Analysis and Closed-Form Solutions of Circular and Rectangular Apertures in the Transverse Plane of a Circular Waveguide

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***Abstract*—An analysis has been carried out to determine the discontinuity susceptance of a circular or rectangular aperture in the transverse plane of a circular waveguide. Closed-form solutions based on the scattered amplitude calculation and aperture coupling theory have been derived. The theoretical results agree very well with the experiments. The results should have many applications in the design of circular waveguide components and circular waveguide-backed aperture antennas.**

I. Introduction

DIFFRACTION of an incident electromagnetic field by a small hole was first investigated by Bethe [1] for small apertures. Later, Collin [2] summarized Bethe's approach and defined coupling coefficients which make it possible to determine the electrical properties of the aperture. The electrical properties of the aperture, such as susceptance, are important in the design of coupled cavity resonant filters [3], [4], directional couplers, and conformal circular waveguide-backed aperture antennas [5].

Apertures in rectangular waveguides have been thoroughly investigated [6]–[8], but apertures in circular waveguides have received only sparse attention [6]. A truly comprehensive study of different aperture shapes embedded in a circular waveguide does not exist. The purpose of this paper is to study various apertures in the transverse plane of a circular waveguide.

Bethe's small-aperture theory applies only to apertures with resonant frequencies at least three times the operating frequency of the waveguide. To overcome this limitation, a frequency correction factor has been derived from Foster's reactance theory. Therefore, moderately sized apertures in a circular waveguide can be investigated without the implementation of numerical methods.

This paper reports an analysis based on the scattered amplitude calculation and aperture coupling theory. A closed-form, easy-to-use formula for the normalized susceptance has been derived for a general aperture discontinuity in the transverse plane of a circular waveguide. The expression is very general in that the aperture polarizability is left as a variable. Example solutions for a small circular aperture and a resonant rectangular aperture are given. The calculated results agree very well with the experiments.

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II. Analysis and Closed-Form Solution

Closed-form solutions have been derived for both circular and rectangular apertures in a circular waveguide. The solutions are based on scattered amplitude calculation and the aperture coupling theory. The results can be used to predict the resonant frequency of an aperture discontinuity or a slot antenna fed by a circular waveguide cavity.

Fig. 1 shows a general aperture discontinuity in the transverse plane of a circular waveguide. The equations linking the incident electric and magnetic fields to the aperture equivalent electric and magnetic dipole moments are

$$\vec{P} = -\epsilon_0 \alpha_e (\hat{n} \cdot \vec{E}) \hat{n} \tag{1}$$

$$\vec{M} = -\alpha_m \vec{H}_t \tag{2}$$

where $\hat{n} \cdot \vec{E}$ represents the incident normal electric field, $\vec{H}_t$ represents the tangential magnetic field [2], and α_e and α_m are the electric and magnetic polarizabilities, respectively. The polarizability is dependent only upon the shape and size of the aperture being excited by the incident electromagnetic fields.

The scattered amplitude coefficient can be calculated by [2]

$$C_n^+ = \frac{j\omega\mu_0}{P_n} \vec{H}^- \cdot \vec{M}_T \tag{3}$$

where

$$P_n = 2\int_s \vec{e}_n \times \vec{h}_n \cdot \hat{z}\, ds.$$

C_n^+ is the nth scattered amplitude coefficient of all possible TE and TM modes in the region of $z > 0$. To determine the exciting field, assume that the aperture is closed [2]. The magnetic field in the region $z < 0$ consists of incident and

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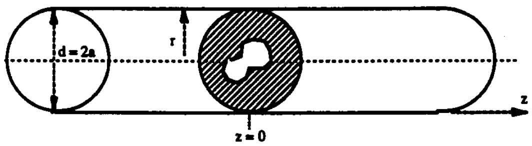

Fig. 1. Circular waveguide with an arbitrary aperture on the transverse conducting wall.

