Free Groups and Subgroups of Finite Index in the Unit Group of an Integral Group Ring*

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Abstract

In this article we construct free groups and subgroups of finite index in the unit group of the integral group ring of a finite non-abelian group G for which every non-linear irreducible complex representation is of degree 2 and with commutator subgroup G' a central elementary abelian 2-group.

1 Introduction

It is well-known from a result of Borel and Harish Chandra that the unit group of the integral group ring $\mathbb{Z}[G]$ of a finite group G is finitely presented [2]. In case G is abelian, Higman showed that $\mathcal{U}(\mathbb{Z}[G]) = \pm G \times F$, a direct product of the trivial units $\pm G$ with a finitely generated free abelian group F. However, when G is non-abelian, there is no general structure theorem.

Hartley and Pickel [5] showed that if the unit group of the integral group ring of a finite non-abelian group is not trivial, then it contains a non-abelian

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free subgroup of rank two. Marciniak and Sehgal constructed in [11] such a subgroup using a non-trivial bicyclic unit $u = 1 + (1 - x)y\widehat{x}$ of $\mathbb{Z}[G]$, where $x, y \in G$ and $\widehat{x} = \sum_{1 \le i \le o(x)} x^i$, with $o(x)$ the order of x. They showed that $\langle u, u^* \rangle$ is a non-abelian free subgroup of $\mathcal{U}(\mathbb{Z}[G])$, where $*$ denotes the classical involution on the rational group algebra $\mathbb{Q}[G]$.

It is thus a natural question to ask whether $\langle u, \varphi(u) \rangle$ is free in case φ is an arbitrary involution on G. We will solve this question for the class $\mathcal{G}$ consisting of the finite non-abelian groups G for which every non-linear irreducible complex representation is of degree 2 and with commutator subgroup G' a central elementary abelian 2-group. Due to a result of Amitsur [1] the former condition is equivalent to either G containing an abelian subgroup of index 2 or $G/\mathcal{Z}(G)$ being an elementary abelian 2-group of order 8.

If e is a primitive central idempotent of $\mathbb{Q}[G]$ with $G \in \mathcal{G}$ such that $\mathbb{Q}[G]e$ is non-commutative, then $H = Ge \in \mathcal{G}$ and clearly $H' \cong C_2$. We denote by C_n the cyclic group of order n. By [10, Lemma 1.4] we know that for an arbitrary finite group G and p a prime, $G/\mathcal{Z}(G) \cong C_p \times C_p$ is equivalent to $|G'| = p$ and every non-linear irreducible complex representation of G has degree p. Thus $H/\mathcal{Z}(H) \cong C_2 \times C_2$. Hence we first will concentrate on groups satisfying the latter property. Furthermore, we also will characterize when two arbitrary bicyclic units generate a free group.

Besides constructing free groups in the unit group of an integral group ring, finding generators for a subgroup of finite index is an important step in understanding the structure of the unit group. When the non-commutative simple components of $\mathbb{Q}[G]$ are of a so-called exceptional type, they are an obstruction to construct in a generic way generators of a subgroup of finite index in the unit group $\mathcal{U}(\mathbb{Z}[G])$ (*Problem 23 in [15]*). For details we refer the reader to [7], [13] and [15].

In [3] φ-unitary units were introduced to overcome this difficulty for finite groups G of type $G/\mathcal{Z}(G) \cong C_2 \times C_2$ (and also for all groups up to order 16). These φ-unitary units together with the Bass cyclic units generate a subgroup of finite index in the unit group of $\mathbb{Z}[G]$ and we will extend this result to groups in the class $\mathcal{G}$. Recall that for $g \in G$ with $o(g) = n$ and $1 < k < n$, $gcd(k, n) = 1$, a Bass cyclic unit of $\mathbb{Z}[G]$ is of the form $b(g, k) = \left(\sum_{j=0}^{k-1} g^j\right)^{\phi(n)} + \frac{1-k^{\phi(n)}}{n}\hat{g}$, where ϕ is the Euler's function.

