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Some wavelets tools for Maxwell's Equations

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Abstract

In recent years wavelets decompositions have been widely used in computational Maxwell's curl equations, to effectively resolve complex problems. In this paper, we review different types of wavelets that we can consider, the Cohen-Daubechies-Feauveau biorthogonal wavelets, the orthogonal Daubechies wavelets and the Deslauriers-Dubuc interpolating wavelets. We summarize the main features of these frameworks and we propose some possible future works.

Key Words. Wavelets, Multiresolution, Maxwell's equations.

1 Introduction

In the last decades, application of the wavelets theory has been extensively investigated in various research fields of science and engineering. The wavelet decompositions yield very efficient algorithms, in terms of accuracy and CPU time, when applied to numerical solutions of differential equations. In particular, three methods of deriving wavelet schemes have been presented so far in the literature to solve electromagnetic problems from Maxwell's equations: Multiresolution Time-Domain (MRTD) scheme, Fast Wavelet Transform-based (FWT) algorithms and Interpolating Wavelets (IW). Although the three methods are time-domain schemes, they differs in how wavelet theory is implemented.

In the MRTD method the electric and magnetic fields are expanded in a wavelet basis and Maxwell'curl equations are discretized using the Garlekin's version of

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the method of moment [30]. Two conflicting requirements for the choice of the wavelet basis are high regularity properties and minimal support. The former reduces numerical dispersion and the latter reduces the algorithmic computational complexity and improves stability. In a first approach, we can find a formulation based on the Haar [15] or Battle-Lemarie [21], [27] orthonormal wavelets. The Haar family have compact support and it yields a simple algorithm but it lacks smoothness, then poor numerical dispersion properties are expected. In contrast, the Battle-Lemarie family have good regularity properties which yields highly linear numerical dispersion behavior, but they have infinite support, thus the MRTD scheme have to be truncated affecting the accuracy of the field computation. As a natural alternative, the orthogonal Daubechies wavelets [10] and the Cohen-Daubechies-Feauveau biorthogonal wavelets [8] have been considered. Both being compactly supported, the CDF biorthogonal wavelet family seems to allow a good balance between regularity and reduced support, while also being symmetric.

Other alternative, which we have named FWT, leading to fast and very efficient algorithms, was first proposed in [31]. It uses the fast wavelet transform and multiscale representation of derivative operators for compactly supported wavelets in the form of [2]. In contrast to the MRTD schemes, no integrals have to be evaluated. Orthogonal Daubechies'wavelets were used in the above mentioned work as well in [23] and [13]. Biorthogonal CDF wavelets were also used in [13].

On the other hand, there exists a fundamental characteristic of some sort of wavelets: the interpolation property. The advantage of interpolating wavelets (IW) is that the coefficients of the associated expansion represent directly the physical values of the electromagnetic fields. The so-called "shifted interpolating property" [25] of Daubechies' wavelets has been used in [3] and [17]. The last authors also proposed in [17] a higher order biorthogonal scheme using Deslauries-Dubuc interpolating functions [11], [14] which are smooth, symmetric and compactly supported. If the basis do not have the interpolating property, then it is necessary considering some means of the neighboring coefficients or reconstruction procedure in order to obtain the physical variables, resulting in a more elaborate scheme and more computational cost.

An interesting question on MRTD investigated in [24] and [22] is that adding wavelets may not result in the expected enhancement of resolution of an scaling zero-order scheme. As a consequence, spurious nonphisical modes can appear, for example. While the FWT method is claimed not suffering from this effect [31], authors of [24] claim this is a gridding-related dispersion effect and derive a necessary condition for the development of MRTD schemes with a consistent accuracy performance. It is also noted in [24] that, under the same condition, both methods become equivalent. Authors of [22] also conclude that MRTD algorithms seem to be superior over standard FDTD only for certain geometries which are

rather candidates for spectral-domain methods.

