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Coupling Nonlinear Sigma-Matter to Yang-Mills Fields: Symmetry Breaking Patterns

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Abstract

We extend the traditional formulation of Gauge Field Theory by incorporating the (non-Abelian) gauge group parameters (traditionally simple spectators) as new dynamical (nonlinear-sigma-model-type) fields. These new fields interact with the usual Yang-Mills fields through a generalized minimal coupling prescription, which resembles the so-called Stueckelberg transformation [1], but for the non-Abelian case. Here we study the case of internal gauge symmetry groups, in particular, unitary groups $U(N)$. We show how to couple standard Yang-Mills Theory to Nonlinear-Sigma Models on cosets of $U(N)$: complex projective, Grassman and flag manifolds. These different couplings lead to distinct (chiral) symmetry breaking patterns and *Higgs-less* mass-generating mechanisms for Yang-Mills fields.

1 Introduction

Although there has been many successful applications Non-Linear Sigma Models (NLSM) in (Quantum Gauge) Field Theory, String Theory and Statistical Mechanics, their basic role in Fundamental Physics is still rather unexplored. Generally speaking, NLSM consists of a set of coupled scalar fields $\varphi^a(x^\mu), a = 1, \ldots, D$, in a d-dimensional Minkowski spacetime $M, \mu = 0, 1, 2, \ldots, d-1$, with the action

$$S_\sigma = \lambda \int_M d^d x g_{ab}(\varphi) \partial^\mu \varphi^a \partial_\mu \varphi^b, \tag{1.1}$$

where $\partial^\mu = \eta^{\mu\nu}\partial_\nu, \partial_\nu = \partial/\partial x^\nu$, $\eta = \mathrm{diag}(+, -, \ldots, -)$ the Minkowski metric and λ a coupling constant. The field theory (1.1) is called the NLSM with metric $g_{ab}(\varphi)$ (usually a positive-definite field-dependent matrix). The fields φ^a themselves can also be considered as the coordinates of an internal Riemannian manifold Σ with metric g_{ab}. In particular, we shall consider the case in which Σ is a (semisimple) Lie group manifold G.

The relevance of NLSM in Quantum (Gauge) Field Theory originates from the paramount importance of symmetry principles in fundamental physics. From the String Theory point of view, the two-dimensional space M represents a string world sheet, whereas g_{ab} is identified with the 'truly' spacetime metric representing the gravitational background where the string propagates. In two dimensions we also have (infinite) conformal symmetry and the possibility of adding new Wess-Zumino terms to our NLSM.

NLSM also provides a useful field-theoretical laboratory for studying some two-dimensional, exactly solvable systems on a lattice, such as the Ising model of the Heisemberg antiferromagnetism, in statistical mechanics. Some particular $O(n)$-invariant two-dimensional NLSM are frequently used in condensed matter physics in connection with antiferromagnetic spin chains and the quantum Hall effect. Also, the effective Lagrangian for superfluid He 3 is described by a NLSM. In four dimensions, pions and nucleons are described by a (Skyrme) NLSM model, as solitonic solutions ('skyrmions').

We shall concentrate in the role that NLSM plays in the *spontaneous symmetry breaking* mechanism, which is crucial for phenomenological applications of QFT like the Higgs-Kibble mechanism in the Standard Model of Strong and Electro-Weak interactions, by means of which some vector bosons acquire mass in a renormalizable way. According to the well known Goldstone theorem (see e.g. [2]), there are as many massless (Nambu-Goldstone) particles as broken symmetry generators. If these Nambu-Goldstone fields are scalars, their low energy effective action often appears to be a NLSM. Usually, Goldstone bosons are eliminated from the theory by gauge fixing.

Despite the undoubted success of the Standard Model in describing strong and electroweak interactions, a real (versus artificial) mechanism of mass generation is still lacking. Needless to say that the discovery of a Higgs boson (a quantum vibration of an abnormal Higgs vacuum) would be of enormous importance; nevertheless, at present, no dynamical basis for the Higgs mechanism exists and, as said, it is purely phenomenological. It is true that there is actually nothing inherently unreasonable in the idea that the state of minimum energy (the vacuum) may be one in which some field quantity has a non-zero expectation value; in fact, many examples in condensed-matter physics display this feature. Nevertheless, it remains conjectural whether something similar actually happens in the weak interaction case. Also, the ad hoc introduction of extra (Higgs) scalar fields in the theory to provide mass to the vector bosons could be seen as our modern equivalent of those earlier mechanical contrivances populating the plenum (the ether), albeit very subtly. As in those days, new perspectives are necessary to explain why it is really not indispensable to look at things in this way at all.

