9] where the development in this section have been carried further – and the Abelian Chern-Simons theory is quantized in section VI. For the sake of clarity, in this part –except in the last section– the analysis is restricted to linear (abelian) fields, even though one of our main motives is to extend, in the future, as much as possible of our results to non-abelian fields. In the last section, we discuss very briefly the difficulties in trying to extend our formalism to non-linear fields (non-abelian current groups).

Since this paper is aimed to present the unifying theory behind some previous or parallel (and to clear the way to future) developments of the GAQ formalism – those which only involve linear fields – the examples has been carried on only up to the point that they provide a link with those developments but do not significantly overlap with them. For more details on how the GAQ formalism is actually applied, the reader may consult the bibliography here provided where diverse applications can be found. In particular ref. [4], where quite a few examples of quantizing groups in configuration space are also given, complements the present analysis in several respects.

## PART 1. THE GENERAL FORMALISM

### II The Geometric Quantization and the Group Approach to Quantization

Before considering the GAQ formalism, we shall briefly describe the basic features of Geometric Quantization (GQ) which is a formalism from which the former derived.

#### A Geometric Quantization

The Geometric Quantization (see for instance [2]) is a formalism which intends to place the familiar canonical quantization rules of Quantum Mechanics in a rigorous setting:

$$\begin{aligned} q^i &\longrightarrow \widehat{q}^i; & (\widehat{q}^i \Psi)(\mathbf{q}) &\equiv q^i \Psi(\mathbf{q}) \\ p_j &\longrightarrow \widehat{p}_j; & (\widehat{p}_j \Psi)(\mathbf{q}) &\equiv -i\hbar \frac{\partial}{\partial q^j} \Psi(\mathbf{q}) \end{aligned} \quad (2.1)$$

where  $q^i, p_j$  fulfil the classical relationships

$$\{p_i, q^j\} = \delta_i^j \quad (2.2)$$

[From here on we shall make  $\hbar = 1$ .]

The basic idea in this formalism is that the quantum theory should be an irreducible representation of the Poisson algebra  $\mathcal{F}(P)$  of observables of the classical phase space  $P$ , which should act in a Hilbert space  $\mathcal{H}$  which is also constructed in a natural manner out of the classical system. Thus, with any function  $f : P \longrightarrow \mathfrak{R}$ , it should be associated a linear self-adjoint operator  $\widehat{f}$ , which acts on  $\mathcal{H}$  and such that,

$$\{\widehat{f}, \widehat{g}\} = [\widehat{f}, \widehat{g}], \quad \forall f, g \in \mathcal{F}(P) \quad (2.3)$$

It is well known that this program cannot be fully executed because obstructions arise, mainly due to ordering problems, which prevent the whole  $\mathcal{F}(P)$  from being represented. These obstructions are not a major problem if one is able a) to represent a subset of  $\mathcal{F}(P)$  which is big enough to generate the whole  $\mathcal{F}(P)$ , and b) to obtain without ambiguities the basic observables of the theory such as the Hamiltonian ( $\equiv$  quantum temporal evolution), the quantum angular momentum operators, etc.

Given a classical phase space with Poisson bracket  $\{, \}$  ( $\equiv$  symplectic form  $\omega$ ), with any  $f \in \mathcal{F}(P)$  we associate a natural operator  $X_f : \mathcal{F}(P) \longrightarrow \mathcal{F}(P)$ , defined through:

$$X_f(g) = \{f, g\}, \quad \forall g \in \mathcal{F}(P) \quad (2.4)$$

Because of the Jacobi identity, these operators also fulfil eq. (2.3). These relationships give us a basic guide to the expected nature of the Hilbert space of the quantum theory,  $\mathcal{H} \sim \mathcal{F}(P)$ , and the quantum operators:  $\widehat{f} \sim X_f$ . The difficulty is that the correspondence  $f \rightarrow X_f$  is not faithful because the constant functions are in its kernel. To overcome this problem a new term has to be added to the operators  $X$  so as to associate the natural constant operators with the constant functions. This is achieved by (non-trivially) enlarging  $P$  with a new parameter  $\zeta \in U(1)$  to give rise to a new manifold  $Q_P$  – which is called a **quantum manifold**– with a structure of  $U(1)$  principal bundle over  $P$ , so that  $Q_P/U(1) = P$ . The dependence of the wave functions with respect to the new co-ordinate  $\zeta \in U(1)$  is fixed by means of the condition

$$\Psi(\zeta p) = \zeta \Psi(p), \quad \forall \zeta \in U(1) \quad (2.5)$$

If  $X_\zeta$  is the vector field which generates the action of  $U(1)$  on  $Q_P$ , the constraint (2.5) reads:

$$X_\zeta \Psi = i\Psi \quad (2.6)$$

This condition together with the natural requirement that the constant functions must be properly represented implies that the new (pre-)quantum operator associated with  $f \in \mathcal{F}(P_Q)$  has the (local) expression:

$$\widetilde{X}_f = -i \left[ X_f - \left( i_{X_f} \lambda - i f \right) X_\zeta \right] \quad (2.7)$$

where  $\lambda$  is a symplectic potential to  $\omega$ .

Let now  $\Theta$  be the connection 1-form on  $Q_P \rightarrow P$ , which is defined by the conditions  $i_{X_\zeta} \Theta = 1$ ,  $i_{X_\zeta} d\Theta = 0$  and  $(Q_P, d\Theta)/U(1) \sim (P, \omega)$ . Then, the operators  $\widetilde{X}_f$  will be defined by the relationships:

$$i_{\widetilde{X}_f} \Theta = f, \quad i_{\widetilde{X}_f} d\Theta = -d f \quad (2.8)$$

(This relationships imply in particular that  $L_{\widetilde{X}_f} \Theta = 0$ .)

With this procedure, we make sure that the correspondence  $f \rightarrow \widetilde{X}_f$  is faithful. However, it will in general be reducible: there are non-trivial operators,  $\widetilde{X}_a$ ,  $a \in I$ , which commute with the basic ones of the representation,  $\widetilde{X}_{q^i}$ ,  $\widetilde{X}_{p_j}$ . The irreducibility has to be achieved by imposing further that (some of) these operators act trivially on the physical Hilbert space:

$$\widetilde{X}_a \Psi = 0, \quad \text{for some } a \in I, \quad \forall \Psi \in \mathcal{H} \quad (2.9)$$

This last condition roughly amounts to requiring that the wave functions depend only on the  $q^i$ 's or the  $p_j$ 's (or a particular combination of these such as the creation/annihilation operators).

## B The GAQ formalism over a connected Lie group

The GAQ formalism was originally conceived [1] to improve GQ by freeing it from several limitations and technical obstructions. Among them we point out the impossibility of considering quantum systems without classical limit, the lack of a proper (and naturally defined) Schrödinger equation in many simple cases and the ineffectiveness in dealing with anomalous systems [10].

The main ingredient which enable GAQ to avoid these limitations is a Lie group structure on the manifold  $\tilde{G}$  replacing the quantum manifold  $Q_P$  of GQ.  $\tilde{G}$  is also a principal bundle with structure group  $U(1)$ , but now  $\tilde{G}/U(1)$  is not forced to wear a symplectic structure. This way, non-symplectic parameters associated with symmetries like time translations, rotations, gauge transformations, etc. are naturally allowed and give rise to relevant operators (Hamiltonian, angular momentum, null charges, etc). Needless to say that the requirement of a group structure in  $\tilde{G}$  represent some drawback, although it is lesser, in practice, than it might seem. In particular constrained quantization (see below and ref. [11]) as well as higher-order polarizations [12, 13] allow GAQ to be applied to phase spaces that do not wear a group structure, thus greatly expanding the range of applicability of the formalism.

Nonetheless, we should remark that the GAQ formalism is not meant to quantize a classical system (a phase space) but, rather, the quantizing group is the primary quantity and in some cases (anomalous groups [13], for instance) it is unclear how to associate a phase space with the quantum theory obtained.

As a general rule, and roughly speaking,  $\tilde{G}$  is a central extension of a group  $G$  which represent a phase space enlarged by the (usually semi-direct) action of additional (non-symplectic) symmetries. As mentioned in the Introduction, GAQ proceeds associating quantum systems with already given groups  $\tilde{G}$ , but also the possibility exists of looking for an appropriate group  $\tilde{G}$  out of a given (classical) Lagrangian  $\mathcal{L}$ . In this case the solution manifold of  $\mathcal{L}$  (as a phase space) should be the starting point to construct the manifold of  $\tilde{G}$ .

The basic structure in the GAQ formalism, is, therefore, a Lie group  $\tilde{G}$  (see next section where generalizations are discussed) which is called the **quantizing group**. In this group, there are naturally defined left-invariant (right-invariant) vector fields,  $\tilde{X}_i^L$  ( $\tilde{X}_i^R$ ) as well as left-invariant (right-invariant) forms  $\theta^{L^i}$  ( $\theta^{R^i}$ ). As in Geometric Quantization a major role is played by the left-invariant form,  $\theta^{L^\zeta}$ , which is dual of the generator of the central subgroup  $U(1)$  after a basis of the Lie algebra has been chosen.

*Definition 2.1:* The 1-form  $\Theta \equiv \theta^{L^\zeta}$  dual to the vertical generator  $\tilde{X}_\zeta$  is called **quantization form**.

The space of wave functions will now be constructed on the functions on  $\tilde{G}$  which fulfil the condition of being  $U(1)$ -functions, which is now written:

$$\Xi\Psi = i\Psi, \quad \forall\Psi \in \mathcal{F}(\tilde{G}) \quad (2.10)$$

where  $\Xi = \tilde{X}_\zeta^L = i\zeta \frac{\partial}{\partial \zeta} = \tilde{X}_\zeta^R$ .