In essence, the goal of a multiresolution algorithm for the fields is to make adding wavelets virtually equivalent to the use of a denser grid. To take advantage of it as regarding memory economy, implementation complexity and execution time as a whole, future work should concentrate on dynamic scale adaptation as concluded in [22]. Two examples of implementation of this idea can be found in [31] and in [26] in the context of the FWT and MRTD schemes, respectively. In this paper we explain the basis of this technique and present the guidelines of an algorithm to be used in the future for solving Maxwell's equations. So, the rest of the paper is organized as follows: the principles of multiresolution analysis is reviewed in next section, then we introduce and compare, in section 3, the different approaches above mentioned for deriving wavelets schemes for solving Maxwell's equations; finally, in section 4, some future developments are presented.

2 Brief review of the multiresolution analysis

A multiresolution approximation is a sequence of nested space $V_j \subset V_{j+1} \subset \cdots$ of $L^2(\mathbb{R})$, such that:

- $\bigcup \bar{V}_j = L^2$
- There exist a scaling function $\varphi \in V_0$ such that

$$\varphi_{j,k}(t) = 2^{j/2}\varphi(2^j t - k), \quad k \in \mathbb{Z},$$

constitute a Riesz basis of V_j (basis (e_n), in some Hilbert space H, such that $||(x_n)||_{l^2} \sim ||\sum x_n e_n||_H$).

The scaling function $\varphi \in V_0 \subset V_1$ should satisfy a two scale equation

$$\varphi(t) = \sum_{n\in\mathbb{Z}} h_n \varphi(2t - n),$$

and from it, we obtain

$$\varphi_{j,k} = \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} h_n \varphi_{j+1,2k+n}.$$

The support of φ and the discrete support of (h_n) have the same length.

Assuming that φ is such that the $(\varphi_{j,k})_{k\in\mathbb{Z}}$ are an orthonormal basis, one builds the wavelet ψ by

$$\psi(t) = \sum_{n\in\mathbb{Z}} g_n \varphi(2t - n)$$

with $g_n = (-1)^n h_{1-n}$.

Then $(\psi_{j,k})_{k\in\mathbb{Z}}$ are an orthonormal basis of the orthogonal complement W_j of V_j into V_{j+1}.

We thus can decompose f in the orthonormal basis of $L^2(\mathbb{R})$

$$f = \sum_{k\in\mathbb{Z}} < f, \varphi_{0,k} > \varphi_{0,k} + \sum_{j\geq 0}\sum_{k\in\mathbb{Z}} < f, \psi_{j,k} > \psi_{j,k}.$$

For the construction of orthonormal scaling function we refer [10]. The orthonormality property give us that $\sum_n h_n h_{n+2k} = 2$ if $k = 0$, and 0 otherwise. For order N, we have $\sum_n h_n = 2$ and $\sum_n (-1)^n n^m h_n = 0$, $m = 0, \ldots, N$.

For the biorthogonal case, we replace the orthogonality assumption by a dual scaling function

$$\hat{\varphi} = \sum_{n\in\mathbb{Z}} \hat{h}_n \hat{\varphi}(2\cdot - n)$$

such that $< \varphi_{j,k}, \hat{\varphi}_{j,l} >= 1$ if $k = l$ and 0 otherwise.

Associated to the dual scaling function we have the dual wavelets

$$\begin{aligned} \psi &= \sum_{n\in\mathbb{Z}} g_n \varphi(2\cdot - n) \\ \hat{\psi} &= \sum_{n\in\mathbb{Z}} \hat{g}_n \hat{\varphi}(2\cdot - n) \end{aligned}$$

with $g_n = (-1)^n \hat{h}_{1-n}$ and $\hat{g}_n = (-1)^n h_{1-n}$.

We have that $< \psi_{j,k}, \hat{\psi}_{j,l} >= 1$ if $k = l$ and 0 otherwise, and that $< \varphi_{j,k}, \hat{\psi}_{j,l} >=< \hat{\varphi}_{j,k}, \psi_{j,l} >= 0$.

The results in a decomposition of f in a biorthogonal basis of $L^2(\mathbb{R})$ are given by

$$f = \sum_{k\in\mathbb{Z}} < f, \hat{\varphi}_{0,k} > \varphi_{0,k} + \sum_{j\geq 0}\sum_{k\in\mathbb{Z}} < f, \hat{\psi}_{j,k} > \psi_{j,k}.$$

For the construction of the dual function, one can prescribe the h_n and therefore the function φ and look for the coefficients $\hat{h}_n$. We remark that the duality property implies that $\sum_n \hat{h}_n h_{n+2k} = 2$ if $k = 0$ and 0 otherwise.