One of the purposes of this paper is to provide a new formulation of gauge theory in which the mass of gauge vector fields enters the theory in a 'natural' way without damaging gauge invariance. In this sense we shall generalize the so-called Stueckelberg model for electrodynamics [1] to account for a Higgs-less mass-generating mechanism for gauge fields. In our new approach we shall nearly restrict the external information to the symmetry group and, therefore, the group parameters, described by Lagrangians of NLSM type, will acquire dynamical content as 'exotic' matter fields.[1]. By the time being, we shall not enter into the possible physical meaning of these σ-matter fields. Just to mention that,

[1]It is worth pointing out that the incorporation of group parameters into some dynamical framework has already been considered in other contexts, for example, in [3]. There, conventional Eulerian fluid mechanics is extended to encompass the possibility of describing a plasma state of quarks and gluons produced as the result of high-energy collisions of heavy nuclei [4] due to the fact that such fluid may posses degrees of freedom indexed by group variables.

when this idea is applied to the Weyl group (Poincaré+dilations), and the corresponding gauge gravitational theory is developed, σ-fields appear to be a natural source to account for some sort of *dark matter* intrinsically related to the gauge-group parameter associated with scale transformations [5].

The underlying mathematical framework relies on the idea of *jet-gauge group* [6] introduced in Sec. 2. In Sec. 3 we revise the Lagrangian formalism on jet-gauge groups and generalize the well known Utiyama theorem [7] which provides a prescription to 'minimally' couple Yang-Mills fields to σ-matter fields. In Sec. 4 we discuss several *chiral* gauge symmetry breaking patterns related to different mass matrices. Sec. 5 is devoted to some comments on the quantization of this model.

2 Jet-Gauge Groups and Nonlinear σ-Fields

Definition 1. (Gauge group) Let G be a (matrix) Lie group (the "rigid" group) and M the Minkowski space-time (or any other orientable space-time manifold). The gauge group $G(M)$ ("local" or current group) is the set of mappings

$$G(M) = \{g : M \to G,\ x \mapsto g(x)\} = \mathrm{Map}(M, G) \tag{2.1}$$

with point-wise multiplication $(gg')(x) = g(x)g'(x)$. The corresponding Lie algebra $\mathcal{G}(M)$ is the tensor product $\mathcal{F}(M)\otimes\mathcal{G} = \{f^a X_a,\ a = 1, \ldots, \dim G\}$, where $\mathcal{F}(M)$ is the multiplicative algebra of (C^∞) differentiable functions f on M, and $\mathcal{G}$ is the Lie algebra of G with generators X_a. The commutation relations of this local algebra are $[f\otimes X, h\otimes Y] = fh\otimes[X, Y]$ since, for internal symmetries, the "rigid" group G does not act on the space-time manifold M.

We shall mainly consider special unitary groups $G = SU(N)$, the Lie algebra of which $\mathcal{G} = su(N) = \langle X_a,\ a = 1, \ldots, N^2 - 1\rangle$ can be expanded in terms of traceless hermitian matrices, X_a, whose Lie-algebra commutators $[X_a, X_b] = C_{ab}^c X_c$ are given in terms of totally antisymmetric structure constants C_{ab}^c. The generators X_a can also be chosen to be orthogonal in the sense $\mathrm{Tr}(X_a X_b) = \delta_{ab}$. A given group element $g \in G$ can be written in terms of a (local) system of canonical coordinates $\{\varphi^a, a = 1, \ldots, \dim(G)\}$ at the identity element as $g = e^{i\varphi^a X_a}$. Thus, the composition group law $g'' = g'g$ can also be locally written as:

$$\varphi''^a = \varphi'^a + \varphi^a + \frac{1}{2} C_{bc}^a \varphi'^b \varphi^c + \text{higher-order terms}, \tag{2.2}$$

by using the Baker-Campbell-Hausdorff formula. Let us denote an element $g(x) \in G(M)$ simply by its coordinates $\varphi^a(x)$ (the **exotic matter σ-fields**) .