It is worth mentioning that from the classification in [12, Theorem 3.3] it follows that the class $\mathcal{G}$ contains for example the finite groups of Kleinian type with central commutators. For the finite groups G of Kleinian type there exist geometrical methods [12] that allow to compute a presentation by generators and relations for a subgroup of finite index in $\mathcal{U}(\mathbb{Z}[G])$. Although, it is very hard to accomplish these calculations, several examples have been calculated in [12]. Hence we need to obtain more algebraic information on the structure of the unit group $\mathcal{U}(\mathbb{Z}[G])$ of such groups G of Kleinian type.

2 Free Subgroups

To investigate the group $\langle u, \varphi(u)\rangle$ where u is a non-trivial bicyclic unit and φ is an arbitrary involution on a finite group G we will make use of the following criterion.

Theorem 2.1. *[14, 9, Proposition 2.4] Let A be a $\mathbb{Q}$-algebra which is a direct product of division rings and 2×2-matrix rings over subfields k of $\mathbb{C}$.*
Let $a, b \in A$ be such that $a^2 = b^2 = 0$, then

1. *if ab is nilpotent, then $\langle 1+a, 1+b\rangle$ is torsion-free abelian,*
2. *if ab is not nilpotent and if for some projection ρ of A onto a simple component $M_2(k)$ we have that $|Tr(\rho(ab))| \geq 4$, then $\langle 1+a, 1+b\rangle$ is free of rank 2, where Tr denotes the ordinary trace function on matrices.*

2.1 Preliminaries

Let G be a finite group that is not Hamiltonian and such that

$$G/\mathcal{Z}(G) \cong C_2 \times C_2.$$

Note that by [4, Proposition III.3.6] the latter is equivalent to G having a unique non-identity commutator s and for $x, y \in G$ one has that $xy = yx$ if and only if $x \in \mathcal{Z}(G)$ or $y \in \mathcal{Z}(G)$ or $xy \in \mathcal{Z}(G)$. The last property is the so called *lack of commutativity property.* Note that s is central of order 2.

Take $x, y \in G$ with $s = (x, y) \notin \langle x\rangle$, then $u = 1 + (1-x)y\widehat{x}$ is a non-trivial bicyclic unit of $\mathbb{Z}[G]$. Clearly $x^2, y^2 \in \mathcal{Z}(G)$ and we can write $G = \langle x, y, \mathcal{Z}(G)\rangle$. It is readily verified that an involution φ on G has to be of one of the following types:

$$\varphi_1 : \begin{cases} x \mapsto z_1 x \\ y \mapsto z_2 y \end{cases} \quad \varphi_2 : \begin{cases} x \mapsto z_1 x \\ y \mapsto z_2 xy \end{cases} \quad \varphi_3 : \begin{cases} x \mapsto z_1 y \\ y \mapsto z_2 x \end{cases} \quad \varphi_4 : \begin{cases} x \mapsto z_1 xy \\ y \mapsto z_2 y \end{cases} \qquad (1)$$

for some $z_1, z_2 \in \mathcal{Z}(G)$. The natural extension of φ to a $\mathbb{Q}$-linear involution on $\mathbb{Q}[G]$ is also denoted by φ. Consider the images of the bicyclic unit u

under the mentioned involutions φ. Since $\widehat{g} = \widehat{g^2}(1+g)$ for a non-central $g \in G$, we obtain that

$$u = 1 + \widehat{x^2}(1-x)y(1-s) \text{ and } \varphi(u) = 1 + \widehat{\varphi(x)^2}(1+\varphi(x))\varphi(y)(1-s).$$

Investigating the structure of $\langle u, \varphi(u)\rangle$ forces us to look at the non- commutative simple components of $\mathbb{Q}[G]$, thus the simple components of $\mathbb{Q}[G]\left(\frac{1-s}{2}\right)$. By [4, Proposition VII.2.1] the primitive central idempotents of $\mathbb{Q}[G]\left(\frac{1-s}{2}\right)$ are precisely the elements of the form $e = \widetilde{H}\left(\frac{1-s}{2}\right)$, where H is a subgroup of $\mathcal{Z}(G)$ not containing s and such that $\mathcal{Z}(G) = \langle H, c\rangle$ for some $1 \neq c \in \mathcal{Z}(G)$. Furthermore, if $\mathcal{Z}(G)/H$ has order m, with $m > 1$ then $\mathcal{Z}(\mathbb{Q}[G])e \cong \mathbb{Q}(\xi_m)$.