The quantum operators are the *right-invariant* vector fields.

Now there are two main points to be taken into account:

a) Some of the parameters of the group are not symplectic; that is, there are left invariant vector fields  $X_i^L$  such that

$$i_{X_i^L}\Theta = 0 = i_{X_i^L}d\Theta \quad (2.11)$$

b) The left-invariant and right-invariant vector fields commute. Therefore, the right-invariant vector fields do not provide an irreducible representation of  $\tilde{G}$  when acting on the space of  $U(1)$ -functions.

*Definition 2.2:* Let  $\tilde{\mathcal{G}}$  be the Lie algebra of  $\tilde{G}$ . The **characteristic subalgebra**  $\mathcal{C}$  of  $\tilde{\mathcal{G}}$  is the subalgebra which is expanded by the vector fields which fulfil eq. (2.11).

*Definition 2.3:* We shall say that a left subspace  $\mathcal{S}$  is **horizontal** iff

$$i_{X^L}\Theta = 0, \quad \forall X^L \in \mathcal{S} \quad (2.12)$$

*Definition 2.4:* A **polarization subalgebra**  $\mathcal{P}$  is a maximal horizontal subalgebra of  $\tilde{\mathcal{G}}$  such that  $\mathcal{C} \subset \mathcal{P}$ .

Points a) and b) are taken into account together by imposing the polarization conditions on the wave functions:

*Definition 2.5:* A wave functions  $\Psi$  is said to be **polarized** iff

$$\tilde{X}\Psi = 0, \quad \forall\tilde{X} \in \mathcal{P} \quad (2.13)$$

where  $\mathcal{P}$  is a polarization. With this requirement, and in the absence of constraints (see below), the quantization procedure is completed if we further specify a  $\tilde{G}$ -invariant integration measure. This measure has, in practice, turned out to be derivable from the natural one  $\theta^{L1} \wedge \theta^{L2} \wedge \dots$  on  $\tilde{G}$ , though the general case has not yet been addressed. The physical Hilbert space  $\mathcal{H}$  is then expanded by the integrable polarized wave functions. The physical operators are the right-invariant vector fields acting in this space and they are unitarily represented.

## Gauge subalgebra

*Definition 2.6:* We shall say that a right-invariant vector field  $\tilde{X}^R$  is **gauge** if

$$i_{\tilde{X}^R}\Theta = 0 \quad (2.14)$$



The subalgebra expanded by all the gauge vector fields will be denoted  $\mathcal{N}$  and will be termed **gauge subalgebra**.

Since for all  $\widetilde{X}^R$  and  $\theta^L$ ,  $L_{\widetilde{X}^R}\theta^L = 0$ , eq. (2.14) implies  $i_{X^R}d\Theta = 0$ . This agrees with the usual description of the gauge symmetries as the ones which are generated by vector fields in the kernel of the presymplectic 2-form (see, for instance, [14] and references therein). Also, in the GAQ formalism, the conserved (Noether) charge associated with  $\widetilde{X}^R$  corresponds to  $i_{\widetilde{X}^R}\Theta$ . Therefore, the definition above is consistent with the well-known fact that gauge symmetries have null conserved charges (see, for instance, ref. [15] for a direct proof).

*Proposition 2.1:* Let  $\mathcal{N}$  be the subspace expanded by the gauge vector fields. Then  $\mathcal{N}$  is an ideal of  $\widetilde{\mathcal{G}}$ .

*Proof:* It follows immediately by making use of the equality  $i_{[X,Y]} = L_X i_Y - i_Y L_X$ .

For  $\widetilde{X}^R$  gauge,  $\widetilde{X}^R \in \text{Ker}\Theta \cap \text{Ker}d\Theta = \mathcal{C}$ . Since  $\mathcal{C}$  is expanded by the characteristic subalgebra,  $\widetilde{X}^R$  must be of the form

$$\widetilde{X}^R = \sum_{j \in \mathcal{C}} f^j \widetilde{X}_j^L \quad (2.15)$$

Therefore the polarized wave functions are automatically gauge invariant:

$$\widetilde{X}^R(\Psi) = 0, \quad \forall \widetilde{X}^R \text{ gauge} \quad (2.16)$$

and no new (right) constraints need to be imposed.

### III The (algebraic) GAQ formalism over a group

In this section the GAQ formalism will be presented in a pure algebraic language. That is, we shall make use of finite quantities and algebraic operations only: composition of group elements, subgroups, etc. A (desired) consequence of this reformulation is that nowhere it is needed a differential structure on the quantizing group, that is, now  $\widetilde{\mathcal{G}}$  need not to be a Lie group. It can be a discrete or even finite group.

We shall consider only the case in which the quantizing group  $\widetilde{\mathcal{G}}$  is provided with a *central* subgroup  $T_0$  which, in this paper, will be called **canonical subgroup**. Natural extensions of the formalism to more general cases have already been discussed in the literature (see for instance [16]) but will not be considered here.

The canonical subgroup is the centre of gravity around which the group quantization formalism is formulated.

The GAQ formalism requires us to singularize, appart from the canonical subgroup, two other subgroups of  $\widetilde{\mathcal{G}}$ : the characteristic subgroup and the polarization subgroup. In addition, the gauge subgroup is also naturally defined.

*Definition 3.1:* We shall say that a subgroup  $H \subset \widetilde{\mathcal{G}}$  is **horizontal** if  $H \cap T_0 = \{1_{\widetilde{\mathcal{G}}}\}$ , where  $1_{\widetilde{\mathcal{G}}}$  is the neutral element of  $\widetilde{\mathcal{G}}$ .

*Definition 3.2:* Given  $g, g' \in \tilde{G}$ , we define the **commutator** of  $g, g'$  as  $[g, g'] = gg'g^{-1}g'^{-1}$ . If  $S, S'$  are two subsets (not necessarily subgroups) of  $\tilde{G}$ , then  $[S, S'] \equiv \{[g, g'] / g \in S, g' \in S'\}$ .

*Definition 3.3:* The **characteristic subgroup**  $C$  of  $\tilde{G}$  is the maximal horizontal subgroup such that  $[C, \tilde{G}] \cap T_0 = \{1_{\tilde{G}}\}$ .

*Definition 3.4:* A **polarization subgroup**  $P$  is a maximal horizontal subgroup of  $\tilde{G}$  such that  $C \subset P$ .

*Definition 3.5:* The **gauge subgroup**  $N$  of  $\tilde{G}$  is the maximal horizontal normal subgroup of  $\tilde{G}$ .

*Note:* Since  $N$  is horizontal and  $[\tilde{G}, N] \subset N$ , then  $N \subset C$ .

When  $\tilde{G}$  is a Lie group the above definitions lead to the ones for the Lie algebras in the previous section. In particular, because of the following proposition, which is the reciprocal of Proposition 2.1, the Definition 3.5 corresponds to the one for a gauge subalgebra:

*Proposition 3.1:* Let  $H$  be a horizontal normal subgroup of a Lie quantizing group  $\tilde{G}$  and let  $\tilde{X}_i^R$  be the right invariant vector fields which generate  $H$ , then

$$i_{\tilde{X}_i^R} \Theta = 0. \quad (3.1)$$

*Proof:* Consider any function  $\Psi : \tilde{G} \rightarrow C$  such that  $\Psi(gh) = \Psi(g)$  for all  $g \in \tilde{G}, h \in H$ . Then, because  $H$  is normal,  $\Psi(hg) = \Psi(g)$  also. This fact requires that, at any  $g \in \tilde{G}$ , any right-invariant vector field  $\tilde{X}_i^R$  which generates the left action of  $H$  can be expressed as a linear combination of the left-invariant vector fields  $\tilde{X}_j^L$  which only involves the vector fields which generate the (right) action of  $H$ , and the other way round. Therefore, since  $H$  is horizontal, the charges which are associated with the invariant vector fields tangent to  $H$  and to  $\Theta \equiv \theta^{L^\zeta}$  are zero.

The proper quantization proceeds as follows:

We start with the space  $\mathcal{F}(\tilde{G})$  of complex functions on  $\tilde{G}$  and pick up a representation  $D_{T_0}$  of  $T_0$ , and a right-representation  $D_P$  of a polarization  $P$ , on  $\mathcal{F}(\tilde{G})$ .

*Definition 3.6:* We shall say that  $\Psi \in \mathcal{F}(\tilde{G})$  is a  $D_{T_0}$ -function iff

$$\Psi(zg) = D_{T_0}(z)\Psi(g), \quad \forall g \in \tilde{G}, \forall z \in T_0 \quad (3.2)$$

*Definition 3.7:* A function  $\Psi \in \mathcal{F}(\tilde{G})$  is called **polarized** ( $D_P$ -polarized) iff

$$\Psi(gp) = D_P(p)\Psi(g), \quad \forall g \in \tilde{G}, \forall p \in P \quad (3.3)$$

In absence of constraints, these conditions fully determines the Hilbert space of the theory: it is given by the set of all (square integrable) polarized  $D_{T_0}$ -functions in  $\mathcal{F}(\tilde{G})$ . The dynamical operators are all the elements in  $\tilde{G}$ , and they act as finite *left* translations on the Hilbert space:

$$(\widehat{g}\Psi)(g') = \Psi(g^{-1}g'), \forall g, g' \in \widetilde{G} \quad (3.4)$$

Therefore the gauge subgroup, which corresponds to gauge constraints which have been solved classically, is automatically and trivially represented.