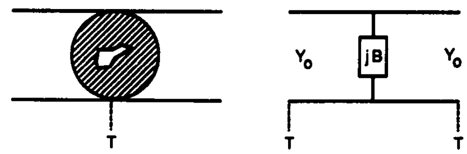

Fig. 2. Equivalent circuit of the aperture on the transverse conducting wall.

scattered fields as given by

$$H_r = H_r^+ + H_r^- = -\frac{j\beta}{k_c} J_n'(k_c r)\cos(n\phi)(e^{-j\beta z} + e^{j\beta z}) \tag{4}$$

$$H_\phi = H_\phi^+ + H_\phi^- = \frac{jn\beta}{k_c^2 r} J_n(k_c r)\sin(n\phi)(e^{-j\beta z} + e^{j\beta z}). \tag{5}$$

The equal amplitude in the incident and scattered fields is due to the fact that a TE_{nm} mode incident field from $z<0$ is reflected by the conducting wall at $z=0$ to produce a standing-wave field in the region $z<0$ [2].

The scattered magnetic field at $z=0$ is

$$\vec{H}^- = H_r^-\hat{r} + H_\phi^-\hat{\phi} = \frac{j\beta}{k_c}\left[-J_n'(k_c r)\cos(n\phi)\hat{r} + \frac{n}{k_c r}J_n(k_c r)\sin(n\phi)\hat{\phi}\right]. \tag{6}$$

The tangential field evaluated at the aperture plane ($z=0$) is given by

$$\vec{H}_t = H_r\hat{r} + H_\phi\hat{\phi} = \frac{2j\beta}{k_c}\left[-J_n'(k_c r)\cos(n\phi)\hat{r} + \frac{n}{k_c r}J_n(k_c r)\sin(n\phi)\hat{\phi}\right]. \tag{7}$$

By removing the conducting transverse wall containing the aperture, image theory dictates that the magnetic dipole must be doubled to preserve the original fields which existed before the aperture wall was removed. Therefore, the total equivalent magnetic dipole moment is given as

$$\vec{M}_T = 2\vec{M} = -2\alpha_m \vec{H}_t. \tag{8}$$

Substituting the scattered $\vec{H}^-$ field from (6) and the magnetic dipole moment $\vec{M}_T$ from (8) into (3), we have

$$C_n^+ = \frac{j4\alpha_m\omega\mu_0}{P_n}\left(\frac{\beta^2}{k_c^2}\right)\left\{\left[-J_n'(k_c r)\cos(n\phi)\right]^2 + \left[\frac{n}{k_c r}J_n(k_c r)\sin(n\phi)\right]^2\right\}. \tag{9}$$

For the dominant mode incident (i.e., TE_{11} mode), the scattering amplitude coefficient can be written by letting $n=1$. If we assume the small aperture to be located at $r=0$, using the recurrence relation for small r gives

$$J_n(k_c r) \approx \frac{1}{n!}\left(\frac{k_c r}{2}\right)^n \tag{10}$$

$$J_n'(k_c r) \approx -\frac{n}{n!}\frac{1}{2^n}(k_c r)^{n-1}. \tag{11}$$

The normalization constant can be written as

$$P_1 = \frac{\pi Z_0 k_0 \beta}{k_c^4}\left[(p_{11}')^2 - 1\right]J_1^2(k_c a). \tag{12}$$

Substituting P_1 and (10) and (11) into (9), we have

$$C_1^+ = \frac{j2\alpha_m (p_{11}')^2}{\lambda_g a^2\left[(p_{11}')^2 - 1\right]J_1^2(k_c a)} \tag{13}$$

where p_{11}' is the first root of the derivative of the Bessel function, $J_1'(k_c r) = 0$.

The reflection coefficient Γ is given by

$$\Gamma = C_1^+ - 1. \tag{14}$$

The effect of an infinitely thin transverse aperture on the dominant mode field of a waveguide can be described by a shunt susceptance across the waveguide as shown in Fig. 2. From circuit theory, the input reflection coefficient is

$$S_{11} = \Gamma = -\frac{j\bar{B}}{2 + j\bar{B}} \tag{15}$$

where $j\bar{B}$ is the normalized susceptance. For a small aperture, $\bar{B}$ is large. The input reflection coefficient can be approximated by

$$\Gamma = -\left(\frac{2 + j\bar{B}}{j\bar{B}}\right)^{-1} = -\left(1 + \frac{2}{j\bar{B}}\right)^{-1} \approx -1 - j\frac{2}{\bar{B}}. \tag{16}$$