Recall that for a subgroup H of a finite group we denote by $\widetilde{H}$ the idempotent $\frac{1}{|H|}\sum_{h\in H} h$ of $\mathbb{Q}[G]$. Recall that $\widetilde{H}$ is central precisely when H is normal in G.

Theorem 2.2. *Let φ be an involution on a finite group G that is not Hamiltonian and such that $G/\mathcal{Z}(G) \cong C_2 \times C_2$. Let $x, y \in G$ be such that $u = 1 + (1-x)y\widehat{x}$ is a non-trivial bicyclic unit (thus $s = (x, y) \notin \langle x\rangle$).*

Put $T = \langle x^2, \varphi(x)^2, \varphi(x)x^{-1}\rangle = \langle x^2, \varphi(x)x^{-1}\rangle$ in case $\varphi(x)x^{-1}$ is central, otherwise put $T = \langle x^2, \varphi(x)^2\rangle$.

Then $\langle u, \varphi(u)\rangle$ is a free group of rank two if and only if $s \notin T$. Otherwise, it is a torsion-free abelian group.

Proof. Let e be an arbitrary primitive central idempotent of $\mathbb{Q}[G]\left(\frac{1-s}{2}\right)$. Then $e = \widetilde{H}\left(\frac{1-s}{2}\right)$ for some subgroup H of G as mentioned above. Put $a = \widehat{x^2}\,(1-x)y(1-s)$ and $b = \widehat{\varphi(x)^2}\,(1+\varphi(x))\varphi(y)(1-s)$. Then

$$ab\left(\frac{1-s}{2}\right) = 4\widehat{x^2}\,\widehat{\varphi(x)^2}\,(1+sx)y(1+\varphi(x))\varphi(y)\left(\frac{1-s}{2}\right).$$

If $\varphi = \varphi_1$ or φ_2, then

$$ab\left(\frac{1-s}{2}\right) = 4\widehat{x^2}\,\widehat{z_1^2x^2}\,z_2y^2(1+z_1)(1+sx)\left(\frac{1-s}{2}\right).$$

If $\varphi = \varphi_3$, then

$$ab\left(\frac{1-s}{2}\right) = 4\widehat{x^2}\,\widehat{z_1^2y^2}\,z_2z_1^{-1}(1+sx)(1+z_1y)(z_1y)x\left(\frac{1-s}{2}\right)$$

If $\varphi = \varphi_4$, then

$$ab\left(\frac{1-s}{2}\right) = 4\widehat{x^2}\,\widehat{(z_1xy)^2}\,z_2y^2(1+sx)(1+z_1y)\left(\frac{1-s}{2}\right).$$

Put

$$d_1 = d_2 = 4\widehat{x^2}\ \widehat{z_1^2x^2}z_2y^2(1+z_1)$$

and

$$d_3 = 4\widehat{x^2}\ \widehat{z_1^2y^2}\ z_2z_1^{-1},\quad d_4 = 4\widehat{x^2}\ \widehat{(z_1xy)^2}\ z_2y^2.$$

Now $\mathcal{Z}(\mathbb{Q}[G])\widetilde{H}\left(\frac{1-s}{2}\right) \cong \mathbb{Q}(\xi_m)$ and thus the central torsion units x^2e, y^2e, z_1e, z_2e, $(xy)^2e$ belong to $\langle\xi_m\rangle$, where m is the order of $\mathcal{Z}(G)/H$. It follows in particular that $d_ie \in \mathbb{Q}(\xi_m)$ for $1 \le i \le 4$. Furthermore, $d_1e \neq 0$ if and only if $\widehat{x^2}e \neq 0$, $\widehat{z_1^2x^2}e \neq 0$ and $z_1e \neq -e$, while d_3e and d_4e are non-zero if and only if $\widehat{x^2}e \neq 0$, $\widehat{\varphi(x)^2}e \neq 0$.