Different settings can be considered depending on the linear discretization operator that produces the data. Classical settings are provided by the sampling operator (point value setting) [14], [11] or the averaging operators (spline settings) [8].

3 Wavelet based time domain approach for Maxwell's equations

Maxwell's curl equations for nonconducting media in time domain are

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1}$$

where $\mathbf{D}, \mathbf{B}, \mathbf{E}, \mathbf{H}$ are called the electric flux density, the magnetic flux density, the electric field and the magnetic field, respectively. Constitutive relations must be written accounting for the electromagnetic properties of the material and for a homogeneous material of linear response, these are

$$\mathbf{D} = \epsilon \mathbf{B}, \quad \mathbf{E} = \mu \mathbf{H} \tag{2}$$

where ϵ, μ are constants named permittivity and permeability, respectively. Then, equations (1) can be written in terms of only $\mathbf{E}$ and $\mathbf{H}$ vectors as follows

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t}, \quad \nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \tag{3}$$

For simplicity, we consider one-dimensional plane wave propagation with components E_x and H_x along the z-direction, resulting the following two scalar cartesian equations

$$\begin{aligned} \frac{\partial E_x}{\partial z} &= -\mu \frac{\partial H_y}{\partial t}, \\ \frac{\partial H_y}{\partial t} &= -\epsilon \frac{\partial E_x}{\partial t} \end{aligned} \tag{4}$$

To incorporate the multiresolution techniques, one first develops expansions of the fields into the basis associated with the wavelet setting considered in space and in pulse functions in time. These expansions are then inserted in Maxwell's curl equations and sampled according to Galerkin method using the same basis and pulse functions as testing functions in space and time, respectively. The final time-evolution equations are then obtained through the orthogonality properties of the basis used.

3.1 MRTD orthogonal and biorthogonal wavelets schemes

Restricting the presentation to one level of resolution for simplicity, the field expansion for orthogonal wavelets can be written as

$$\begin{aligned} E_x(z,t) &= \sum_{k,m=-\infty}^{+\infty} [E^{\varphi}_{k,m}\varphi_m(z) + E^{\psi}_{k,m}\psi_m(z)]h_k(t) \\ H_y(z,t) &= \sum_{k,m=-\infty}^{+\infty} [H^{\varphi}_{k+\frac{1}{2},m+\frac{1}{2}}\varphi_{m+\frac{1}{2}}(z) + H^{\psi}_{k+\frac{1}{2},m+\frac{1}{2}}\psi_{m+\frac{1}{2}}(z)]h_{k+\frac{1}{2}}(t) \end{aligned} \tag{5}$$

and for biorthogonal ones

$$\begin{aligned} E_x(z,t) &= \sum_{k,m=-\infty}^{+\infty} [E^{\varphi}_{k,m}\hat{\varphi}_m(z) + E^{\psi}_{k,m}\hat{\psi}_m(z)]h_k(t) \\ H_y(z,t) &= \sum_{k,m=-\infty}^{+\infty} [H^{\varphi}_{k+\frac{1}{2},m+\frac{1}{2}}\hat{\varphi}_{m+\frac{1}{2}}(z) + H^{\psi}_{k+\frac{1}{2},m+\frac{1}{2}}\hat{\psi}_{m+\frac{1}{2}}(z)]h_{k+\frac{1}{2}}(t) \end{aligned} \tag{6}$$

Integers m and k indicate the discrete lattice indexes in space and time grids related to the space and time coordinates via $x = m\triangle x$ and $t = k\triangle t$, where $\triangle x$ and $\triangle t$ represents the space and time discretization intervals. The φ_m and ψ_m are the scaling and wavelet orthogonal functions, respectively, displaced by m units,

$$\begin{aligned} \varphi_m(x) &= \varphi(\frac{x}{\triangle x} - m), \\ \psi_m(x) &= \psi(\frac{x}{\triangle x} - m) \end{aligned} \tag{7}$$

being $\hat{\varphi}_m$ and $\hat{\psi}_m$ their dual functions. For time discretization, one uses rectangular pulses (Haar scaling functions) $h_k(t)$, where k represents shift in time units, defined as