### Constraint quantization and good operators

As is well known (see basis references in [17], see also [18]), there is a close relationship between constraints and gauge symmetries. Loosely speaking, the existence of a gauge symmetry suffices to have a constrained system, and first-class constraints generate gauge symmetries. Constraints are not, however, always due to the presence of gauge symmetries in the system: the former are more general than the latter.

Here we shall consider only the case in which the constraints close into a subgroup  $\widetilde{T} \subset \widetilde{G}$ . The constraint subgroup  $\widetilde{T}$  is required to be a fibre group of  $\widetilde{G}$ , i.e.,  $\widetilde{G} \longrightarrow \widetilde{G}/\widetilde{T}$  is a principal bundle and to contain  $T_0$  as a fibre group, i.e.  $\widetilde{T} \longrightarrow \widetilde{T}/T_0$  is also a principal bundle. In particular,  $\widetilde{T}$  should be regarded as a quantizing group with the same canonical subgroup  $T_0$  as  $\widetilde{G}$ .

When there are constraints the procedure described above has to be completed with additional conditions on the wave functions. Now the physical Hilbert space is made up of all the polarized  $T_0$ -functions which are constrained:

*Definition 3.8.* A wave function  $\Psi : \widetilde{G} \longrightarrow C$  is termed **constrained** iff

$$\Psi(t * g) = D_{\widetilde{T}}(t)\Psi(g), \quad \forall t \in \widetilde{T}, g \in \widetilde{G} \quad (3.5)$$

where  $D_{\widetilde{T}}$  is an irreducible representation of  $\widetilde{T}$ .

Representations of  $\widetilde{T}$  which are compatible with  $D_{T_0}$  are naturally found by applying the GAQ formalism to  $\widetilde{T}$ . We, therefore, need the same collection of subgroups of  $\widetilde{T}$  in relation to  $T_0$  as just described for  $\widetilde{G}$ . When there is danger of confusion, these subgroups of  $\widetilde{T}$  will be signalled by placing a prefix  $\widetilde{T}$  before them. Thus, we shall have the  $\widetilde{T}$ -characteristic subgroup, the  $\widetilde{T}$ -polarization subgroup and so on.

Clearly not all the operators in  $\widetilde{G}$  will preserve the representation  $D_{\widetilde{T}}$  of  $\widetilde{T}$ ; for the dynamical operators that do we shall use the name **good operators** [5]. The group of all the good operators thus constitutes the natural generalization of the concept of normalizer of  $\widetilde{T}$ . This is the manner in which the concept of gauge subgroup (gauge symmetries) is incorporated into the quantum level.

In some cases (where  $\widetilde{T}$  is connected and is not a direct product  $\widetilde{T} \neq T_0 \otimes T$ ) the  $\widetilde{T}$ -function condition (3.5) may not be compatible with the representation  $D_{T_0}$  for  $T_0$ . Then we must soften that requirement and consider, rather than the whole  $\widetilde{T}$ , a subgroup  $T_0 \oplus P_T$ , where  $P_T$  is a polarization subgroup of  $\widetilde{T}$ . This subtlety does not arise, however, in the models we shall consider in the present paper, in which the whole  $\widetilde{T}$  can be represented in a way compatible with the  $T_0$ -function condition.

When  $\tilde{T}$  is a non-trivial central extension, it is sometimes said that the gauge symmetries are “anomalous”. Nonetheless, these “anomalies” do not necessarily imply obstruction to quantization, and do not particularly when the condition (3.5) can be imposed for the entire  $\tilde{T}$ .

## PART 2. LINEAR FIELDS

### IV Linear fields

Throughout this section, we shall consider a theory with fields  $\varphi^a$ ,  $a = 1, \dots, N$ , and action

$$S = \int_{\mathcal{M}} \mu \mathcal{L}(\varphi^a, \partial_\mu \varphi^a) \quad (4.1)$$

The space-time manifold  $\mathcal{M}$ , with volume element  $\mu = d^{D+1}x$ , will always be homeomorphic to  $\Sigma \times \mathfrak{R}$  where  $\mathfrak{R}$  represents the time-like directions and  $\Sigma$  is any ( $D$ -dimensional) spacelike hypersurface. When picking up a particular Lagrangian, we shall make use, if necessary, of the indetermination under a total divergence.

The set of all fields, irrespective of whether or not they satisfy the Euler-Lagrange equations of motion will be denoted by  $\mathcal{F}$ . We shall term any solution of the (classical) equations of motion as **trajectory**, or classical trajectory.  $\mathcal{T}$  will be the set of all the trajectories of the system.

If a (classical) theory of fields,  $(S, \mathcal{F})$  is linear, the space  $\mathcal{T}$  of all the solutions of the equations of motion is a vector space. That is, if  $\varphi$  and  $\phi$  are solutions, so is  $\lambda\varphi + \beta\phi$  for any  $\lambda, \beta \in \mathfrak{R}$ . Therefore  $\mathcal{T}$  can be regarded as an (abelian) group of symmetries of the theory with composition law:

$$\varphi'' = \varphi' + \varphi \quad (4.2)$$

This group will be denoted as  $G_{(\varphi)}$ .

*Theorem 4.1:* If  $(S, \mathcal{F})$  is a classical theory of linear fields, with Euler-Lagrange equations of motion  $([E - L]\varphi)_a = 0$ , then

a)  $\mathcal{L}(\varphi + \phi) = \mathcal{L}(\varphi) + \mathcal{L}(\phi) + ([E - L]\phi)_a \varphi^a + \partial_\mu J^\mu(\varphi, \phi), \forall \varphi, \phi \in \mathcal{F}$

b) A Lagrangian is given by:

$$\mathcal{L}(\varphi) = \frac{1}{2} ([E - L]\varphi)_a \varphi^a \quad (4.3)$$

Therefore, there exists a Lagrangian which vanishes “on-shell”, i.e.  $\mathcal{L}(\varphi) = 0$  for any classical trajectory  $\varphi$ .

*Proof:* The point a) follows immediately if we look at  $\mathcal{L}(\varphi + \phi)$  as a variation of the Lagrangian, a variation similar to the one which gives the Euler-Lagrange equations of motion.

If in the equality a) we make  $\varphi = \phi = \frac{1}{2}\kappa$ , we obtain:

$$\mathcal{L}(\kappa) = \frac{1}{2} ([E - L]\kappa)_a \kappa^a + \frac{1}{2} \partial_\mu J^\mu(\kappa, \kappa) \quad (4.4)$$

The new Lagrangian

$$\widehat{\mathcal{L}}(\kappa) = \mathcal{L}(\kappa) - \frac{1}{2} \partial_\mu J^\mu(\kappa, \kappa) \quad (4.5)$$

fulfils part b).

*Corollary 4.1.1:* Since the current  $J^\mu(\varphi, \phi)$  is bilinear, it can be chosen to be:

a) Divergenceless on trajectories.

and

b) Antisymmetric.

*Proof:*

a) This is a consequence of part a) of Theorem 4.1 if a Lagrangian that vanishes on shell is chosen.

b) It is sufficient to show that  $J^\mu(\kappa, \kappa)$  is identically null. If  $J^\mu(\kappa, \kappa)$  were not identically null, then  $\widetilde{J}^\mu(\kappa, \kappa)$ , where  $\widetilde{J}^\mu(\varphi, \phi) = J^\mu(\varphi, \phi) - \frac{1}{2}J^\mu(\varphi, \varphi) - \frac{1}{2}J^\mu(\phi, \phi)$  is also an admissible current, would. However, both  $J^\mu$  and  $\widetilde{J}^\mu$  have to be bilinear. Therefore,  $J^\mu(\kappa, \kappa) = 0 \forall \kappa$  and  $\widetilde{J}^\mu = J^\mu$ .

*Definition 4.1:* The current  $J^\mu$  for which a) and b) hold will be called the **canonical current** of  $(S, \mathcal{F})$ .

*Note:* There is, in fact, a shorter but equivalent way of obtaining the canonical  $J^\mu$ . If in Theorem 4.1.a) we exchange  $\varphi$  and  $\phi$  and then antisymmetrize, we get:

$$\partial_\mu \left( \frac{1}{2} J^\mu(\varphi, \phi) - \frac{1}{2} J^\mu(\phi, \varphi) \right) = \frac{1}{2} ([E - L]\varphi)_a \phi^a - \frac{1}{2} ([E - L]\phi)_a \varphi^a \quad (4.6)$$

The current  $\left( \frac{1}{2} J^\mu(\varphi, \phi) - \frac{1}{2} J^\mu(\phi, \varphi) \right)$  is the canonical current of  $(S, \mathcal{F})$ .

*Corollary 4.1.2:* If  $\widetilde{\mathcal{L}} = \mathcal{L} + \partial_\mu \Lambda$ , then  $\widetilde{J}^\mu = J^\mu$

*Definition 4.2:* For all  $\varphi, \phi \in \mathcal{T}$  we define the **canonical product**  $\Omega(\varphi, \phi)$  by means of:

$$\Omega(\varphi, \phi) = \int_\Sigma d\sigma_\mu J^\mu(\varphi, \phi) \quad (4.7)$$

where  $\Sigma$  is any Cauchy hypersurface in  $\mathcal{M}$ . Therefore it is bilinear, antisymmetric and independent on the  $\Sigma$  hypersurface.

The canonical product of two solutions  $\varphi$  and  $\phi$  is nothing other than the Noether charge associated with the symmetry generated by  $\varphi$  in the point  $\phi \in \mathcal{T}$ , or (minus) the other way round. It measures the degree to which the classical trajectories  $\varphi, \phi$  are coordinate-momentum conjugate to each other.