Therefore, by comparing (14) and (16), the normalized susceptance is

$$\bar{B} = -j\frac{2}{C_1^+} = -\frac{\lambda_g a^2\left[(p_{11}')^2 - 1\right]J_1^2(k_c a)}{\alpha_m (p_{11}')^2}. \tag{17}$$

III. Small Circular Aperture

Substituting $p_{11}' = 1.841$ and $J_1(p_{11}') = 0.58$ into (17), we have

$$\bar{B} = -\frac{0.238\lambda_g a^2}{\alpha_m}. \tag{18}$$

The magnetic polarizability, α_m, for a small circular aperture is given by [1], [2], [7]

$$\alpha_m = \frac{4}{3}r_0^3 \tag{19}$$

where r_0 is the radius of the aperture. Therefore the normal-

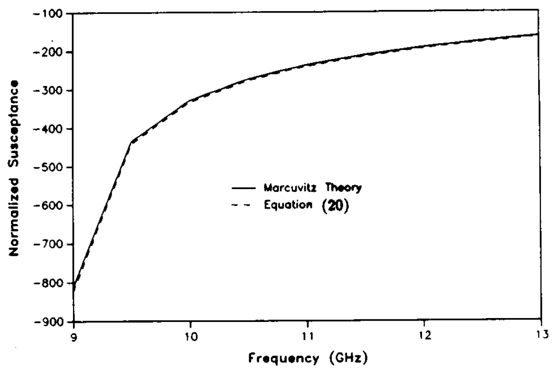

Fig. 3. Comparison of theoretical results with Marcuvitz's.

ized susceptance becomes

$$\bar{B} = -\frac{0.178\lambda_g a^2}{r_0^3}. \tag{20}$$

Assuming that the circular waveguide is operating above cutoff, which for a TE_{11} mode in a circular waveguide of radius $a = 1.0$ cm is $f_c = 8.79$ GHz, a comparison between Marcuvitz's theoretical susceptance [6] for a circular aperture and the normalized susceptance of (20) is shown in Fig. 3. The aperture radius is 0.15 cm. The agreement is excellent. A maximum difference between the two theoretical values is only of the order of 0.1%. It should be emphasized that (20) is obtained from the analysis described in Section II and is much simpler than the equation given by Marcuvitz.

IV. Experimental Verification

A measurement procedure was also devised to verify the theoretical expression. The setup consists of a two-port circuit with the aperture sandwiched by two circular waveguide sections. Each port comprises a coaxial to rectangular waveguide transition, a rectangular waveguide to circular waveguide transition, and a circular waveguide section, as shown in Fig. 4. Associated with both A and B are all the discontinuities from the coaxial connectors to the rectangular to circular waveguide transitions and waveguide flanges. The DUT (device under test) is the transverse aperture. An HP 8510 network analyzer was used to measure the field scattering parameters S_{11}, S_{12}, S_{21}, and S_{22}.

The scattering parameter of the DUT (transverse aperture) can be expressed as [9]

$$S_{21}^D = \frac{S_{21}}{S_{21}^\circ} \tag{21}$$

where S_{21} and S_{21}° are the measured values from the setup of Fig. 4 with and without the aperture discontinuity, respectively. For transitions of VSWR = 1.3, the measurement error using (21) is less than 1.7%.

The scattering parameter S_{21}^D is related to the normalized admittance $\bar{Y}$ of the aperture discontinuity by the following expression:

$$S_{21}^D = \frac{2}{2+\bar{Y}} = |S_{21}^D| e^{j\psi}. \tag{22}$$

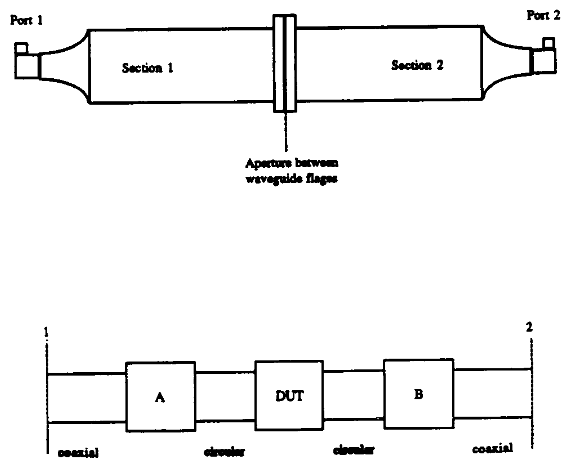

Fig. 4. Bisection circular waveguide in which a transverse aperture is embedded.