$$h_k(x) = h(\frac{x}{\triangle x} - k) \tag{8}$$

According to the standard Yee's leap-frog approach [29] of the traditional finite difference time-domain (FDTD) method, the magnetic-field components are usually shifted by half a discretization interval in the space and time domains with respect to the electric-field components. Nevertheless, other offsets should be investigated in order to improve numerical dispersion and stability characteristics [26], [31]. After inserting the expansions (6) in Maxwell's equations, equation (4) is sampled with $h_k(t)$ in time and with $\varphi_{m+\frac{1}{2}}(z)$ and $\psi_{m+\frac{1}{2}}(z)$ in space while equation (4) is sampled with $h_{k+\frac{1}{2}}(t)$ in time and with $\varphi_m(z)$ and $\psi_m(z)$ in space.

From the the orthogonality properties of scaling, wavelet and pulse functions, and from

$$\langle h_m(z), \frac{\partial h_{m'+\frac{1}{2}}(z)}{\partial z} \rangle = \delta_{m,m'} - \delta_{m,m'+1} \tag{9}$$

the MRTD update equations are

$$\begin{aligned}
H^{\varphi}_{k+\frac{1}{2},m+\frac{1}{2}} &= H^{\varphi}_{k-\frac{1}{2},m+\frac{1}{2}} - \frac{\triangle t}{\mu \triangle z}(\sum_{m'} E^{\varphi}_{k,m'} a_{m'} + \sum_{m'} E^{\psi}_{k,m'} c_{m'}) \\
H^{\psi}_{k+\frac{1}{2},m+\frac{1}{2}} &= H^{\psi}_{k-\frac{1}{2},m+\frac{1}{2}} - \frac{\triangle t}{\mu \triangle z}(\sum_{m'} E^{\varphi}_{k,m'} d_{m'} + \sum_{m'} E^{\psi}_{k,m'} b_{m'}) \\
E^{\varphi}_{k+1,m} &= E^{\varphi}_{k,m} - \frac{\triangle t}{\epsilon \triangle z}(\sum_{m'} H^{\varphi}_{k+\frac{1}{2},m'+\frac{1}{2}} a_{m'} + \sum_{m'} H^{\psi}_{k+\frac{1}{2},m'+\frac{1}{2}} c_{m'}) \\
E^{\psi}_{k+1,m} &= E^{\psi}_{k,m} - \frac{\triangle t}{\epsilon \triangle z}(\sum_{m'} H^{\varphi}_{k+\frac{1}{2},m'+\frac{1}{2}} d_{m'} + \sum_{m'} H^{\psi}_{k+\frac{1}{2},m'+\frac{1}{2}} b_{m'})
\end{aligned} \tag{10}$$

The coefficients a, b, c and d are integrals connecting the scaling or wavelets functions with their derivatives as follows

$$\begin{aligned}
a_{m'} &= \langle \varphi_m(z), \frac{\partial \varphi_{m'+\frac{1}{2}}(z)}{\partial z} \rangle \\
b_{m'} &= \langle \psi_m(z), \frac{\partial \psi_{m'+\frac{1}{2}}(z)}{\partial z} \rangle \\
c_{m'} &= \langle \varphi_m(z), \frac{\partial \psi_{m'+\frac{1}{2}}(z)}{\partial z} \rangle \\
d_{m'} &= \langle \psi_m(z), \frac{\partial \varphi_{m'+\frac{1}{2}}(z)}{\partial z} \rangle
\end{aligned} \tag{11}$$

for orthogonal functions, or with dual functions under the derivative operator if biorthogonal. These integrals are calculated numerically from explicit computation of the scaling and the wavelet functions (or their Fourier counterparts) and $m^{'}$-index is updated from orthogonality properties of the results. If Battle-Lemarie functions are used, the sums in (10) become of infinite extend and are to be truncated by virtue of the exponential decay of the Battle-Lemarie functions. On the other hand, orthogonal Daubechies or biorthogonal CDF families of wavelets yield compact support and this means that the resulting sequence of coefficients is rigorously finite (no truncation is needed). CDF family may also been made symmetric using spline functions as dual wavelets, thus we impose smoothness as they appears are differentiated in (11). Moreover, vanishing moments of the non dual functions (used to form the inner products) are maximized for a given extend of

their support, thus moment suppression is imposed. In any case, the performances of orthogonal Daubechies and biorthogonal CDF wavelets in the MRTD methods are shown to be similar in terms of dispersion errors and computational efficiency [13].