Definition 2. (Jet prolongations) Given a gauge group $G(M)$, we define the group $J^1(G(M))$ of the 1-jets of $G(M)$ as the quotient:

$$J^1(G(M)) \equiv G(M) \times M / \sim^1$$

where the equivalence relation $\sim^1$ is defined as follows:

$$(\varphi, x) \sim^1 (\varphi', x') \Longleftrightarrow \begin{cases} x = x', \\ \varphi(x) = \varphi'(x), \\ \partial_\mu \varphi(x) = \partial_\mu \varphi'(x) \end{cases}$$

for all (φ, x), (φ', x') belonging to $G(M) \times M$. This definition may be easily extended from order $r = 1$ to r-th order. A coordinate system for $J^1(G(M))$ is $\{x^\mu, \varphi^a, \varphi^a_\mu\}$.

The formal definition of $J^1(G(M))$ is fully analogous to that of the $(q^i, \dot{q}^j)$ phase-space in Lagrangian Mechanics, or $(\psi^\alpha, \psi^\beta_\mu)$ in Lagrangian Field Theory, when one desires to vary independently coordinates and velocities (momenta) according to the modified Hamilton principle.

Definition 3. (Jet-gauge group) We define the (infinite-dimensional) jet-gauge group $G^1(M)$ as the set of mappings from M into $J^1(G(M))$:

$$G^1(M) \equiv \mathrm{Map}(M, J^1(G(M))).$$

It is parametrized by the coordinate system $\{\varphi^a(x), \varphi^a_\mu(x)\}$ and has the composition group law (2.2), at each point $x \in M$, together with:

$$\varphi''^a_\mu = \varphi'^a_\mu + \varphi^a_\mu + \frac{1}{2} C^a_{bc} \varphi'^b_\mu \varphi^c + \frac{1}{2} C^a_{bc} \varphi'^b \varphi^c_\mu + \text{higher order.} \tag{2.3}$$

In this formalism, φ^a_μ are essentially the standard gauge vector potentials (**Yang-Mills fields**) A^a_μ or connections, the actual relationship being:

$$A^a_\mu \equiv \theta^a_b(\varphi) \varphi^b_\mu \,, \tag{2.4}$$

where $\theta^a_b(\varphi)$ is the (non-constant) invertible matrix defining the (left-) invariant canonical 1-form on the group

$$\theta^{L^a} = \theta^a_b(\varphi) d\varphi^b = \mathrm{Tr}(i g^{-1} dg X_a), \tag{2.5}$$

dual to the (left-invariant) vector fields

$$X^L_a = X^b_a(\varphi) \frac{\partial}{\partial \varphi^b}, \quad X^b_a(\varphi) \equiv \frac{\partial \varphi''^b(\varphi', \varphi)}{\partial \varphi^a}|_{\varphi=0, \varphi'=\varphi}, \tag{2.6}$$

that is: $\theta^a_b X^b_c = \delta^a_c$. Writing then $A_\mu = i g^{-1} g_\mu$, the group law (2.3) for Yang-Mills fields is simply:

$$A''_\mu(x) = g^{-1}(x) A'_\mu(x) g(x) + A_\mu(x).$$

Note that φ^a_μ comprises all possible values of derivatives of φ^a, but in general $\varphi^a_\mu \neq \partial_\mu \varphi^a$. That is, not all Yang-Mills fields A_μ are "pure gauge", $\theta_\mu = i g^{-1} \partial_\mu g$, except for the particular inmersion (1-jet-extension) of the gauge group $G(M)$ into the jet-gauge group:

$$j^1 : G(M) \to G^1(M), \ \varphi \mapsto j^1(\varphi) = (\varphi^a, \partial_\mu \varphi^a). \tag{2.7}$$

3 Lagrangian Formalism on Jet-Gauge Groups: Generalized Utiyama Theorem

In the standard formulation of gauge theories, the well-known Minimal Coupling Principle (or Utiyama theorem [7], see also [6]) for internal gauge symmetries establishes that if the action of some matter fields ψ^α, $\alpha = 1, ..., n$

$$S = \int \mathcal{L}_{\mathrm{m}}(\psi^\alpha, \partial_\mu \psi^\alpha) d^4x,$$

is invariant under a rigid internal Lie group G, then the modified action

$$\widehat{S} = \int [\mathcal{L}_{\rm m}(\psi^\alpha, D_\mu\psi^\alpha) + \mathcal{L}_0(F^a_{\mu\nu})]d^4x$$

is invariant under the gauge group $G(M)$, where

$$D_\mu\psi^\alpha \equiv \partial_\mu\psi^\alpha - eA^a_\mu(X_a)^\alpha_\beta\psi^\beta$$

is usually known as the covariant derivative (e is a coupling constant), and

$$F^a_{\mu\nu} \equiv \frac{1}{e}[D_\mu,\, D_\nu]^a = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \frac{e}{2}C^a_{bc}(A^b_\mu A^c_\nu - A^b_\nu A^c_\mu)$$

is known as curvature of the connection A^a_μ.