Write $x^2e = \xi_m^i$ for some $i \ge 0$. Hence

$$\widehat{x^2}e = k\widehat{\xi_m^i},$$

where $k = o(x^2)/o(\xi_m^i)$ and $\widehat{\xi_m^i} = \sum_{j=0}^{o(\xi_m^i)-1}\xi_m^{ij}$. Now $\widehat{\xi_m^i} \neq 0$ if and only if $\xi_m^i = 1$. Hence $\widehat{x^2}e \neq 0$ if and only if $x^2e = e$. If this is the case, then

$$\widehat{x^2}e = o(x^2) = \frac{o(x)}{2}.$$

Similarly, we deduce that $\widehat{\varphi(x)^2}e \neq 0$ if and only if $\varphi(x)^2e = e$. If this is the case, then

$$\widehat{\varphi(x)^2}e = o(x^2) = \frac{o(x)}{2}.$$

Hence

$$d_1e \neq 0 \text{ if and only if } x^2e = e,\ z_1^2x^2e = e \text{ and } z_1e \neq -e,$$

which is equivalent to $x^2e = e$ and $z_1e = \varphi(x)x^{-1}e = e$. Thus for $i = 1,2,3,4$

$$d_ie \neq 0 \text{ if and only if } T \subseteq H, \tag{2}$$

where T is as in the statement of the Theorem.

If $s \notin T$, then there exists a primitive central idempotent $e = \widetilde{H}\left(\frac{1-s}{2}\right)$ of $\mathbb{Q}[G]$ so that H contains T. In particular all $d_ie \neq 0$ and thus $\mathbb{Q}[G]e$ is not a division ring. Since it is four dimensional over its center and simple, the algebra $\mathbb{Q}[G]e$ is a two-by-two matrix ring over its center. It is readily verified, using $x^2e = e$, that $\mathbb{Q}[G]e$ has the following set of matrix units

$$E_{11} = \tfrac{1+x}{2}e \qquad E_{12} = y^{-2}\tfrac{1+x}{2}y\tfrac{1-x}{2}e$$
$$E_{21} = \tfrac{1-x}{2}y\tfrac{1+x}{2}e \qquad E_{22} = \tfrac{1-x}{2}e$$

With respect to these matrix units one verifies that

$$xe = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad ye = \begin{pmatrix} 0 & \xi_m^j \\ 1 & 0 \end{pmatrix} \qquad se = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

for some $j \geq 0$.

Since $ze \in \langle \xi_m \rangle$ for any $z \in \mathcal{Z}(G)$ we also have that $|ze| = 1$. It follows that

$$\begin{aligned}
&|Tr(d_1(1+sx)e)| = |4\tfrac{o(x)^2}{4}\, 2\, Tr \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} | \ = \ 4o(x)^2, \\
&|Tr(d_3(1+sx)(1+z_1y)(z_1y)xe)| \\
&= |4\tfrac{o(x)^2}{4} Tr \left(\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & \xi_m^{k+j} \\ \xi_m^k & 1 \end{pmatrix} \begin{pmatrix} 0 & \xi_m^{k+j} \\ \xi_m^k & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) | \\
&= |o(x)^2 Tr \begin{pmatrix} 0 & 0 \\ 2\xi_m^k & -2\xi_m^{2k+j} \end{pmatrix} | \\
&= 2o(x)^2
\end{aligned}$$

and

$$\begin{aligned}
|Tr(d_4(1+sx)(1+z_1y)e)| &= |4\frac{o(x)^2}{4} Tr \left(\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & \xi_m^{k+j} \\ \xi_m^k & 1 \end{pmatrix} \right) | \\
&= |o(x)^2 Tr \begin{pmatrix} 0 & 0 \\ 2\xi_m^k & 2 \end{pmatrix} | \\
&= 2o(x)^2.
\end{aligned}$$

As $o(x) \geq 2$, Theorem 2.1 gives us that $\langle u, \varphi(u) \rangle$ is free.