3.2 FWT wavelets schemes

This scheme was first suggested in [31] for compactly supported orthonormal wavelets. In this work, time derivatives are approximated using central differences of second order to obtain

$$\begin{aligned} \mathbf{E}^{k} &= \mathbf{E}^{k-1} + \frac{\triangle t}{\epsilon} \text{ curl } \mathbf{H}^{k-\frac{1}{2}} \\ \mathbf{H}^{k+\frac{1}{2}} &= \mathbf{H}^{k-\frac{1}{2}} - \frac{\triangle t}{\mu} \text{ curl } \mathbf{E}^{k} \end{aligned} \tag{12}$$

Then the electromagnetic field is expanded into a system of orthogonal functions as previously. According to the theory of the representation of operators in bases of compactly supported wavelets [2], a curl operator is introduced to express (3) in terms of scaling and wavelets functions, obtaining a matricial equation in the form

$$[E]_{w}^{k} = [E]_{w}^{k-1} + \frac{\triangle t}{\epsilon} C [H]_{w}^{k-\frac{1}{2}} \tag{13}$$

$$[H]_{w}^{k+\frac{1}{2}} = [H]_{w}^{k-\frac{1}{2}} - \frac{\triangle t}{\mu} C [E]_{w}^{k} \tag{14}$$

where H_w and E_w represents the coefficients of the electric and magnetic field expansion, respectively, and C denotes the curl operator. Expressions (13)-(14) are a time evolution explicit algorithm to be started from applying the fast wavelet transform to the initial fields. It has been seen that the method does not suffer from the excitation of spurious modes like the MRTD. As basis functions, compactly supported Daubechies wavelets and scaling functions with two vanishing moments were used at three different scales. By changing the number of vanishing moments, schemes of different order of accuracy can be realized.

It must be noted that this method can be incorporated into the MRTD scheme, as shown in [13] for orthogonal Daubechies and biorthogonal CDF families of wavelets. In fact, integrals expressed by (11) can be solved analytically in the case of compactly supported wavelet systems from the scaling/wavelet filter coefficients according to the algorithms presented in [31], without the need to explicitly compute the scaling/wavelet functions (or their Fourier counterparts).

3.3 IW schemes

A wavelet-Garlekin interpolating scheme based on the so-called "shifted interpolation property " [25] of Daubechies' wavelet family was proposed in [3]. Here, Daubechies' scaling functions with two vanishing moments (D_2) were used. The formulation is similar to the analogous scaling-function-based multiresolution time-domain method (S-MRTD) [21]. The resulting algorithm can be extracted directly from expressions (10) by eliminating all wavelet-related content. But the interpolation property adopted in [3] enables local sampling of the field, leading to a more versatile and simple algorithms despite the large support and asymmetry of the Daubechies' scaling functions. To make use of this property, (7) is modified to

$$\varphi_m(z) = \varphi(\frac{z}{\triangle z} - m + M_1) \tag{15}$$

where M_1 is the first-order moment of the scaling function, and the interpolation property is written as

$$\varphi(i + M_1) = \delta_{i,0} \tag{16}$$

for i integer. This approach is as better as smaller is M_1 and it is almost satisfied for Daubechies' scaling functions. The scaling function behaving as a delta makes the field sample at one given grid point equals the scaling coefficient corresponding to that point, so that the expansion coefficients represent direct physical values of the field. Otherwise, it becomes necessary obtaining the fields at the point of interest by taking a weighted sum of neighboring coefficients (MRTD) or through a reconstruction procedure based on discrete wavelet transform methods (FWT), both resulting in a more complicated algorithm and large computational overhead.