Here we shall treat the gauge group parameters $\varphi^a \in G(M)$ as "exotic matter" σ-fields, so that our configuration space is now $J^1(G(M))$, with coordinates $\{x^\mu, \varphi^a, A^a_\mu\}$, and Lagrangians are accordingly functions

$$\mathcal{L}(x^\mu, \varphi^a, A^a_\mu; \partial_\nu\varphi^a, \partial_\nu A^a_\mu).$$

We shall proceed to formulate some sort of Minimal Coupling Principle on $J^1(G(M))$:

Theorem 1. ***(Generalized Utiyama's Theorem)*** *If the action*

$$S_\sigma = \int \mathcal{L}_\sigma(\varphi^a, \partial_\mu\varphi^a)d^4x,$$

of the "exotic matter" σ-fields φ^a, $a = 1, ..., \dim G$ is invariant under the global (rigid) internal Lie group G, i.e.

$$\delta^{\rm global}_a \mathcal{L}_\sigma(\varphi^b, \partial_\mu\varphi^b) \equiv X^b_a \frac{\partial\mathcal{L}_\sigma}{\partial\varphi^b} + \frac{\partial X^b_a}{\partial\varphi^c}\partial_\mu\varphi^c \frac{\partial\mathcal{L}_\sigma}{\partial(\partial_\mu\varphi^b)} = 0,$$

then the modified action $S_{\rm tot} = \widetilde{S}_\sigma + S_0$, with

$$\widetilde{S}_\sigma \equiv \int \mathcal{L}_\sigma(\varphi^a, \widetilde{D}_\mu\varphi^a)d^4x, \;\; S_0 = \int \mathcal{L}_0(F^a_{\mu\nu})d^4x, \tag{3.1}$$

is invariant under the gauge (local) group $G(M)$, where

$$\widetilde{D}_\mu\varphi^a \equiv \partial_\mu\varphi^a - eA^b_\mu X^a_b$$

is the "covariant derivative" for σ-fields.

Proof. As the local invariance of S_0 is already well-known in the standard gauge theory, we shall focus on the local invariance of $\widetilde{S}_\sigma$. We must prove that the new Lagrangian describing the gauge-group parameters as well as their interaction with the gauge fields A^a_μ (according to the prescription of Minimal Coupling, i.e. supposing that the group parameters interact only with the gauge fields and not with their derivatives),

$$\widetilde{\mathcal{L}}_\sigma(\varphi^a, \partial_\mu\varphi^a, A^a_\mu) \equiv \mathcal{L}_\sigma(\varphi^a, \partial_\mu\varphi^a - eA^b_\mu X^a_b),$$

is invariant under the gauge group $G(M)$ in the sense that

$$\delta\widetilde{\mathcal{L}}_\sigma(\varphi^a, \partial_\mu\varphi^a, A^a_\mu) \equiv f^a X^b_a \frac{\partial\widetilde{\mathcal{L}}_\sigma}{\partial\varphi^b} + \left(f^a \frac{\partial X^b_a}{\partial\varphi^c}\partial_\mu\varphi^c + X^b_a \frac{\partial f^a}{\partial x^\mu}\right)\frac{\partial\widetilde{\mathcal{L}}_\sigma}{\partial(\partial_\mu\varphi^b)}$$
$$+ \left(gf^b C^a_{bc} A^c_\mu + \frac{\partial f^a}{\partial x^\mu}\right)\frac{\partial\widetilde{\mathcal{L}}_\sigma}{\partial A^a_\mu} = 0,$$

where f^a denote gauge-algebra parameters.