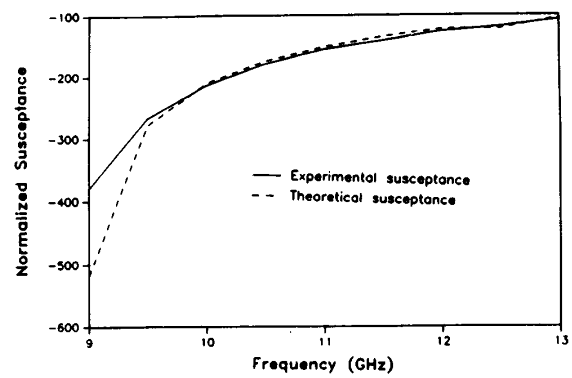

Fig. 5. Theoretical and experimental susceptance results for a circular aperture on the transverse plane of a circular waveguide.

Therefore, we have

$$\bar{B} = \frac{2}{|S_{21}^D|}\sin\psi \tag{23}$$

$$\bar{G} = 2\left(\frac{\cos\psi}{|S_{21}^D|} - 1\right). \tag{24}$$

Fig. 5 shows the theoretical results compared with the experimental results for a circular aperture of radius equal to 0.175 cm in a circular waveguide with a radius of 1 cm. The agreement is fairly good except at frequencies near the cutoff frequency. The resonant frequency is well above the operating frequency range of the circular waveguide.

By examining (20) it is evident that the susceptance will vary inversely to the cube of the apertures radius, r_0. The sensitivity of the theoretical normalized susceptance to the radius of the aperture is shown in Fig. 6 for four different sizes of circular apertures. It can be seen that a minor error in the radius measurement could have a dramatic effect on the susceptance.

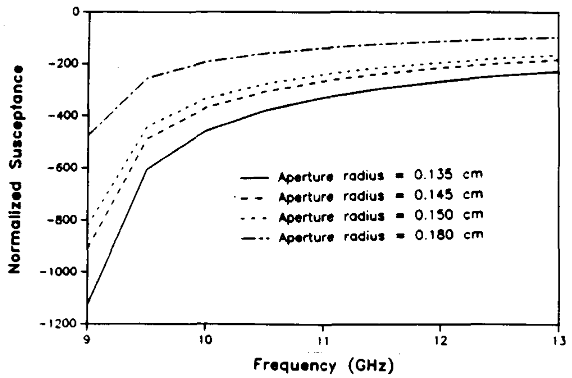

Fig. 6. Comparison of the theoretical normalized susceptances for different aperture radii.

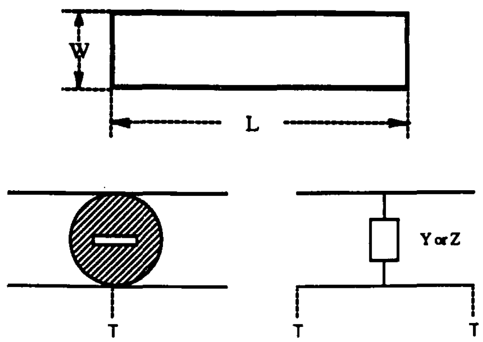

Fig. 7. Rectangular aperture with length L and width W.

V. Narrow Rectangular Aperture

The small-aperture approximation discussed earlier can be modified and adapted to an aperture whose resonant frequency is close to the operating frequency of the waveguide. Foster's reactance theorem is implemented to derive a frequency correction term for the normalized susceptance of the aperture. Applying this theory to a resonant rectangular aperture with a small aspect ratio (i.e., $W/L \ll 1$) in a circular waveguide, the susceptance can be obtained with a simple closed-form expression. The small aspect ratio is necessary so that the incident fields are not significantly perturbed and the frequency correction factor can be applied. Using this method, a theory is devised to determine the susceptance of a transverse aperture whose resonant frequency is in the operating frequency range of the circular waveguide in which it is embedded without having to resort to numerical solutions.