Although the numerical dispersion of this technique is larger than that of the MRTD method using Battle-Lemarie functions, it has advantages over MRTD in that the Daubechies' scaling function has compact support. Moreover, by using basis functions of higher regularity and minimum support, such as Daubechies' scaling functions with three (D_3) and four (D_4) vanishing wavelet moments [17], better accuracy and minimum stencil sizes can be expected, resulting in an optimally efficient algorithm.

On the other hand, CDF [13] and Deslauriers-Dubuc [18], [15] biorthogonal interpolating schemes have also been applied. In the first case, the interpolating property of the CDF dual scaling functions is used as in Cheong's method [3]. In the second case, the Deslauriers-Dubuc interpolating function [11] is adopted as the fundamental scaling basis. As can be read in, the Deslauriers-Dubuc interpolating function φ of order $2p-1$ is given by an autocorrelation function of the Daubechies

compactly supported orthogonal scaling function φ_0 of p vanishing moments as

$$\varphi(x) = \int_{-\infty}^{+\infty} \varphi_0(u)\varphi_0(u-x)du \tag{17}$$

As the additional wavelet basis, a shifted and contracted version of the scaling function can be chosen. Dirac delta functions or their linear combinations are chosen as dual functions and used for testing, which implies the interpolatory property. These functions constitute non-L^2 biorthogonal bases that are smooth, symmetric, compactly supported and exactly interpolating. Unlike the Daubechies orthogonal wavelets, of which interpolation property is limite to the bases of low regularity [16], the proposed basis set yields a scheme of desired order of regularity. Moreover, by adding wavelets to this interpolating scheme multiresolution analysis can be generated while saving the computational overhead of total field reconstruction, as both the scaling and the wavelet functions are exactly interpolating. The time evolution equations are obtained in a form similar to the MRTD scheme but the calculation of the integrals (11) are simple due to the dual delta functions appearing as test functions.

So far we have considered homogeneous media being ϵ and μ constants in the entire domain. When applied to time-domain inhomogeneous electromagnetic problems, the interpolating scheme have demonstrated higher versatility and simplicity when compared to FWT or MRTD methods. The treatment of inhomogeneous configuration in the context of MRTD poses significant problems, because the material properties (ϵ and μ), as functions of space, introduce coupling between adjacent basis. In [21], [26], the inhomogeneities are treated rigorously through a matrix formulation obtained applying the standard Galerkin scheme starting from the general form of Maxwell equations (1) and incorporating discretization of the constitutive relations (2). On the other hand, a new material operator accounting for the material distribution is introduced in the context of the FWT method. Both procedures result in more complex algorithms than those obtained from applying interpolating wavelets which allow to extract the local media value by virtue of the interpolation property of the basis used, thus neglecting material operator. Authors in [22] claim that this approach yields worse results compared to the conventional FDTD.

4 Future developments

In this section we present two ways in order to improve in some cases the above approaches. The first one is related with the boundary conditions and the second one with the size of the considered wavelet's expansions.

4.1 Wavelets on the interval

To simulate an electromagnetic problem, specific boundary conditions for the fields must be introduced to account for the finite domain of the simulation. Usually, perfect electric conductor (PEC) or perfect magnetic conductor (PMC), meaning zero tangential electric or magnetic fields, respectively, at the conductors positions, must be modelled. In all the mentioned papers, nonlocalized basis functions are considered and problems arise to an exact localization of the specific boundary conditions. Then, the image principle is used which means that PEC or PMC are replaced by an open structure with proper symmetry conditions for the electromagnetic field. Namely, the electric and magnetic field components tangential to the PEC must have odd and even symmetry, respectively, and conversely for a magnetic conductor. Alternatively, we propose using wavelets defined on a interval according to the guidelines we give in next section.

It is telling that, despite the promise of orthogonal quantum mechanical multiresolution bases, the numerical solution of even such simple problems as the particle in a box have previously resisted solution via wavelets because of the difficulty in imposing boundary conditions.

The difficulties for the construction of the multiresolution and the wavelets focuses naturally on the edges. Different strategies may be adopted that can be adapted to the construction of spaces of functions satisfying homogeneous boundary conditions.

A $[0, N]$ supported signal can be represented as the product of a general signal with the characteristic function of $[0, N]$. The discontinuities of this function require special attention. Three methods are known to handle them, the last one being the most efficient.