If $s \in T$, then for every primitive central idempotent $e = \widetilde{H}\left(\frac{1-s}{2}\right)$ of $\mathbb{Q}[G]\left(\frac{1-s}{2}\right)$ (so H is a subgroup of $\mathcal{Z}(G)$ with $s \notin H$ and $\mathcal{Z}(G)/H$ is cyclic) the group H cannot contain T. Hence, by (2), $d_i e = 0$ for $i = 1,2,3,4$ and thus $ab\left(\frac{1-s}{2}\right) = 0$, so $ab = 0$. Therefore, by Theorem 2.1, $\langle u, \varphi(u) \rangle$ is torsion-free abelian.

□

Remark.

We note that in the proof of the Theorem it is not essential that φ is an involution. The result actually characterizes when the non-trivial bicyclic unit $u_{x,y} = 1 + (1-x)y\widehat{x}$ and $u'_{x',y'} = 1 + \widehat{x'}y'(1-x')$ generate a free group, where $x', y' \in G$ are such that $G/\mathcal{Z}(G) = \langle x'\mathcal{Z}(G), y'\mathcal{Z}(G) \rangle$.

Note that there are six cases to be dealt with; when $x' = \varphi(x)$ and $y' = \varphi(y)$ with φ an involution on G then the cases reduce to the four listed in (1). Hence to characterize when $\langle u_{x,y}, u'_{x',y'}\rangle$ is free we also have to deal with $x' = z_1 xy$, $y' = z_2 x$ and $x' = z_1 y$, $y' = z_2 xy$. These are handled in a similar manner.

Since $u_{x',y'} = u'_{sx',y'}$ we then know when any two bicyclic (of both types) generate a free group.

Theorem 2.3. *Let G be a finite group that is not Hamiltonian and such that $G/\mathcal{Z}(G) \cong C_2 \times C_2$ and let $u_{x,y}$ and $u_{x',y'}$ be non-trivial bicyclic units.*

Denote by $s = (x, y) = (x', y')$. Put $T = \langle x^2, sx'x^{-1}\rangle$ in case $x'x^{-1}$ is central, otherwise put $T = \langle x^2, x'^2\rangle$.

Then $\langle u_{x,y}, u_{x',y'}\rangle$ is a free group of rank two if and only if $s \notin T$. Otherwise, it is a torsion-free abelian group.

2.2 The class $\mathcal{G}$

Recall that the class $\mathcal{G}$ consists of the finite groups G for which every non-linear irreducible complex representation is of degree 2 and with commutator subgroup G' a central elementary abelian 2-group.

Let $G \in \mathcal{G}$ and let $x, y \in G$ be such that $\langle x\rangle$ is not normal in $\langle x, y\rangle$ and thus $u = 1 + (1 - x)y\widehat{x}$ is a non-trivial bicyclic unit. Let S be a hyperplane of the elementary abelian 2-group G' not containing $t = (x, y)$. Obviously $|(G/S)'| = 2$ and thus by [10, Lemma 1.4] $(G/S)/\mathcal{Z}(G/S) \cong C_2 \times C_2$ and thus the primitive central idempotents of $\mathbb{Q}[G/S]\left(\frac{1-t}{2}\right)$ are given by

$$e = \widetilde{D}\left(\frac{1-t}{2}\right),$$

where D is a subgroup of G containing S such that $D/S \subseteq \mathcal{Z}(G/S)$ and $\mathcal{Z}(G/S)/(D/S)$ is cyclic and $t \notin D$.

We now can deduce the structure of the group $\langle u, \varphi(u)\rangle$, where φ is an arbitrary involution on G.