Fig. 7 shows a rectangular aperture with length L and width W. The magnetic polarizability of a rectangular aperture was derived by McDonald and De Smedt [10]–[12] in the following form:

$$\alpha_m = \frac{0.132}{\ln(1+0.66/\alpha)} L^3 \qquad (25)$$

where α is the aspect ratio, defined by W/L.

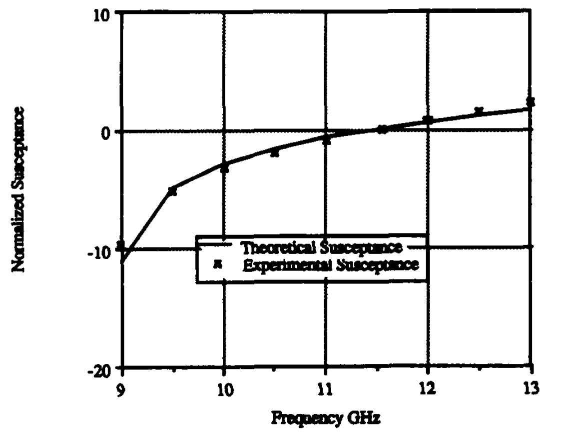

Fig. 8. Comparison of theoretical and experimental susceptances for a resonant rectangular aperture on the transverse plane of a circular waveguide.

By considering a second-order quadratic approximation, (17) can be modified by including a $(1-f^2/f_m^2)$ factor derived from Foster reactance theorem [13]. The normalized susceptance is

$$\bar{B} = -\frac{\lambda_g a^2\left[(p'_{11})^2-1\right]J_1^2(k_c a)}{\alpha_m (p'_{11})^2}\left(1-\frac{f^2}{f_m^2}\right). \qquad (26)$$

Here f_m is the resonant frequency of the aperture, which can be approximated from its cutoff frequency or determined experimentally.

By way of example, if an aperture resonant frequency of 11.55 GHz is assumed and the circular waveguide has a radius of 10 mm, the resonant length, L, of the rectangular aperture is approximately equal to 12 mm. Using an aspect ratio of 6, the measured and calculated normalized susceptances as a function of frequency are shown in Fig. 8 for comparison. The agreement is very good.

The analysis described here is based on the assumption that the wall containing the aperture has zero thickness. For an aperture with finite thickness, the polarizabilities can be modified by the thick-aperture coupling theory [14], [15]. The polarizabilities are functions of frequency and thickness. A T-network equivalent circuit should be used to replace the shunt susceptance for the aperture.

VI. Slot Aperture Antennas Backed by Circular Waveguides

The analysis for apertures can be applied to calculate the resonance frequency of a resonant slot aperture antenna backed by a circular waveguide. The normalized susceptance obtained from the transverse aperture embedded in a waveguide is halved to determine the susceptance of the resonant slot antenna [8].

Fig. 9 shows a radiating rectangular aperture backed by a circular waveguide. The quantity G is the conductance of the

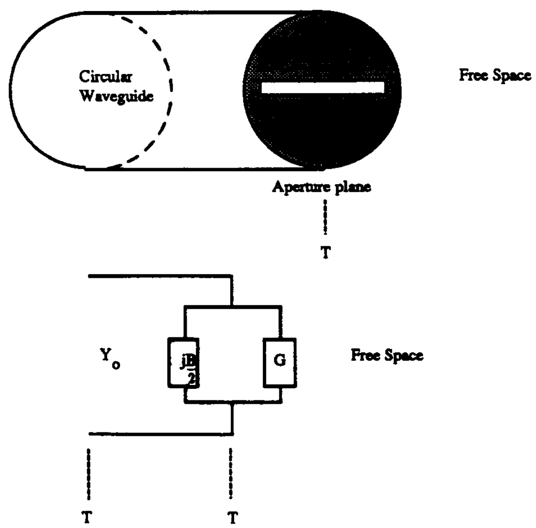

Fig. 9. Radiating resonant aperture antenna and its equivalent circuit.