Let us consider the following change of variables:

$$\begin{aligned} \phi^a &= \varphi^a, \\ \phi^a_\mu &= \partial_\mu\varphi^a - eA^b_\mu X^a_b, \\ B^a_\mu &= A^a_\mu. \end{aligned}$$

Then, the partial derivatives related to the old variables can be expressed in terms of the new ones:

$$\begin{aligned} \frac{\partial}{\partial\varphi^a} &= \frac{\partial}{\partial\phi^a} - B^c_\mu \frac{\partial X^b_c}{\partial\phi^a}\frac{\partial}{\partial(\phi^b_\mu)}, \\ \frac{\partial}{\partial(\partial_\mu\varphi^a)} &= \frac{\partial}{\partial(\phi^a_\mu)}, \\ \frac{\partial}{\partial A^a_\mu} &= \frac{\partial}{\partial B^a_\mu} - X^b_a \frac{\partial}{\partial(\phi^b_\mu)}. \end{aligned}$$

After this change of variables it is now straightforward to arrive at

$$\delta\widetilde{\mathcal{L}}_\sigma = f^a \delta^{\rm global}_a \mathcal{L}_\sigma(\phi^a, \phi^a_\mu) = 0,$$

equality which follows from the hypothesis of invariance of the σ-matter action under the global group. ■

As a consequence, the new "minimal coupling" $\partial_\mu\varphi^a \to \partial_\mu\varphi^a - eA^b_\mu X^a_b$ now occurs in an affine manner. Indeed, the matrix X^a_b is invertible, and therefore the minimal coupling above is proportional to

$$\theta^a_b[\partial_\mu\varphi^b - eX^b_c A^c_\mu] \equiv \theta^{La}_\mu - eA^a_\mu,$$

where θ^{La} is the canonical (left-)invariant 1-form in (2.5). The new minimal coupling, when written in the form $\theta^L - A$, strongly suggests the introduction of "exotic matter" of the σ-model type:

$$\mathcal{L}_\sigma = \frac{\lambda^2}{2}{\rm Tr}_G(\theta^L_\mu \theta^{L\mu}) = -\frac{\lambda^2}{2}{\rm Tr}_G(g^{-1}\partial_\mu g g^{-1}\partial^\mu g)\,. \quad (3.2)$$

This Lagrangian is G-invariant (left- and right-invariant, that is, chiral) and the *new* minimal coupling gives rise to

$$\widetilde{\mathcal{L}}_\sigma = \frac{\lambda^2}{2}\mathrm{Tr}_G[(\theta^L_\mu - eA_\mu)(\theta^{L\mu} - eA^\mu)], \tag{3.3}$$

which is *gauge-invariant* even though it contains *mass terms* $\frac{\lambda^2}{2}e^2\mathrm{Tr}_G[A_\mu A^\mu]$ for the A_μ fields, a piece which spoils gauge invariance in the traditional framework of Yang-Mills theories.

4 Chiral Symmetry Breaking Patterns

The virtue of a kinetic term like (3.2) is the two-side symmetry, that is, *chirality*. In fact, any function of θ^L is of course left-invariant, but only a scalar sum on all the group indices $a = 1, \ldots, \dim(G)$ can provide also right invariance. Therefore, several chiral symmetry breaking patterns are possible by considering a *partial trace* σ-Lagrangian

$$\mathcal{L}^{(\lambda)}_\sigma = \frac{1}{2}\mathrm{Tr}^{(\lambda)}_G(\theta^L_\mu\theta^{L\mu}) \equiv \frac{1}{2}\mathrm{Tr}_G(\theta^{L\mu}_\lambda\theta^L_{\lambda\,\mu}), \tag{4.1}$$

where we have defined

$$\theta^L_\lambda \equiv [\theta^L, \lambda] \tag{4.2}$$

the ‘projection’ of θ^L by the *mass matrix* $\lambda = i\lambda^a H_a$, with $\lambda^a \in \mathbb{R}$ and H_a the Lie algebra generators of the toral (Cartan, maximal Abelian) subgroup H of G (see later on this section for an example). Defining

$$\Lambda \equiv g\lambda g^{-1},\ \ g \in G,$$

(the adjoint action of G on its Lie algebra) we have an alternative way of writing (4.1) as

$$\mathcal{L}^{(\lambda)}_\sigma = \frac{1}{2}\mathrm{Tr}_G(\partial_\mu\Lambda\partial^\mu\Lambda),$$

which is singular due to the constraint $\mathrm{Tr}_G(\Lambda^2) = \mathrm{Tr}_G(\lambda^2) = \lambda_a\lambda^a =$constant. Introducing Lagrange multipliers, the equations of motion read:

$$\partial_\mu\partial^\mu\Lambda = -\frac{\mathrm{Tr}_G(\partial_\mu\Lambda\partial^\mu\Lambda)}{\mathrm{Tr}_G(\Lambda^2)}\Lambda, \tag{4.3}$$

which describe a set of coupled Klein-Gordon-like fields $\phi^a = \mathrm{Tr}_G(\Lambda X_a)$ with variable mass $m^2 = \mathrm{Tr}_G(\partial_\mu\Lambda\partial^\mu\Lambda)/\mathrm{Tr}_G(\Lambda^2)$.

Let us explicitly consider the case of the unitary group $G = U(N)$. We shall take, as the Lie algebra generators X_a, the step operators $X_{\alpha\beta}$ defined by the usual matrix elements:

$$(X_{\alpha\beta})_{\gamma\rho} = \delta_{\alpha\gamma}\delta_{\beta\rho},\ \ \alpha,\beta,\gamma,\rho = 1,\ldots,N, \tag{4.4}$$

fulfilling the commutation relations:

$$[X_{\alpha\beta}, X_{\gamma\rho}] = \delta_{\gamma\beta}X_{\alpha\rho} - \delta_{\alpha\rho}X_{\gamma\beta}, \tag{4.5}$$

and the usual orthogonallity relations:

$$\mathrm{Tr}(X_{\alpha\beta}X_{\gamma\rho}) = \delta_{\alpha\rho}\delta_{\gamma\beta}. \tag{4.6}$$

Note that the step generators $X_{\alpha\beta}$ are not hermitian but $X^\dagger_{\alpha\beta} = X_{\beta\alpha}$, where $X^\dagger$ denotes hermitian conjugate. This fact introduces some minor modifications with respect to the general theory exposed before. For example, the canonical left-invariant 1-form θ^L can be written in this Lie-algebra basis as (we shall drop the upper-script L for convenience):

$$\theta_\mu = \sum_{\alpha,\beta=1}^{N} \theta_\mu^{\alpha\beta} X_{\alpha\beta}, \tag{4.7}$$

with $\theta^{\alpha\beta} = \bar\theta^{\beta\alpha}$ in order to make $\theta^\dagger = \theta$ (hermitian). The mass matrix λ is now

$$\lambda = i\sum_{\alpha=1}^{N} \lambda^\alpha X_{\alpha\alpha}, \tag{4.8}$$

where the complex i has been introduced in order to make the projected 1-form

$$\theta_\lambda = [\theta, \lambda] = -i\sum_{\alpha,\beta=1}^{N} \theta^{\alpha\beta}(\lambda_\alpha - \lambda_\beta)X_{\alpha\beta} \tag{4.9}$$

hermitian too.

When minimally coupled, like in (3.3), the partial trace σ-Lagrangian (4.1) only assigns mass $m_{\alpha\beta} = e^2(\lambda_\alpha - \lambda_\beta)^2$ to those Yang-Mills fields $A^{\alpha\beta}$ living on a certain coset G/G_λ of the group G, where G_λ represents the 'unbroken' chiral symmetry subgroup. Indeed, the Lagrangian $\mathcal{L}_\sigma^{(\lambda)}$ is *left*-invariant under the whole group G, but right-invariant under the unbroken subgroup G_λ only.

For $G = U(N)$ we can consider several symmetry breaking patterns according to distinct mass matrix λ choices:

1. For the case $\lambda_\alpha \neq \lambda_\beta, \forall \alpha, \beta = 1\ldots, N$ the unbroken symmetry is $G_\lambda = U(1)^N$, so that we give mass to all of $N(N-1)/2$ charged (complex) Yang-Mills fields $A^{\alpha\beta}, \alpha > \beta$ (the analogue of $W_\pm$ in $U(2)$ invariant electro-weak model [2]) living on the *flag manifold* (coset) $\mathbb{F}_N = G/G_\lambda = U(N)/U(1)^N$. The neutral (not charged) vector bosons $A^{\alpha\alpha}$ remain massless.