*Note:* Notice that the potential current of the theory is  $j^\mu = J^\mu(\varphi, \delta\varphi)$ . The symplectic form is therefore given by [19, 14]:

$$\omega = - \int_\Sigma d\sigma_\mu \delta j^\mu \quad (4.8)$$

*Theorem 4.2:* With  $\Omega$  defined above, the following composition law defines a central extension of  $G_{(\varphi)}$  which will be denoted  $\widetilde{G}_{(\varphi)}$ :

$$\varphi''(x) = \varphi'(x) + \varphi(x) \quad (4.9)$$

$$\zeta'' = \zeta' \zeta \exp i\Omega(\varphi', \varphi) \quad (4.10)$$

where the fields  $\varphi, \varphi' \dots$  are trajectories and  $\zeta, \zeta' \dots \in U(1)$ .

## A Space-time and internal symmetries

In addition to the symmetries in  $G_{(\varphi)}$ , which act additively, there are in general other symmetries, such as space-time or internal ones, which act multiplicatively. In this section, we shall study the conditions under which the group  $\tilde{G}_{(\varphi)}$  can be enlarged with these other symmetries.

First of all we note that since the composition of two symmetries is another symmetry, any two groups of symmetries  $U_1$  and  $U_2$ , can be enlarged to obtain a new group  $U_3$  such that  $U_1, U_2 \subset U_3$ . Therefore, without loss of generality, we can consider a single group of symmetries  $U = \{u, v, \dots\}$ . The requirement of being symmetries is that, if  $\varphi \in \mathcal{T}$ , then  $u(\varphi) \in \mathcal{T}$ .

These symmetries (which should be thought of as being like  $SU(2)$ , the Poincaré, the conformal or the Virasoro groups) usually act on  $\mathcal{F}$  ( $\mathcal{T}$ ) through a previous representation in the space-time.

For any field  $X$  which generates the action of  $U$  on  $\mathcal{F}$ , we have

$$L_X \mathcal{L}\mu = d \Lambda_X \quad (4.11)$$

with  $\Lambda_X$  a space-time  $D$ -form.

Eq. (4.11) together with Corollary 4.1.2 imply that the following lemma holds:

*Lemma 4.1:* Let  $U_0$  be the component of  $U$  which is connected to the identity, then  $\Omega(u(\varphi), u(\phi)) = \Omega(\varphi, \phi)$ ,  $\forall \varphi, \phi \in \mathcal{T}$ ,  $\forall u \in U_0$ .

For symmetries which are not connected to the identity, such as parity or temporal inversion, this lemma has to be relaxed, as we can have **anticanonical symmetries**, that is, symmetries  $u$  for which  $\Omega(u(\varphi), u(\phi)) = -\Omega(\varphi, \phi)$ . In general, the action of  $U$  on  $\Omega$  defines a representation  $\epsilon$  of  $U$  on  $Z_2 = \{+, -\}$ . Then we shall have:

*Theorem 4.3:* With the fields as defined above, the following composition law is a group:

$$\begin{aligned} u'' &= u' * u, \quad u, u', u'' \in U \\ \varphi''(x) &= \varphi'(x) + (u'(\varphi))(x), \quad \varphi, \varphi', \varphi'' \in \mathcal{T} \\ \zeta'' &= \zeta' \zeta^{\epsilon(u')} \exp i\Omega(\varphi', (u'(\varphi))), \quad \zeta, \zeta', \zeta'' \in U(1) \end{aligned} \quad (4.12)$$

This group will be denoted  $\tilde{G}_{(S, \mathcal{F})}$ . Note that when there are anticanonical symmetries in  $U$ , it is no longer a central extension.

For the sake of brevity we shall consider only canonical symmetries; that is, symmetries for which  $\epsilon(u) = 1$ . Anticanonical transformations, which give rise to interesting subtleties, will be the subject of a separate study [20].

Depending on the context, expressions for the group  $\tilde{G}_{(S, \mathcal{F})}$  which are different from eq. (4.12) –where the symmetry group  $U$  acts from the left– and which are obtained from it by means of a change of variables, may appear to be more natural ones. For instance

$$\begin{aligned} u'' &= u' * u, \quad u, u', u'' \in U \\ \varphi''(x) &= \left(u^{-1}(\varphi')\right)(x) + \varphi(x), \quad \varphi, \varphi', \varphi'' \in \mathcal{T} \\ \zeta'' &= \zeta'^{\epsilon(u)} \zeta \exp i\Omega\left(\left(u^{-1}(\varphi'), \varphi\right)\right), \quad \zeta, \zeta', \zeta'' \in U(1) \end{aligned} \quad (4.13)$$

where the symmetry group  $U$  acts on the left instead. In the rest of this paper, we shall make use of combinations of these two presentations in which some subgroups of  $U$  act from the left and others from the right.

### Example: the non-relativistic free particle and the Galilei group

As a first example of the construction above, let us consider the non-relativistic free particle –regarded as a  $(0 + 1)$ -dimensional field theory – and construct the quantizing group for it. In spite of its simplicity, we follow the same steps as for a standard field in contrast with the quantum-mechanical treatment of the free particle [1]. For more examples, see below and ref. [4] where the harmonic oscillator, which provides an useful link between mechanics and field theory, is also considered.

A Lagrangian for the non-relativistic free-particle is:

$$L'_{FP}(\mathbf{x}) = \frac{m}{2} \dot{\mathbf{x}}^2 \quad (4.14)$$

We have

$$L'_{FP}(\mathbf{A} + \mathbf{B}) = \frac{m}{2} \left[ \dot{\mathbf{A}}^2 + \dot{\mathbf{B}}^2 - 2\dot{\mathbf{B}}\mathbf{A} + 2\frac{d}{dt}(\dot{\mathbf{B}}\mathbf{A}) \right] \quad (4.15)$$

Thus, the associated on-shell-vanishing Lagrangian, the equations of motion and the canonical product are, respectively:

$$\begin{aligned} L_{FP} &= -\frac{m}{2} \ddot{\mathbf{x}}\mathbf{x} \\ -\frac{m}{2} \ddot{\mathbf{x}} &= 0 \\ \Omega_{FP}(\mathbf{A}, \mathbf{B}) &= \frac{m}{2} [\dot{\mathbf{A}}\mathbf{B} - \mathbf{A}\dot{\mathbf{B}}] \end{aligned} \quad (4.16)$$

Now we can consider the spatial rotations and time translations as the group of space-time symmetries. These act on  $\mathcal{F}_{FP}$  as follows



$$\begin{aligned}
(R\mathbf{A})^i(t) &= R^i_j A^j(t), \quad R^i_j \in O(3) \\
(T_b(\mathbf{A}))(t) &= \mathbf{A}(t-b), \quad b \in \mathfrak{R}
\end{aligned} \tag{4.17}$$

Now the general solution to the equations of motion is:

$$\mathbf{x}(t) = \mathbf{Q} + \mathbf{V}t, \quad \mathbf{Q}, \mathbf{V} \in \mathfrak{R}^3 \tag{4.18}$$

and  $\mathbf{Q}, \mathbf{V}$  can be taken as the coordinates in  $\mathcal{T}_{FP}$ . It is simple to see that

$$\begin{aligned}
R(\mathbf{Q})^i &= R^i_j Q^j, & R(\mathbf{V})^i &= R^i_j V^j \\
T_b(\mathbf{Q}) &= \mathbf{Q} - \mathbf{V}b, & T_b \mathbf{V} &= \mathbf{V}
\end{aligned}$$

The group  $\tilde{G}_{FP}$  is therefore given by

$$\begin{aligned}
b'' &= b' + b \\
\mathbf{Q}'' &= \mathbf{Q}' + \mathbf{V}'b + R'(\mathbf{Q}) \\
\mathbf{V}'' &= \mathbf{V}' + R'(\mathbf{V}) \\
\zeta'' &= \zeta' \zeta \exp \frac{i}{2} m [(\mathbf{Q}' + \mathbf{V}'b) R'(\mathbf{V}) - \mathbf{V}' R'(\mathbf{Q})]
\end{aligned} \tag{4.19}$$

which is the Galileo group (extended by the Bargmann cocycle [1]).

## B Quantization

Now that we have found out the quantizing group  $\tilde{G}_{(S, \mathcal{F})}$ , we shall apply to it the GAQ formalism presented in Part 1.

To identify the characteristic subgroup, we have to construct the commutator of two generic elements  $g = (u, \varphi, \zeta) \in \tilde{G}_{(S, \mathcal{F})}$ ,  $g' = (u', \varphi', 1) \in C$ .  $C$  will be the maximal subgroup such that  $[g, g'] = (1_U, 0, \zeta)$  implies  $\zeta = 1$ .

We have

$$\begin{aligned}
g'g &= (u'u, u^{-1}(\varphi') + \varphi, \zeta' \zeta \exp \frac{i}{2} \Omega(u^{-1}(\varphi'), \varphi)) \\
g^{-1} &= (u^{-1}, -u(\varphi), \zeta^{-1}) \\
gg' &= (uu', u'^{-1}(\varphi) + \varphi', \zeta \zeta' \exp \frac{i}{2} \Omega(u'^{-1}(\varphi), \varphi')) \\
g'g(gg')^{-1} &= \left( (u'u(uu')^{-1}, uu'[u^{-1}(\varphi') + \varphi] - uu'[u'^{-1}(\varphi) + \varphi'], \right. \\
&\quad \exp \frac{i}{2} \left[ \Omega(u^{-1}(\varphi'), \varphi) - \Omega(u'^{-1}(\varphi), \varphi') \right. \\
&\quad \left. \left. - \Omega(uu'[u^{-1}(\varphi') + \varphi], uu'[u'^{-1}(\varphi) + \varphi']) \right] \right)
\end{aligned} \tag{4.20}$$

Therefore,  $g' = (u', \varphi', 1)$  has to fulfil

$$\Omega(\varphi', u(\varphi) + u'^{-1}(\varphi)) = 0 \quad \forall g = (u, \varphi, \zeta) \in \tilde{G}_{(S, F)} \quad (4.21)$$

This implies

$$C = U \oplus N \quad (4.22)$$

with  $N = \text{gauge subgroup} = \{(1_U, \varphi', 1) / \Omega(\varphi', \varphi) = 0 \quad \forall g = (u, \varphi, \zeta) \in \tilde{G}(u, \varphi, \zeta)\}$ .  
 $[U \oplus N$  stands for the subgroup generated by  $U \cup N$  and it also means  $U \cap N = \{1_{\tilde{G}}\}$ .]