Theorem 2.4. *Let $G \in \mathcal{G}$ and let $u_{x,y}$ and $u_{x',y'}$ be non-trivial bicyclic units. Then $\langle u_{x,y}, u_{x',y'}\rangle$ is a free group if and only if there exists a hyperplane S of G' such that*

1. *$t = (x, y) \notin (\langle x\rangle \cap G')S$,*
2. *$t' = (x', y') \notin (\langle x'\rangle \cap G')S$,*
3. *t is not in T_S modulo S, where $T_S = \langle x^2, tx'x^{-1}\rangle$ if $x'x^{-1}$ is central modulo S, and $T_S = \langle x^2, x'^2\rangle$ otherwise.*

Otherwise, $\langle u_{x,y}, u_{x',y'}\rangle$ *is a torsion-free abelian group.*

Proof. Let S be a hyperplane of G' satisfying conditions (1) to (3). Condition (1) says that $\langle xS\rangle$ is not normalized by yS and thus the natural image of $u_{x,y}$ is a power of a non-trivial bicyclic unit in $\mathbb{Z}[G/S]$. Similarly, condition (2) says that the natural image of $u_{x',y'}$ is a power of a non-trivial bicyclic unit in $\mathbb{Z}[G/S]$. Also $G/S = \langle xS, yS, \mathcal{Z}(G/S)\rangle = \langle x'S, y'S, \mathcal{Z}(G/S)\rangle$ and $(G/S)/(\mathcal{Z}(G/S)) \cong C_2 \times C_2$.

It follows that $x'S$ equals an element of the from z_1xS, z_1yS or z_1xyS for some $z_1 \in G$ so that $z_1S \in \mathcal{Z}(G/S)$. If, for example $x'S = z_1xS$ then since $x'S$ and $y'S$ do not commute, the lack of commutativity in G/S implies that $y'S = z_2yS$ or $y'S = z_2xyS$ for some $z_2 \in G$ so that $z_2S \in \mathcal{Z}(G/S)$. The other cases are dealt with similarly. Hence, because of Theorem 2.3 the result follows.

If there does not exist a hyperplane S of G' with conditions (1) to (3), then for every hyperplane S of G' either $u_{x,y}$ becomes trivial modulo S, or $u_{x',y'}$ becomes trivial modulo S or the natural images of $u_{x,y}$ and $u_{x',y'}$ commute in $\mathbb{Z}[G/S]$. It follows that in every non-commutative simple component $\mathbb{Q}[G]e$ of $\mathbb{Q}[G]$, $\langle u_{x,y}, u_{x',y'}\rangle$ is abelian and hence $(u_{x,y}-1)(u_{x',y'}-1)$ is nilpotent. It then follows easily from Theorem 2.1 that $\langle u_{x,y}, u_{x',y'}\rangle$ is a torsion-free abelian group. □

Examples.

1. We recover the result of Marciniak and Sehgal for the class of finite groups G which are not Hamiltonian and such that $G/\mathcal{Z}(G) \cong C_2 \times C_2$. Take $x, y \in G$ such that $s = (x, y) \notin \langle x\rangle$, then $u_{x,y} = 1 + (1-x)y\widehat{x}$ is a non-trivial bicyclic unit. For the classical involution $*$, $x^*x^{-1} = x^{-2}$ is central. Hence $T = \langle x^2, x^{-2}\rangle$, which does not contain s by assumption. Therefore, by Theorem 2.3 $\langle u_{x,y}, u^*_{x,y} = u_{sx^{-1},y^{-1}}\rangle$ is free.

2. Consider $u_{b,a} = 1 + (1-b)a(1+b)$ in $\mathbb{Z}[D^+_{16}]$. Let $\varphi(b) = b$ and $\varphi(a) = a^5$, then $T = \{1\}$ and hence $\langle u, \varphi(u)\rangle$ is free. For $\psi(b) = a^4b$ and $\psi(a) = a^3$, we have that $s \in T = \{1, a^4\}$ and hence $\langle u, \psi(u)\rangle$ is torsion-free abelian.