Wavelet periodization

The wavelets are periodized by the following transformation:

$$\psi_{j,k}^{per}(t) = \frac{1}{\sqrt{2^j}} \sum_{k=-\infty}^{\infty} \psi\left(\frac{t - 2^j k + kN}{2^j}\right)$$

with $j <= log_2 N$. This is equivalent to a signal periodization.

This procedure creates large wavelet coefficients when the periodized signal is not itself continuous.

Wavelet folding

To bypass this problem, the signal is symmetrically folded around the right edge of the interval and periodized over the double sized interval.

This yields a continuous periodic signal.

Porting the signal transformation to the wavelet basis shows that the vector family is a basis of $L^2([0, N])$ if the wavelet is symmetric or antisymmetric. This

puts orthogonal bases asides.

In fact, the continuity problem reappears at the next derivative. The following approach takes the problem at the root, which is how to make wavelets over an interval with vanishing moments.

Edge wavelets

Boundary effects are explicitly handled. Consider an Daubechies orthogonal basis with p vanishing moments.

From the Strang et Fix conditions, it appears that there exists a polynomial θ_k of degree k such that:

$$\sum_{n=-\infty}^{\infty} n^k \varphi(t-n) = \theta_k(t)$$

for $k < p$.

This equation is multiplied by the characteristic function of $[0, N]$. Assuming that the support of φ is $[-p+1, p]$, scaling functions with indices $p <= k < N-p$ are not changed by this restriction. To recover the Strang and Fix condition on the interval, p "left" edge scaling function and p "right" edge scaling functions are to be found such that

$$\theta_k(t)\aleph_{[0,N]}(t) = \sum_{n=0}^{p-1} a[n]\varphi_n^{left}(t) + \sum_{n=p}^{N-p-1} n^k \varphi(t-n) + \sum_{n=0}^{p-1} b[n]\varphi_n^{right}(t)$$

If this equation is satisfied, it remains valid after re-scaling since the n^k, up to a power of 2, are the scaling coefficients of θ_k at all resolutions.

Interpolatory wavelets on the interval are defined in [28], where explicit formulations of the resulting decomposition and reconstruction algorithms are calculated. The associated interpolatory subdivision scheme for finite sequences is shown to be convergent.

The case of orthogonal and biorthogonal wavelets on the interval is most classical and can be found in [9].

4.2 Nonlinear multiresolution representations

There is a significant evolution since 90's, strongly connect to the development of appropriated mathematical representations and approximation theory associated to these representations, of schemes from linear to nonlinear and from uniform to adaptive.

The goal of a multiresolution algorithm for the numerical approximation of the solution of PDEs is to design a framework that are optimal giving a prescribed accuracy in the shortest computational time. The recent studies on wavelets and

nonlinear approximation have provided new strategies, which allow us, to build algorithms that have such characteristics [7], [4], [5], [6], [1]. The choice of an appropriated representation of the signal can be fundamental to solve an specific task.

Given a wavelet basis of $L^2(\Omega)$, $\{\psi_\lambda, \lambda \in \Lambda\}$, we define the nonlinear space

$$\Sigma_N = \{u = \sum_\lambda c_\lambda \psi_\lambda \ : \ \overline{c} = \{c_\lambda\}_{\lambda\in\Lambda} \in \sigma_N\}$$

with $\sigma_N = \{\overline{c} \in l^2(\Lambda) \ : \ \#\{\lambda \in \Lambda \ : \ c_\lambda \neq 0\} \leq N\}$.

There are two central points in approximation theory

- Characterize the function that has a certain rate of approximation

$$f \in X^r \Leftrightarrow \sigma_N(f) \leq CN^{-r}$$

- Practical realization of $f \to g \in \Sigma_N$ such that

$$||f - g||_X \leq C\sigma_N(f)$$

Two examples of approximation are given by

- Linear

$$\Sigma_N := Vect(e_1, e_2, \ldots, e_N)$$

  with $(e_k)_{k>0}$ a functional basis.

- Nonlinear

$$\Sigma_N := \{\Sigma_{\lambda\in E} d_\lambda \psi_\lambda \ : \ \#(E) \leq N\}$$

  set of all N-terms combination of a basis (ψ_λ).