We recall now that a polarization subgroup is a maximal horizontal subgroup  $P$  such that  $C \subset P$ . Thus, any  $P$  is generated by

$$P = C \cup P_\varphi \quad (4.23)$$

where  $P_\varphi$  is the maximal horizontal subgroup such that  $\Omega(v(\varphi), \varphi') = 0 \quad \forall g = (1_U, \varphi, 1)$ ,  
 $g' = (1_U, \varphi', 1) \in P_\varphi, \quad \forall v \in U$

**Definition 4.3:** A **Lagrangian subgroup** is any subgroup  $L = \{(1_U, \varphi, 1)\}$  such that  $\Omega(\varphi, \varphi') = 0$ , for any  $(1_U, \varphi, 1), (1_U, \varphi', 1) \in L$ . If  $U(L) \subset L$  it will be called **invariant Lagrangian subgroup**.

We, therefore, have:

**Proposition 4.1:** Any polarization subgroup  $P$  is generated by  $U \cup N \cup L$ , where  $L$  is a maximal invariant Lagrangian subgroup.

## C Holomorphic quantization

We now consider the case when there are two subgroups  $L, \bar{L} \subset \tilde{G}$  which fulfil

- a)  $\bar{L}$  is a Lagrangian subgroup (not necessarily invariant),
- b)  $L$  is an invariant Lagrangian subgroup,
- c)  $\tilde{G}_{(S, \mathcal{F})} = U \oplus L \oplus \bar{L} \oplus U(1)$ .

Therefore, any trajectory  $\varphi$  has a unique decomposition

$$\varphi = a + \bar{a}, \quad \text{where} \quad (1_U, a, 1) \in L, (1_U, \bar{a}, 1) \in \bar{L} \quad (4.24)$$

*Note:* In general, to find  $L$  and  $\bar{L}$  with the properties above, it is necessary to go to  $\bar{\mathcal{F}}$ , the complexified  $\mathcal{F}$ , and to consider instead the group  $\tilde{G}_{(S, \bar{\mathcal{F}})} \supset \tilde{G}_{(S, \mathcal{F})}$  over that complexified space. In this case, the third condition above takes the form:

$$c') \tilde{G}_{(S, \mathcal{F})} \subset U \oplus L \oplus \bar{L} \oplus U(1) = \tilde{G}_{(S, \bar{\mathcal{F}})}.$$

If we take  $\varphi = \bar{a} + a$ , the polarization  $P = U \oplus L$ , and we pick up the trivial representation for it, one of the  $D_P$ -polarization conditions reads:

$$\Psi(u, \bar{a} + a, \zeta \exp i\Omega(\bar{a}, a)) = \Psi(u, \bar{a}, \zeta) \quad (4.25)$$

This equality, together with the  $U(1)$ -function condition on  $\Psi(u, \varphi, \zeta)$ , implies:

$$\Psi(u, \varphi, \zeta) = \zeta \Phi(u, \bar{a}) \exp[-i\Omega(\bar{a}, a)] \quad (4.26)$$

The rest of the polarization conditions reads:

$$\Psi(u'u, u^{-1}(\varphi), \zeta) = \Psi(u, \varphi, \zeta) \quad (4.27)$$

Therefore

$$\Phi(u'u, u^{-1}(\bar{a})) = \Phi(u', \bar{a}) \quad (4.28)$$

where we have made use of the fact that  $L$  is an invariant Lagrangian subgroup. Since  $\bar{L}$  may not be invariant,  $u^{-1}(\bar{a})$  is not in general in  $\bar{L}$ . However, whatever the case, eq. (4.28) gives the (finite) action of the space-time and internal symmetries in the wave functions. The infinitesimal action, and in particular the Schrödinger equation, can be obtained as the first-order terms in the power series in the parameters of the symmetries.

In the quantum theory of relativistic fields a splitting which fulfils the requirements above – and where both  $\bar{L}$  and  $L$  are invariant under the (proper) Poincaré group – is the usual one into negative- and positive-frequency parts. On the other hand, the non-relativistic free particle provides an interesting and simple example in which the trajectories  $\mathbf{x}$  split as  $\mathbf{x} = \mathbf{a} + \bar{\mathbf{a}}$  where  $\mathbf{a}$  is invariant under  $U$  whereas  $\bar{\mathbf{a}}$  is not. Here  $U$  is generated by the time translations and the spatial rotations, the trajectory  $\mathbf{a}$  is defined by  $\mathbf{a}(t) = \mathbf{x}(t_0)$  and the trajectory  $\bar{\mathbf{a}}$  is defined by  $\bar{\mathbf{a}}(t) = \mathbf{x}(t) - \mathbf{x}(t_0)$ , for all  $t \in \mathfrak{R}$  and a fixed  $t_0 \in \mathfrak{R}$ . This splitting corresponds to the familiar parametrization of the phase space with position and momenta. The fact that the subspace of positions – that is, the subset of trajectories with null momentum – is invariant whereas the one of momenta – that is, the subset of trajectories with null initial position – is not invariant only apparently contradicts the usual transformation of the corresponding classical and quantum operators.

## V The Maxwell theory in Minkowsky space

From here on in the present paper we shall illustrate over the Maxwell field and the abelian Chern-Simon models some aspects of the GAQ formalism we have theorized about in the previous sections. The quantization of the electromagnetic field has been carried further in several papers. In particular, refs. [9] and [8] can both be regarded as natural continuation of the present section. Ref. [7], where the Klein-Gordon field as well as the Proca field are quantized, may also be consulted.

The usual action for the Maxwell field is:

$$S'_{em} = \int d^4x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right\} \quad (5.1)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (5.2)$$

It is, however, more natural, and the best for our purposes, to consider  $F^{\mu\nu}$  and  $A_\mu$  as independent fields, related only by the (now *equations of motion*) eq. (5.2). The action which mirrors this point of view is:

$$S_{em} = \int d^4x \left\{ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} F^{\mu\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) \right\} \quad (5.3)$$

As is well known, the Maxwell action is invariant under the conformal group, which is made up of compositions of the following operations on the space-time:

- a) Space-time translations:  $(ux)^\alpha = x^\alpha + a^\alpha$
- b) Lorentz transformations:  $(ux)^\alpha = \Lambda_\mu^\alpha x^\mu$
- c) Dilatations:  $(ux)^\alpha = e^\lambda x^\alpha$
- d) Special conformal transformations:  $(ux)^\alpha = \frac{x^\alpha + c^\alpha x^2}{1 + 2cx + c^2 x^2}$

The quantizing group for the electromagnetic group is therefore [4]

$$u'' = u' * u \quad \text{Conformal (sub)group}$$

$$\begin{aligned} A''_\mu(x) &= \frac{\partial u^\alpha}{\partial x^\mu} A'_\alpha(ux) + A_\mu(x) \\ &\equiv (S(u^{-1})A')_\mu(x) + A_\mu(x) \end{aligned} \quad (5.4)$$

$$\begin{aligned} F''_{\mu\nu}(x) &= \frac{\partial u^\alpha}{\partial x^\mu} \frac{\partial u^\beta}{\partial x^\nu} F'_{\alpha\beta}(ux) + F_{\mu\nu}(x) + \\ &\equiv (S(u^{-1})F')_{\mu\nu}(x) + F_{\mu\nu}(x) \end{aligned} \quad (5.5)$$

$$\zeta'' = \zeta' \zeta \exp i\Omega_{em} (S(u^{-1})(A'), A) \quad (5.6)$$

where  $S$  is the representation of the conformal group that acts on the electromagnetic vector field. This action is the natural one and means that the potential vector has null conformal weight.

The canonical current is

$$\mathcal{J}_{em}^\mu(g', g)(x) = \frac{1}{2} [F'^{\mu\nu}(x)A_\nu(x) - A'_\nu(x)F^{\mu\nu}(x)] \quad (5.7)$$

## A Non-covariant approach

Let us write down the action (5.3) in terms of the electric field  $\mathbf{E}$  and the potentials  $A_\mu = (A_0, \mathbf{A})$ . In doing so we solve the constraint  $\mathbf{B} = \nabla \times \mathbf{A}$  and place it back into the Lagrangian. This takes the form (save for total derivatives)

$$\mathcal{L}_M = E^i \dot{A}_i - \frac{1}{2} \{ \mathbf{E}^2 + (\nabla \times \mathbf{A})^2 \} + A_0 \partial_i E^i \quad (5.8)$$

The Lagrangian is constrained with  $A_0$  as a Lagrange multiplier and constraint

$$\partial_i E^i = 0 \quad (5.9)$$

The gauge symmetry of this constrained Lagrangian is the usual one:  $A_\mu \longrightarrow A_\mu + \partial_\mu \Lambda$ .