3 Subgroups of finite index

In this section we construct a subgroup of finite index in $\mathcal{U}(\mathbb{Z}[G])$ for $G \in \mathcal{G}$. In order to do so we recall the following definition.

Definition. *[3] For an involution φ of G, put*

$$\mathcal{U}_\varphi(\mathbb{Q}[G]) = \{u \in \mathcal{U}(\mathbb{Q}[G]) \mid u\varphi(u) = 1\}$$

and

$$\mathcal{U}_\varphi(\mathbb{Z}[G]) = \mathcal{U}_\varphi(\mathbb{Q}[G]) \cap \mathbb{Z}[G],$$

these units are called φ-unitary*. If $\varphi_1, \ldots, \varphi_n$ all are involutions on G, then we put*

$$\mathcal{U}_{\varphi_1,\ldots,\varphi_n}(\mathbb{Z}[G]) = \langle \mathcal{U}_{\varphi_i}(\mathbb{Z}[G]) \mid i = 1, \ldots, n \rangle.$$

We will prove that for each non-commutative Wedderburn component $\mathbb{Q}[G]e_i$ $(i = 1, \ldots n)$ of $\mathbb{Q}[G]$ there exists an involution φ_i on G such that the group generated by the Bass cyclic units and $\mathcal{U}_{\varphi_1,\ldots,\varphi_n}(\mathbb{Z}[G])$ is of finite index in $\mathcal{U}(\mathbb{Z}[G])$. The first part of the proof is done following the same lines of [3], where this result is proved for groups of order 16. For completeness' sake we give a compact version of the argument.

Theorem 3.1. *Let $G \in \mathcal{G}$. Denote by B_G the group generated by the Bass cyclic units of $\mathbb{Z}[G]$. Then there exist involutions $\varphi_1, \ldots, \varphi_n$ on G such that*

$$\langle B_G,\, \mathcal{U}_{\varphi_1,\ldots,\varphi_n}(\mathbb{Z}[G]) \rangle$$

is a subgroup of finite index in $\mathcal{U}(\mathbb{Z}[G])$.

Proof. First, let G be such that $G/\mathcal{Z}(G) \cong C_2 \times C_2$ and let $x, y \in G$ be such that $G = \langle x, y, \mathcal{Z}(G) \rangle$. Denote by s the unique commutator of G. Then by [4, Theorem III.3.3] G has an involution φ defined by

$$\varphi(g) = \begin{cases} g & \text{if } g \text{ is central,} \\ sg & \text{otherwise.} \end{cases} \tag{3}$$

By [4, Corollary VI.4.8] $\mathbb{Q}[G] \cong \bigoplus_i D_i$, a direct sum of fields and generalized quaternion algebras over fields. Let e_i be a primitive central idempotent of $\mathbb{Q}[G]$ such that $D_i = \mathbb{Q}[G]e_i$ and let O_i be a $\mathbb{Z}$-order in D_i. Because G is nilpotent, by [8] the group generated by the Bass cyclic units contains a subgroup of finite index in $\bigoplus_i \mathcal{Z}(\mathcal{U}(O_i))$. Hence to prove the result it is sufficient to search for a subgroup (of φ-unitary units) that contains a subgroup of finite index in $SL_1(O_i)$, provided D_i is a generalized quaternion algebra. Recall that by definition $SL_1(O_i) = SL_1(D_i) \cap O_i$, where $SL_1(D_i)$ is the group of elements q of reduced norm $nr(q) = q\overline{q} = 1$, where $-$ denotes the standard involution with respect to the basis $\{e_i, xe_i, ye_i, xye_i\}$ of this generalized quaternion algebra. Now for each such D_i we have that $\varphi(e_i) = e_i$ because the support of e_i is central and

$$\varphi(ge_i) = \overline{ge_i},$$

where $g \in G$. Because $-$ is linear, we get that $\varphi(q) = \overline{q}$ for all $q \in D_i$. Hence $SL_1(D_i)$ equals the image in D_i of the φ-unitary units of $\mathbb{Q}[G]$. Since general order theory gives us that $\mathcal{U}(\mathbb{Z}[G])$ and $\bigoplus_i GL_1(O_i)$ have a common subgroup of finite index, we have that $\mathcal{U}(\mathbb{Z}[G])$ contains a subgroup of finite index in each $(1-e_i)+GL_1(O_i)$, where e_i is the unity of D_i. Consequently, the φ-unitary units of $\mathbb{Z}[G]$ contain a subgroup of finite index in each $(1-e_i)+SL_1(O_i)$, as desired.