We can perform fast computation algorithms using the inter-scale relations.

It is easy to define a nonlinear projection operator $\mathbb{P}_N : H^s(\Omega) \to \Sigma_N$ which minimizes the error $||u - \mathbb{P}_N u||_{H^s}$ in an equivalent $H^s(\Omega)$ norm.

The decay of wavelet coefficients is influenced by the local smoothness of the function. For instance, if $f \in C^1$ on $I_{j,k}$ an estimate is

$$|d_{j,k}| \leq 2^{-3j/2} \sup_{t\in I_{j,k}} |f^{'}(t)|.$$

The rate of convergence is linked to the Besov regularity.

In order to obtain adaptive nonlinear methods, one possibility is to look for iterative approximation schemes in which, by definition, the iterates belong to the space Σ_N.

We can consider three equivalent measurements of sparsity

- weak spaces: $\text{Card}\{\lambda \ : \ |d_\lambda| > \eta\} \leq C\eta^{-p}$

- best N-term nonlinear approximation: if $p < 2$, $s = 1/p - 1/2$,

$$||f - \sum_{N \ largest \ |d_\lambda|} d_\lambda \psi_\lambda|| \leq CN^{-s}$$

- approximation by thresholding algorithms: if $p < 2$,

$$||f - \sum_{|d_\lambda| \geq \eta} || \leq C\eta^{1-p/2}$$

For the most error norm X, $L^p, W^{s,p}, B^s_{p,q}$, a near optimal approximation is obtained by thresholding: if $f = \sum_\lambda d_\lambda \psi_\lambda$, and $f_N := \sum_{N \ largest \ |d_\lambda|} d_\lambda \psi_\lambda$, we then have

$$||f - f_N|| \leq C in f_{g \in \Sigma_N} ||f - g||_X$$

with C independent of f and N.

Denoting by M the direct multiresolution algorithm and by M^{-1} the inverse one, we present an algorithm to find the multiresolution representation $V_M^{n+1,j}$ of $V^{n+1,j}$ from the multiresolution representation $V_M^{n,j}$ of $V^{n,j}$. It is based in a thresholding procedure and we refer [20] for more theoretical and computational aspects. Let us define

$$\mathcal{D}_\epsilon(V) = \{(j,k) \ : \ |d_{j,k}| > \epsilon_j\}.$$

First, we present a way to compute a set $\tilde{\mathcal{D}}^{n+1}$ such that $\tilde{\mathcal{D}}^{n+1} \supseteq \mathcal{D}_\epsilon(V^n)\mathcal{D}_\epsilon(V^{n+1})$.

Calculation of $\tilde{\mathcal{D}}^{n+1}$ and thresholding,

Set

$$\hat{\imath}(j,k) = 0, \quad 1 \leq k \leq N_j, \quad 1 \leq j \leq L.$$

```
for   j = 1, ..., L
for   k = 1, ..., N_j
  if   (|d(j,k)(v^n)| ≤ ε_j)
     d(j,k)(v^n) = 0

  else
     î(j, k − l) = 1,  −K ≤ l ≤ K

     if   (|d(j,k)(v^n)| ≥ 2^{p+1} ε_j,  k > 1)
        î(j − 1, 2k − 1) = 1
        î(j − 1, 2k) = 1
     end
  end
end
end
```

where K is related with the propagation speed and the support of the numerical scheme and p is its order of accuracy.

Define $\mathcal{D}^{n+1}$ by

$$\mathcal{D}^{n+1} = \{(j,k) \ : \ \hat{\imath}(j,k) = 1\}$$

The full algorithm

- Truncate
$$\hat{V}_M^n = tr_\epsilon(V_M^n)$$
and calculate $\tilde{\mathcal{D}}^{n+1} \supseteq \mathcal{D}_\epsilon(V^n)\mathcal{D}_\epsilon(V^{n+1})$.

- Prepare fine-grid
$$\hat{V}^n = M^{-1}\hat{V}_M^n.$$

- Coarsest grid calculations in an usual way.

- Computation of $\{d_{j,k}(V^{n+1})\}$

```
if   ((j,k) ∈ D̃^{n+1})
     Compute d_{j,k}
else
     d_{j,k} = 0
end
```

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