If space-time symmetries are not considered, the quantization of this system with our formalism is straightforward – it amounts to the quantization of three Klein-Gordon fields in a fixed reference frame – and reproduces the quantum theory of the electromagnetic field in the (non explicitly covariant) radiation gauge. The quantizing group is

$$\begin{aligned} (A''_0 &= A'_0 + A_0) \\ \mathbf{A}'' &= \mathbf{A}' + \mathbf{A} \\ \mathbf{E}'' &= \mathbf{E}' + \mathbf{E} \\ \zeta'' &= \zeta \zeta' \exp \frac{i}{2} \int d^3x \sum_{i=1,2,3} \{A'_i E^i - E'^i A_i\} \end{aligned} \quad (5.10)$$

and the subgroup of constraints is  $\tilde{T} = \{(\mathbf{A}, 0, \zeta) / \mathbf{A} = \nabla \Lambda \text{ for some } \Lambda\}$ .

## B Covariant gauge fixing, ghost term and bosonic BRST symmetry

In this section, we construct the quantizing group for the covariant gauge-fixed Maxwell Lagrangian and show how the (bosonic) BRST transformation arises as a one-parameter group of internal symmetries (in ref. [9] the present development was carried further; see [8] for a thorough and unified treatment of the electromagnetic and Proca fields).

Let us therefore consider the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \varphi \partial_\mu A^\mu + \frac{1}{2\lambda} \varphi^2 + \partial^\mu c \partial_\mu c \quad (5.11)$$

where  $\varphi$  is a gauge-fixing Lagrange multiplier and  $c$  are ghost fields. It is straightforward to show that this Lagrangian is invariant under the following (bosonic BRST) symmetry with parameter  $\Lambda$ :

$$\begin{aligned} \delta A_\mu &= \Lambda \partial_\mu c \\ \delta c &= -\frac{1}{2} \varphi \Lambda \\ \delta \varphi &= 0 \end{aligned} \quad (5.12)$$

The finite transformations are given by

$$u_\Lambda(A)_\mu = A_\mu + \partial_\mu c \Lambda - \frac{1}{2} \partial_\mu \varphi \Lambda^2 \quad (5.13)$$

$$u_\Lambda(c) = c - \frac{1}{2} \varphi \Lambda \quad (5.14)$$

The general theory shows us that the quantizing group, which includes the BRST bosonic symmetry but no space-time or internal symmetries, is ( $b \equiv \Lambda$ ):

$$\begin{aligned} A''_\mu &= A'_\mu + A_\mu - \partial_\mu c' b - \frac{1}{2} \partial_\mu \varphi' b^2 \\ \varphi'' &= \varphi' + \varphi \\ c'' &= c' + c + \frac{1}{2} \varphi' b \\ b'' &= b' + b \\ \zeta'' &= \zeta' \zeta \exp i \int_\Sigma d\sigma_\mu J^\mu \end{aligned} \quad (5.15)$$

with

$$\begin{aligned} J^\mu &= \frac{1}{2} \left( (A'_\nu - \partial_\nu c' b - \frac{1}{2} \partial_\nu \varphi' b^2) F^{\mu\nu}(x) - A_\nu(x) F'^{\mu\nu}(x) \right) \\ &+ \frac{1}{2} \left( A^\nu \varphi' - (A'^\nu - \partial^\nu c' b - \frac{1}{2} \partial^\nu \varphi' b^2) \varphi \right) \\ &+ \left( (c' + \frac{1}{2} \varphi' b) \partial^\mu c - (\partial^\mu c' + \frac{1}{2} \partial^\mu \varphi' b) c \right) \end{aligned} \quad (5.16)$$

Now, if we Fourier transform the fields and make use of the equation of motion

$$\varphi = \lambda \partial_\mu A^\mu \quad (5.17)$$

we shall obtain the group law in ref. [9].

## VI The abelian Chern-Simons theory

Let  $\mathcal{M}$  be a three-dimensional manifold which can be decomposed into the form  $\mathcal{M} = \Sigma \times \mathfrak{K}$  with  $\Sigma$  an orientable two-dimensional surface.

The action for an abelian Chern-Simon model is given by [21]:

$$S_{ACS} = \frac{k}{4\pi} \int_{\mathcal{M}} (A \wedge dA) \quad (6.1)$$

where  $A$  is a one-form with takes values on the Lie algebra  $\mathcal{G}$  of some abelian lie group  $G$  [There is in fact a direct generalization of the abelian Chern-Simons theories to higher (odd) dimensions. In these generalization,  $\Sigma$  is a  $2D$  manifold and  $A$  a  $D$ -form for arbitrary natural number  $D$ . Many of the results we present here can be extended to these theories, with one-dimensional quantities replaced with higher dimensional ones]. It is simple to show that  $S_{ACS}$  is invariant under gauge transformation  $A \rightarrow A + d\Lambda$  for any  $\Lambda : \mathcal{M} \rightarrow \mathfrak{g}$ .

It is straightforward to show that the equations of motion and the canonical product are, respectively:

$$dA \equiv F = 0 \quad (6.2)$$

$$\Omega_{ACS}(A', A) = \int_{\Sigma} J = \frac{k}{4\pi} \int_{\Sigma} A' \wedge A \quad (6.3)$$

Thus,  $\mathcal{T}_{ACS} \equiv \mathcal{F}_C$  where  $\mathcal{F}_C$  is the set of all flat connections over  $\mathcal{M}$ .

The exterior derivative commutes with the pullback operator  $*$ . Therefore, if  $f$  is a diffeomorphism of  $M$  and  $A$  and  $A'$  are solutions of the equation of motion (6.2), then  $A' + f^*A$  is also a solution.

All this, together with the general theory, implies that the following composition law defines a group,  $\tilde{G}_{CS}$ , the quantizing group for the abelian Chern-Simons model:

$$\begin{aligned} f'' &= f' \circ f, \quad f, f', f'' \in \text{Diff}_0(\mathcal{M}) \\ A'' &= f^{-1*}A' + A \\ \zeta'' &= \zeta\zeta' \exp \Omega_{CS}(f^{-1*}A', A) \end{aligned} \quad (6.4)$$

The general theory shows that the characteristic subgroup is  $C_{CS} \equiv N_{CS} = \{(f, A, 1) / A = d\Lambda \text{ for some } \Lambda\}$ . The quantum conditions (3.3) imply then that the quantum wave functions should be functions of topological and gauge invariant quantities only. To best deal with these conditions let us remind the reader that all the gauge invariant information of a connection can be extracted from the Wilson loops. These are quantities defined by

$$W(A, \gamma) = \exp \int_{\gamma} A \equiv A(\gamma) \quad (6.5)$$

for any loop  $\gamma$  on  $\mathcal{M}$ . Therefore (the gauge invariant part of) a connection can be seen as an application

$$A : \mathcal{L}_{\mathcal{M}} \rightarrow G / A(\gamma' \circ \gamma) = A(\gamma')A(\gamma) \quad (6.6)$$

where  $\mathcal{L}_{\mathcal{M}}$  is the group of loops on  $\mathcal{M}$  [With a slight abuse of notation, we shall use the same letter for the connection 1-forms as for the applications they define]. Eq. (6.2) also

implies that the diffeomorphisms of  $\mathcal{M}$  which are connected with the identity act trivially on the applications  $A$ . This is not the case with the non-connected diffeomorphisms which give rise to a non-trivial actuation of the modular group  $\text{Diff}(\mathcal{M})/\text{Diff}_0(\mathcal{M})$ . This and others aspects of diffeomorphisms will not be further developed here but rather in a separate study.

For any abelian group  $G$ , there is a natural group structure in the set of all  $A$ :

$$(A' * A)(\gamma) = A'(\gamma)A(\gamma) \quad (6.7)$$

This, of course, is just another expression for the composition law for  $A$  in eq. (6.4).

Now, the equation of motion  $F = 0$  implies that any  $A$  can be considered a function on the homotopy classes  $\{[\gamma]\} = \pi_1(\mathfrak{R} \times \Sigma)$ . Since any loop on  $\mathfrak{R} \times \Sigma$  can be continuously projected onto  $\Sigma$ , we have  $\pi_1(\mathfrak{R} \times \Sigma) = \pi_1(\Sigma)$ .

Any application, and in particular any connection, is completely characterized by its graph. Thus, since any connection is required to satisfy the condition (6.6), it is completely characterized by the images of the elements of a generating subgroup of  $\pi_1(\Sigma)$ . Therefore we have

$$G_{(A)} \equiv G \otimes G \otimes \dots \otimes G \quad (6.8)$$

where  $2g$  is the cardinal of  $\pi_1(\Sigma)$ .

As is well known, the fundamental group  $\pi_1(\Sigma) \equiv \{[\alpha]\}$  of any closed surface  $\Sigma$  is generated by a finite-dimensional subset  $P_\Sigma$ . The generator subset  $P_\Sigma$  can be decomposed into two non-intersecting subsets  $P, \bar{P}$  such that to any  $[\alpha] \in P$  there is associated a unique  $[\bar{\alpha}] \in \bar{P}$  (and the other way round) so that there exists a representative  $\alpha$  of  $[\alpha] \in P$  and a representative  $\bar{\alpha}$  of  $[\bar{\alpha}] \in \bar{P}$  which intersects the one with the other exactly once. This property gives in fact a natural Poisson structure to the fundamental group of orientable surfaces [Although as far as we know this analysis has not been considered in the literature, it would be useful to study, by also considering improper loops; that is, loops that begin and end in the punctures, how much of our analysis can be extend to surfaces  $\Sigma$  with punctures].