2. For $\lambda_\alpha = \lambda_\beta, \forall \alpha, \beta = 2, \ldots, N$ the unbroken symmetry is $G_\lambda = U(N-1) \times U(1)$, so that we have $N-1$ massive charged Yang-Mills fields $A^{1\alpha}, \alpha > 1$ living on the *complex projective* space $\mathbb{C}P^{N-1} = U(N)/U(N-1) \times U(1)$, in addition to $(N-1)(N-2)/2$ massless charged vector bosons $A^{\alpha\beta}, \alpha \neq \beta \neq 1$ and N massless neutral vector bosons $A^{\alpha\alpha}$.

3. For other choices like:

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{N_1} \neq \lambda_{N_1+1} \neq \cdots \neq \lambda_{N-N_2} = \cdots = \lambda_N$$

   the unbroken symmetry group is $G_\lambda = U(N_1) \times U(N_2) \times U(1)$ giving $N_1(N_1 - 1)/2 + N_2(N_2 - 1)/2$ massless charged vector bosons, N massless neutral vector bosons and massive charged vector bosons corresponding to the *complex Grasmannian* $\mathbb{C}G(N_1, N_2) = U(N)/U(N_1) \times U(N_2) \times U(1)$ (see [8] for suitable coordinate systems on these coset spaces).

Note that this 'partial trace' mechanism always keeps the N neutral vector bosons $A^{\alpha\alpha}$ massless. However, we could always supply mass to the neutral vector boson Z_0, related to the central generator $H_0 = \sum_{\alpha=1}^{N} X_{\alpha\alpha}$ (which commutes with everything), without spoiling the previous mechanism, using the conventional Stueckelberg model for the Abelian case $G = U(1)$.

These whole scheme agrees with nature, where we find just one intermediate massive neutral vector boson Z_0 (inside weak currents); the rest of intermediate neutral vector bosons (photon and gluons) remain massless.

5 Comments and Outlook

The fact that both the gauge functions φ and the vector potentials A themselves may be considered as parameters of a group, $G^1(M)$, which constitutes the basic symmetry group of the theory (in the sense that the corresponding Noether invariants parametrizes the solution manifold), permits to face the quantum theory under the perspective of a non-perturbative group-theoretical framework (according to the scheme outlined in Ref. [9]) where questions such as renormalizability, finiteness, unitarity, etc., are much better addressed. The Hilbert space of our theory will be the carrier space of unitary irreducible representations of a centrally extended infinite-dimensional Lie group $\tilde{G}$, incorporating $G^1(M)$ and the phase space of our theory.

Let us make a brief discussion of the physical field degrees of freedom of our theory. This analysis of the dynamical content of the theory can be achieved without the need of writing down the explicit expression of the (linearized) field equations of motion. Instead, we shall resort again to a group-representation viewpoint at the Lie algebra level (see [9] for more precise details on a Group Approach to Quantization of Yang-Mills theories). In fact, for pure, massless, $SU(N)$-Yang-Mills theory we can fix the (Weyl) gauge and set the temporal part $A_0^a = 0, a = 1, \ldots, N^2 - 1$. The equal-time Lie algebra commutators between non-Abelian vector potentials $A_j^a, j = 1, 2, 3; a = 1, \ldots, N^2 - 1$, electric field E_j^a and gauge-group generators φ^a (in natural $\hbar = 1 = c$ unities) turn out to be (see e.g. the Reference [10]):

$$
\begin{aligned}
\left[A_j^a(x), E_k^b(y)\right] &= i\delta_{jk}\delta^{ab}\delta(x-y), \\
\left[\vec{E}^a(x), \varphi^b(y)\right] &= -iC_c^{ab}\vec{E}^c(x)\delta(x-y), \\
\left[\vec{A}^a(x), \varphi^b(y)\right] &= -iC_c^{ab}\vec{A}^c(x)\delta(x-y) - \frac{i}{e}\delta^{ab}\vec{\nabla}_x\delta(x-y), \\
\left[\varphi^a(x), \varphi^b(y)\right] &= -iC_c^{ab}\varphi^c(x)\delta(x-y).
\end{aligned}
\tag{5.1}
$$

From the first commutator we see that A_j^a and E_j^a are conjugated variables, so that we have in principle three field degrees of freedom for each "colour" index $a = 1, \ldots, N^2 - 1$, that is, $f = 3(N^2 - 1)$ original field degrees of freedom. All σ-fields $\varphi^a, a = 1, \ldots, N^2 - 1$, do not have dynamics this time, so that we can impose all of them as constraints $\varphi^a(x)\Psi = 0$

(the *Gauss law*) on wave functionals Ψ in the corresponding quantum field theory. This operation takes away $c = N^2 - 1$ field degrees of freedom out of the original f, leaving $f' = f - c = 2 \times (N^2 - 1)$. These field degrees of freedom correspond to $(N^2 - 1)$ massless vector bosons (remember that transversal fields have two polarizations only).