Now, let $G \in \mathcal{G}$ and let e_k be a primitive central idempotent of the rational group algebra $\mathbb{Q}[G]$ determining a non-commutative Wedderburn component. We will show that there exists an involution φ_k on G that induces the involution (3) on $H = Ge_k$, in particular $\varphi_k(e_k) = e_k$. Since the simple components of $\mathbb{Q}[H]$ are simple components of $\mathbb{Q}[G]$ and $H/\mathcal{Z}(H) \cong C_2 \times C_2$, the case above and again order theory, yield the result.

Let $H = \langle x_1, x_2, \mathcal{Z}(H)\rangle$, for some $x_1, x_2 \in G$ with x_1^2 and x_2^2 central in G. Let S be a hyperplane of the elementary abelian 2-group G' that does not contain $t = (x_1, x_2)$. Then $e_k = \widetilde{D}\left(\frac{1-t}{2}\right)$, where D is a subgroup of G containing S such that $D/S \subseteq \mathcal{Z}(G/S)$ and $\mathcal{Z}(G/S)/(D/S)$ is cyclic and $t \notin D$. As $G/\mathcal{Z}(G)$ is an elementary abelian 2-group, say of rank n, we can write $G = \langle x_1, x_2, \ldots, x_n, \mathcal{Z}(G)\rangle$ with $x_i^2 \in \mathcal{Z}(G), 1 \le i \le n$ and x_i central modulo S for $3 \le i \le n$.

Any element $g \in G$ can be written uniquely as

$$g = zx_1^{a_1}x_2^{a_2}\ldots x_n^{a_n},$$

with $z \in \mathcal{Z}(G)$, $a_i \in \{0,1\}, 1 \le i \le n$. Put $t_{ij} = (x_i, x_j)$. Since G' is an elementary abelian 2-group we have that $t_{ij} = t_{ji}$. Let $\varphi_k : G \to G$ be given by

$$\varphi_k(zx_1^{a_1}x_2^{a_2}\ldots x_n^{a_n}) = zt_{12}^{a_1+a_2}\prod_{i\ge 1}(\prod_{j\ge 2, j>i} t_{ij}^{a_ia_j})x_i^{a_i}$$

Notice that the map φ_k is defined on the generators as $\varphi_k(x_1) = t_{12}x_1$, $\varphi_k(x_2) = t_{12}x_2$ and $\varphi_k(x_i) = x_i$ for all $i \ge 3$. Also $\varphi_k(z) = z$ for $z \in \mathcal{Z}(G)$. Note that the support of e_k is $D \cup Dt$. Suppose that $x_1^{a_1}x_2^{a_2}x \in D$ with $x \in \langle \mathcal{Z}(G), x_3, \ldots, x_n\rangle$ and $a_1, a_2 \in \{0,1\}$ (but not both equal to zero) then $x_1^{a_1}x_2^{a_2}xe_k = e_k$, thus $x_1^{a_1}x_2^{a_2}e_k \in \mathcal{Z}(H)$ and therefore $te_k = e_k$, a contradiction. A similar reasoning holds for $x_1^{a_1}x_2^{a_2}x \in Dt$. So the support of e_k is contained in $\langle \mathcal{Z}(G), x_3, \ldots, x_n \rangle$. Hence $\varphi_k(e_k) = e_k$. Using the fact that G' is of exponent 2 we easily can see that φ_k is an anti-automorphism and that $\varphi_k^2 = 1$. Furthermore, if we restrict the involution φ_k to the simple component $\mathbb{Q}[G]e_k$ it induces (3). □

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