For the sake of clarity we shall restrict ourselves to the groups  $\mathfrak{R}$  and  $U(1)$ . In both case,  $\mathfrak{R}$  or  $U(1)$ , any connection is identified with a pair of vector  $\mathbf{a}, \bar{\mathbf{a}}$

$$\mathbf{a} = (a_1, a_2, \dots, a_g), \quad \bar{\mathbf{a}} = (\bar{a}_1, \bar{a}_2, \dots, \bar{a}_g) \quad (6.9)$$

where

$$\begin{aligned} A([\alpha_i]) &= e^{2\pi a_i}, & \text{if } G = \mathfrak{R} \\ A([\alpha_i]) &= e^{i2\pi a_i}, & \text{if } G = U(1) \end{aligned} \quad (6.10)$$

The numbers  $a_i, \bar{a}_i$  are (local) parameterizations of the connection.



In the non-compact case,  $G = \mathfrak{R}$ , there are no constraints. The quantizing group is simply

$$\begin{aligned} \mathbf{a}'' &= \mathbf{a}' + \mathbf{a} \\ \bar{\mathbf{a}}'' &= \bar{\mathbf{a}}' + \bar{\mathbf{a}} \\ \zeta'' &= \zeta' \zeta \exp i\Omega((\mathbf{a}', \bar{\mathbf{a}}'), (\mathbf{a}, \bar{\mathbf{a}})) \end{aligned} \tag{6.11}$$

with

$$\Omega((\mathbf{a}', \bar{\mathbf{a}}'), (\mathbf{a}, \bar{\mathbf{a}})) = \pi k \sum_{i \in P} (\mathbf{a}' \cdot \bar{\mathbf{a}} - \mathbf{a} \cdot \bar{\mathbf{a}}') \tag{6.12}$$

It is merely a Heisenberg-Weyl-like group whose quantization is straightforward.

## A Quantization of the $U(1)$ Chern-Simons model

The quantizing group for the  $U(1)$  Chern-Simons theory is also given by (6.11) with a canonical product of the form:

$$\Omega((\mathbf{a}', \bar{\mathbf{a}}'), (\mathbf{a}, \bar{\mathbf{a}})) = -\pi k \sum_{i \in P} (\mathbf{a}' \cdot \bar{\mathbf{a}} - \mathbf{a} \cdot \bar{\mathbf{a}}') \tag{6.13}$$

This case is more involved and more subtle due to the non-trivial topology of the group  $U(1)$ . This non-trivial topology requires, in the present case, that two numbers  $a_i$  ( $\bar{a}_i$ ) that differ by an integer  $n_i$  ( $\bar{n}_i$ ) have to be considered as equivalent. The equivalence

$$a_i \sim a_i + n_i \quad \bar{a}_i \sim \bar{a}_i + \bar{n}_i, \quad n_i, \bar{n}_i \in Z \tag{6.14}$$

should be seen as a symmetry of the theory under gauge transformations which are not connected to the identity. The commutator of two group elements is given by:

$$[(\mathbf{a}', \bar{\mathbf{a}}', \zeta'), (\mathbf{a}, \bar{\mathbf{a}}, \zeta)] = (0, 0, \exp\{-i2\pi k(\mathbf{a}' \cdot \bar{\mathbf{a}} - \mathbf{a} \cdot \bar{\mathbf{a}}')\}) \tag{6.15}$$

From now on, and for the sake of simplicity, we shall deal with a single coordinate-momentum pair  $(a_i, \bar{a}_i)$  or, what is the same, we shall restrict ourselves to one of the handles ( $g = 1$ ) of the surface. The total Hilbert space  $\mathcal{H}$  will clearly be:

$$\mathcal{H} = \otimes_{i=1, \dots, g} \mathcal{H}_i \tag{6.16}$$

where  $\mathcal{H}_i$  is the Hilbert space associated with the  $i$ th coordinate-momentum pair ( $\equiv$  handle).

The gauge invariance (6.14) is incorporated into the quantum theory by considering the constraint subgroup  $\tilde{T}$  to be the following one:

$$\tilde{T} = \{(n, \bar{n}, \zeta), \quad n, \bar{n} \in Z\} \quad (6.17)$$

We shall consider only the case in which  $k$  is a rational number;  $k = \frac{p}{d}$  with  $p$  and  $d$  relative prime integers,  $d > 0$ .

### Representing the constraint subgroup $\tilde{T}$

The ( $\tilde{T}$ -)characteristic subgroup is

$$C = \{(dn, d\bar{n}, 1), \quad n, \bar{n} \in Z\} \quad (6.18)$$

and it is easy to show that any ( $\tilde{T}$ -)polarization subgroup  $P$  can be written in the form:

$$P \equiv P_{p/q\bar{q}} = \{(qn, \bar{q}\bar{n}, 1), \quad n, \bar{n} \in Z\} \quad (6.19)$$

where  $q, \bar{q}$  are any two natural numbers such that  $q\bar{q} = d$ .

To impose the polarization conditions properly we need to know the general representation of the polarization subgroup. Since these (sub)groups are abelian and finitely generated, its irreducible representations are given by:

$$D((qn, \bar{q}\bar{n}, 1)) = e^{-i2\pi\bar{r}n} e^{i2\pi r\bar{n}}, \quad r \in [0, 1), \quad \bar{r} \in [0, 1) \quad (6.20)$$

The polarization conditions are:

$$\begin{aligned} \Psi_{p/q\bar{q}}(a + qn, \bar{a}, \zeta \exp\{ik\pi qn\bar{a}\}) &= e^{-i2\pi\bar{r}n} \Psi_{p/q\bar{q}}(a, \bar{a}, \zeta), \quad r \in [0, 1) \\ \Psi_{p/q\bar{q}}(a, \bar{a} + \bar{q}\bar{n}, \zeta \exp\{-ik\pi\bar{q}\bar{n}a\}) &= e^{i2\pi r\bar{n}} \Psi_{p/q\bar{q}}(a, \bar{a}, \zeta), \quad r \in [0, 1) \end{aligned} \quad (6.21)$$

These conditions imply that there are only  $q \times \bar{q} = d$  independent wave functions; that is, the Hilbert space has dimension  $d$ . A natural basis is given by:

$$B_{p/q\bar{q}} = \{|l, \bar{l}\rangle\}_{l=0, \dots, q-1, \bar{l}=0, \dots, \bar{q}-1} \quad (6.22)$$

where

$$|l, \bar{l}\rangle (n, \bar{n}, \zeta) = \zeta \delta_{l,n} \delta_{\bar{l}, \bar{n}}, \quad \forall n = 0, \dots, q-1, \bar{n} = 0, \dots, \bar{q}-1. \quad (6.23)$$

The action of the group operators  $P_{(n, \bar{n}, \zeta)}$  in this basis is generated by the following ones:

$$\begin{aligned} P_{(n, 0, 1)} |l, \bar{l}\rangle &= e^{-i\pi kn\bar{l}} |l - n, \bar{l}\rangle, \quad \forall n \in Z \\ P_{(0, \bar{n}, 1)} |l, \bar{l}\rangle &= e^{i\pi k\bar{n}l} |l, \bar{l} - \bar{n}\rangle, \quad \forall \bar{n} \in Z \\ P_{(0, 0, \zeta)} |l, \bar{l}\rangle &= \zeta |l, \bar{l}\rangle, \quad \zeta \in U(1) \end{aligned} \quad (6.24)$$

where the following equivalence conditions have to be taken into account:

$$\begin{aligned} |l - qn, \bar{l}\rangle &= e^{-i\pi k\bar{l}qn} e^{-i2\pi r n} |l, \bar{l}\rangle, \quad \forall n \in Z \\ |l, \bar{l} - \bar{q}\bar{n}\rangle &= e^{i\pi k l \bar{q}\bar{n}} e^{i2\pi r \bar{n}} |l, \bar{l}\rangle, \quad \forall \bar{n} \in Z \end{aligned} \quad (6.25)$$

### Constraint quantization

Once we know the irreducible representations of  $\tilde{T}$  we can carry out the (constraint) quantization of the  $U(1)$  Chern-Simons model.

Let us choose as polarization the subgroup

$$P = \{(a, \bar{a}, 1)/a = 0\} \quad (6.26)$$

The  $P$ -polarized  $U(1)$ -functions are given by:

$$\Psi(a, \bar{a}, \zeta) = \zeta \exp\{ik\pi a \bar{a}\} \varphi(a) \quad (6.27)$$

Now we are ready to impose the constraining conditions. As we already know the irreducible representations of  $\tilde{T}$ , we can straightforwardly impose the constraining conditions on our wave functions. However, since  $\mathcal{H}_{|00\rangle}$ , the vacuum subspace of the representations of  $\tilde{T}$ , is, by construction, invariant under the ( $\tilde{T}$ -)polarization subgroup  $P_{p/q\bar{q}}$  in eq. (6.19), we shall firstly consider the action of this subgroup on the polarized wave functions.

Moreover, since the operators

$$P_{(n, \bar{n}, 1)}, n = 1, \dots, q, \quad \bar{n} = 1, \dots, \bar{q} \quad (6.28)$$

behave, in the representation of  $\tilde{T}$ , as step operators, we can limit ourselves to the vacuum subspace of the  $\tilde{T}$ -representation and generate, afterwards, the whole Hilbert space by repeated application of these step operators.

Therefore, the constraining conditions, which are produced by the ( $\tilde{T}$ -)polarization subgroup (6.19), together with eq. (6.24,6.25), provide us with the full Hilbert space of  $\tilde{T}$ -constrained wave functions.