When we give dynamics to some of the σ-fields φ^a through a partial-trace σ-Lagrangian like (4.1), and perform minimal coupling $\theta_\mu \to \theta_\mu - eA_\mu$, we introduce new conjugated variables given by the new Lie-algebra commutators:

$$\left[A_0^a(x), \varphi^b(y)\right] = -iC_c^{ab} A_0^c(x)\delta(x-y) - \frac{i}{e} C_c^{ab} \lambda^c \delta(x-y), \tag{5.2}$$

where (in the hope that no confusion arises) we mean here by $A_0^a(x)$ the generator of translations in the temporal component of the vector potential. It should be stressed that the central term proportional to λ^c in the previous commutator can also be considered as associated with some sort of "symmetry breaking" in the sense that it can be hidden into a redefinition, $A_0^c \to A_0^c + \frac{\lambda^c}{e}$, of A_0^c, which now acquires a non-zero vacuum expectation value proportional to the mass λ^c, that is:

$$\langle 0|A_0^c|0\rangle = 0 \longrightarrow \langle 0|A_0^c|0\rangle = -\frac{\lambda^c}{e}.$$

This is one of the differences between the vacua of the massless and the massive theory. Moreover, σ-fields could also acquire non-zero vacuum expectation values, $\langle 0|\varphi^c|0\rangle = \omega^c$, which could be mimicked by new central terms in the last commutator of (5.1). See Ref. [9] for the physical consequences of this particular case.

Let us proceed by counting the new physical field degrees of freedom of the massive theory. We shall restrict ourselves to $G = SU(2)$ (i.e., $N = 2$), for the sake of simplicity, and take $\lambda = i\lambda^3 T_3$ (the "isospin" charge). Let us also use the Cartan basis $\langle T_\pm = T_1 \pm iT_2, T_0 = T_3\rangle$, with commutation relations:

$$[T_\pm, T_0] = \mp T_\pm, \quad [T_+, T_-] = 2T_0.$$

The commutation relations (5.2) say that the temporal part $W_0^\pm \equiv A_0^1 \pm iA_0^2$ (we adopt the usual notation in the Standard Model of electro-weak interactions for charged weak vector bosons) are conjugated fields of $\varphi^\mp \equiv \varphi^1 \mp \varphi^2$, since they give central terms proportional to the mass matrix element λ^3. On the contrary, the temporal part $B_0 \equiv A_0^3$ and the σ-field $\varphi^0 \equiv \varphi^3$ remain without dynamics. Thus, in addition to the original $f = 3(N^2 - 1) = 9$ field degrees of freedom connected to the spatial part $\vec{A}^a, a = 1, 2, 3$, we have two additional field degrees of freedom attached to the temporal part $W_0^\pm$, which results in $\tilde{f} = f + 2 = 11$ field degrees of freedom. If we wished to be consistent with the massless case, we should gauge fix σ-fields to zero as constraints $\varphi^a(x)\Psi = 0$ in the quantum theory. This operation would take away $c = (N^2 - 1) = 3$ field degrees of freedom out of the original $\tilde{f} = 11$, leaving $\tilde{f}' = \tilde{f} - c = 8 = 2 \times 1 + 3 \times 2$. These field degrees of freedom correspond to 1 massless vector boson B (two polarizations) plus 2 massive vector bosons $W^\pm$. We could say that the dynamics of the σ-fields $\varphi^\pm$ has been transferred to the vector potentials $W^\pm$ (the longitudinal part) to conform massive vector bosons. Hence, we do not need an extra (Higgs) field to give mass to vector bosons but it is the gauge group itself which acquires dynamics and transfers it to Yang-Mills fields.

A deeper (Lagrangian) analysis of the particular case of electro-weak gauge group $SU(2) \otimes U(1)$, in the framework of the Standard Model, is in preparation [11]. Also, a proper Group Approach to Quantization of this theory, clarifying the vacuum and including interaction with fermions and comparisons with the Standard Model, as well as a deeper discussion on the physical status of σ-fields, is being investigated by the authors.

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