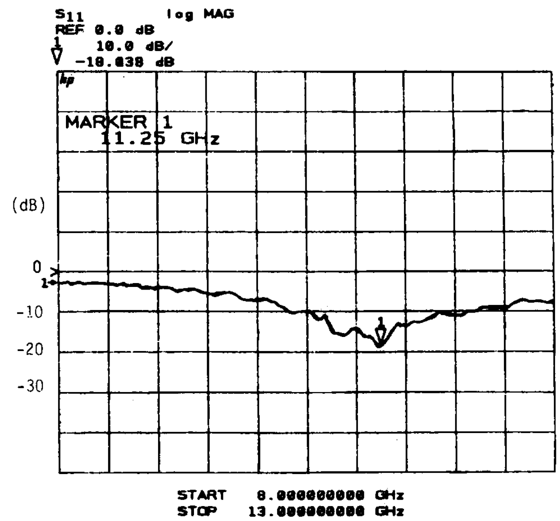

Fig. 10. Input reflection coefficient measurement of a resonant aperture antenna backed by a circular waveguide.

aperture, which is the radiation conductance of the antenna. The susceptance is equal to half of that of the aperture embedded in waveguide; thus the resonant frequency is the same.

A slot antenna with an aperture length of 12 mm and a width of 2 mm was fabricated and tested. To minimize the edge effects, an aluminum metal plate was used as a ground plane for the aperture antenna. Fig. 10 shows the results of input reflection coefficient (S_{11}) measurements, indicating a resonance at 11.25 GHz, which is only a 2% difference compared with the embedded aperture that resonates at 11.55 GHz, thereby justifying the equivalent circuit model shown in Fig. 9. The *E*- and *H*-plane patterns of this antenna are illustrated in Fig. 11. These results compare favorably with the published patterns for slot antennas backed by a rectangular waveguide [16].

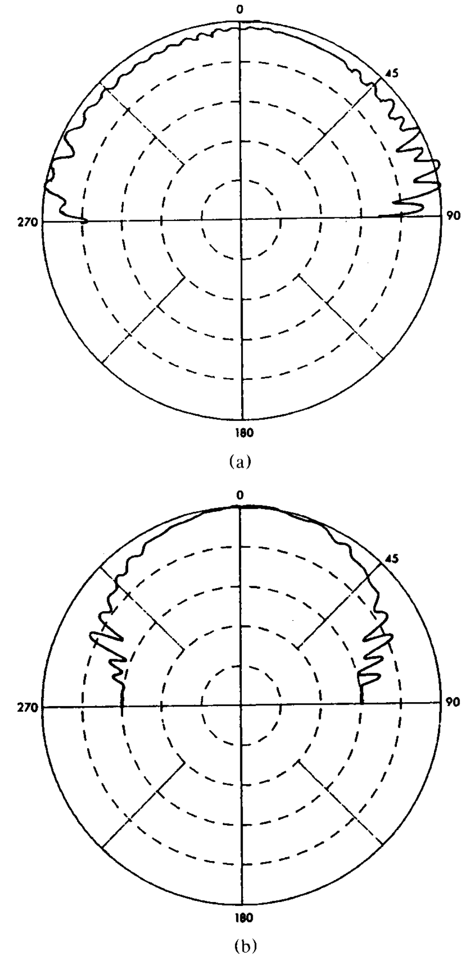

Fig. 11. *E*- and *H*-plane field patterns: (a) *E*-plane; (b) *H*-plane. Scale: 6 dB/division.

VII. Conclusions

In conclusion, closed-form expressions for the discontinuity susceptance of an aperture in the transverse plane of a circular waveguide have been derived. The theoretical results agree very well with the experiments for both a small circular aperture and a resonant rectangular aperture. The results are useful for the design of circular waveguide components and antennas.

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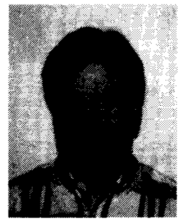

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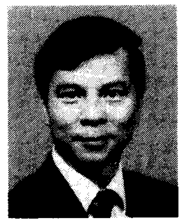

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