Thus, let us consider the action, from the left, of the ( $\tilde{T}$ -)polarization subgroup  $P_{p/q\bar{q}}$  on the functions in the vacuum subspace of the representation of  $\tilde{T}$ . This gives rise to the following two conditions:

$$\begin{aligned} \Psi(qn + a, \bar{a}, \zeta \exp\{-ik\pi qn\bar{a}\}) &= e^{-i2\pi r n} \Psi(a, \bar{a}, \zeta), \quad r \in [0, 1) \\ \Psi(a, \bar{q}\bar{n} + \bar{a}, \zeta \exp\{ik\pi \bar{q}\bar{n}a\}) &= e^{i2\pi r \bar{n}} \Psi(a, \bar{a}, \zeta), \quad r \in [0, 1) \end{aligned} \quad (6.29)$$

The first condition implies for polarized wave functions

$$\varphi(a + qn) = e^{-i2\pi\bar{r}n}\varphi(a) \quad (6.30)$$

The other condition implies that the wave functions  $\varphi$  are supported only on the connections  $a$  that obey

$$\frac{p}{q}a - r \in Z \quad (6.31)$$

Therefore the wave functions  $\varphi$  are of the form

$$\varphi(a) = \sum_{s \in Z} B_s \delta\left(\frac{p}{q}a - r - s\right) \quad (6.32)$$

where the numbers  $B_s$  are not arbitrary but are required to satisfy the quasiperiodicity condition

$$B_{s+p} = e^{-i2\pi\bar{r}} B_s \quad (6.33)$$

Therefore, in the sum (6.32) there are only  $p$  independent complex numbers.

Thus, the Hilbert subspace  $\mathcal{H}_{|00\rangle}$  has dimension  $p$ . Now if we repeatedly apply to this subspace the operators  $P_{(n,\bar{n},1)}$ , which generate the whole  $T$ , we generate a Hilbert space  $\mathcal{H}_{\frac{p}{q\bar{q}}}$  with finite dimension  $p \times q \times \bar{q} = p \times d$ . We have thus recovered the well known fact that compact phase spaces give rise to finite-dimensional Hilbert spaces [22].

The good operators split naturally into two subgroups: firstly, the subgroup  $\tilde{B}_{|00\rangle}$  which is made with the operators that preserve the subspace  $\mathcal{H}_{|00\rangle}$ , and, secondly, the subgroup  $\tilde{T}$  which transforms the subspace  $\mathcal{H}_{|00\rangle}$  into the subspaces  $\mathcal{H}_{|l,\bar{l}\rangle}$ .

It is easy to show that the subgroup  $\tilde{B}_{|00\rangle}$  is the maximal subgroup of  $\tilde{G}$  which obeys

$$Ad(\tilde{G})[P_{\tilde{T}}, \tilde{B}_{|00\rangle}] \subset P \quad (6.34)$$

In the particular case at hand this condition reduces to

$$[P_{\tilde{T}}, \tilde{B}_{|00\rangle}] = \{1_{\tilde{G}}\} \quad (6.35)$$

and implies

$$\tilde{B}_{|00\rangle} = \left\{ \left( \frac{q}{p}n, \frac{\bar{q}}{p}\bar{n}, \zeta \right) / n, \bar{n} \in Z \right\} \quad (6.36)$$

Therefore, the subgroup  $\tilde{B}$  of good operators is given by

$$\begin{aligned} \tilde{B} &\equiv \tilde{B}_{|00\rangle} + \tilde{T} \\ &= \left\{ \left( \frac{q}{p}n, \frac{\bar{q}}{p}\bar{n}, \zeta \right) / n, \bar{n} \in Z \right\} \oplus \left\{ (m, \bar{m}, \zeta) / m, \bar{m} \in Z \right\} \\ &= \left\{ \left( \frac{n}{p}, \frac{\bar{n}}{p}, \zeta \right) / n, \bar{n} \in Z \right\} \end{aligned} \quad (6.37)$$

Therefore, imposing the condition that the Hilbert space must be in a single irreducible representation of  $\tilde{T}$  forces us to only represent a subgroup  $\tilde{B}$  (in the present case, discrete) of the whole  $\tilde{G}$ . Applying to this Hilbert space operators which are not in  $\tilde{B}$  will produce states in different representations of  $\tilde{T}$ .

The operators which are not in  $\tilde{B}$  can be classified as

$$P_{(s', \bar{s}', \zeta)} \quad \text{with } s', \bar{s}' \in (0, \frac{1}{p}) \quad (6.38)$$

Now, it is easy to show that

$$P_{(s', \bar{s}', 1)} \mathcal{H}_{\frac{p}{q\bar{q}}}^{r, \bar{r}} = \mathcal{H}_{\frac{p}{q\bar{q}}}^{r + \frac{p}{q}s', \bar{r} + \frac{p}{q}\bar{s}'} \quad (6.39)$$

Therefore, the Hilbert space  $\mathcal{H}_{\frac{p}{q\bar{q}}}$  which represents the whole  $\tilde{G}$  splits into a (continuum) sum

$$\mathcal{H} = \bigoplus_{s, \bar{s}} \mathcal{H}_{\frac{p}{q\bar{q}}}^{r+s, \bar{r}+\bar{s}}, \quad s \in (0, \frac{1}{q}), \quad \bar{s} \in (0, \frac{1}{\bar{q}}) \quad (6.40)$$

Finally, there is a noteworthy point to be discussed. The approach to the quantum theory in the present subsection has led us to an irreducible representation  $D_{\frac{p}{q\bar{q}}}^{r, \bar{r}}$  of a subgroup of good operators  $\tilde{B}$ . Instead, we could have determined this subgroup  $\tilde{B}$  firstly, and have quantized it afterwards (by applying the algebraic GAQ formalism). It is interesting to point out that in this way we would have obtained representations of  $\tilde{B}$  which would be different from the ones we have actually obtained. These representations can arise, for instance, by taking as  $\tilde{T}$ -polarization  $P_{\frac{p}{q\bar{q}}} = \{(qn, \bar{q}\bar{n}, 1)/n, \bar{n} \in Z\}$  and as polarization

$$P_{\frac{p}{q', \bar{q}'}}^{u, \bar{u}} = \{(q' \frac{n}{u}, \bar{q}' \frac{\bar{n}}{\bar{u}}, 1)/n, \bar{n} \in Z\} \quad (6.41)$$

where  $u \in N, \bar{u} \in Z/u\bar{u} = p, q', \bar{q}' \in N/q'\bar{q}' = d$  and, in general,  $q'$  ( $\bar{q}'$ ) might be taken to be different from  $q$  ( $\bar{q}$ ) (the representations we have found in the present subsection are the ones with  $u = p, \bar{u} = 1$  and  $q' = q, \bar{q}' = \bar{q}$ ). This way of proceeding would constitute a refined version of the approaches in which the constraints are imposed firstly and the quantization is carried out afterwards.

## VII Final comments. Perspectives

We have further developed the algebraic and configuration-space pictures of the GAQ formalism of group quantization. We have combined both in order to make a comprehensive and completely general analysis of the theory of linear fields. We have shown that, for linear fields, the formalism is extremely powerful and this power is best employed when

the pictures just mentioned are combined. It has also been shown that the formalism is specially well suited to deal with topological issues (in this respect see also [11]).

We would like to remark here that the GAQ formalism can, in principle, be applied to any group. It gives as a result a quantum dynamical system. However, for an arbitrary group, it is unclear what physical interpretation, if any, the resulting dynamical system will have. On the other hand, classical systems with a clear physical interpretation are commonly described, not by a group, but by a Lagrangian or a set of differential equations. How to go from Lagrangian ( $\sim$  differential equations) to a quantizing group (and the other way round) is an important question in the GAQ formalism but not much is known yet about its general answer. The present paper, however, addresses this question for the case of linear fields. It turns out that for linear fields the set of solutions of the equations of motion – that is, the (covariant) phase space of the theory [19, 14] –, when extended, is a suitable quantizing group.

A particularly attractive direction of development is, therefore, towards non-linear fields. However, there appear to be obstructions for the phase space of non-linear fields to have a group structure. In particular, ref. [4] presented indications that for non-abelian current groups with group law of a pointwise type, any equation of motion which preserves the group structure would have to be first order in derivatives of the space-time co-ordinates. A rigorous theorem is, however, still lacking and, after all, first-order equations may give plenty of room for interesting developments as recent studies, relevant to our approach, indicate [23]. On the other hand, constraint quantization might be used to circumvent the problem of not having a group structure in the phase space of the theory. In addition to all this, it was also shown in ref. [4] that for some current groups with group laws of a non-pointwise type, we can actually find higher-order differential equations which preserve them.

Let us finally consider the case of non-linear gauge fields. For linear gauge fields, if  $A, A' : \mathcal{L}_M \rightarrow G$  are connection and we define a composition law  $*$  by means of the equality

$$(A' * A)(\gamma) = A'(\gamma)A(\gamma), \quad (7.1)$$

then  $A' * A$  is also a connection. As we have shown the composition law  $*$  is also compatible with the equations of motion, and thus defines the natural group law for the theory. However, when  $G$  is non-abelian,  $A'' = A' * A$  defined by eq. (7.1) does not satisfy the condition

$$A''(\gamma' \circ \gamma) = A''(\gamma')A''(\gamma) \quad (7.2)$$

and thus  $A''$  is not a connection. Therefore, a “naive” extension of the configuration-space approach to non-abelian gauge fields is problematic even before the equations of motion are considered.

Summarizing we would say that, because of obstructions which arise, the analysis we

have performed in this paper for linear fields cannot be straightforwardly extended to non-linear fields. However, the real importance of the obstructions is still not clear and further investigations are in order.

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