Received: 25 October 2018 | Revised: 24 April 2019 | Accepted: 28 May 2019
DOI: 10.1002/cmm4.1048

RESEARCH ARTICLE

WILEY

Conformable Euler's theorem on homogeneous functions

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Recently, the conformable derivative and its properties have been introduced. In this paper, we propose and prove some new results on the conformable multivariable fractional calculus. We introduce a conformable version of classical Euler's theorem on homogeneous functions. Furthermore, we are extending the aforementioned result for higher-order partial derivatives.

KEYWORDS
conformable Euler's theorem, conformable fractional derivative, multivariate conformable fractional calculus

1 | INTRODUCTION

For many years, many definitions of fractional derivative have been introduced by various authors. Commonly, these definitions are not known to many researchers and, until recent years, it has been used in a purely mathematical context only. However, during the last decades, these kinds of derivative operators have been applied to many science context due, in part, to their frequent appearance in various applications in the fields of viscoelasticity, fluid mechanic, biology, physic entropy theory, and engineering.[1-8] Nowadays, many definitions of fractional derivative have been introduced, but most of them are in integral form, which is more complicated to manage. The most known are the Riemann-Liouville definition and the Caputo definition, those are both defined using integral forms as follows.

(i) Riemann-Liouville definition. For $\alpha \in [n-1, n)$, the α derivative of f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx.$$

(ii) Caputo definition. For $\alpha \in [n-1, n)$, the α derivative of f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)}) \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx,$$

where $\Gamma(x)$ is the Gamma function.

Of course, all definitions attempt to satisfy the usual properties of the classical derivative; however, the only property inherited for all of the given definitions is the linearity property, for example, the following are setbacks of one or another definition.

1. The Riemann-Liouville derivative does not satisfy $D_a^{\alpha}(1) = 0$ if α is not a natural number.

2. The fractional derivatives proposed until now lose some of the basic properties that usual derivatives have, such as the product rule, the quotient rule, or the chain rule.
3. In general, the fractional derivatives proposed until now do not satisfy $D^{\alpha}D^{\beta}f = D^{\alpha+\beta}f$.
4. The Caputo definition implies that function f must be differentiable.

Recently, Khalil et al[9] introduced a new definition of fractional derivative called the conformable fractional derivative. Unlike other definitions, this new definition satisfies the rules for the derivative of the product or the quotient of two functions and has a simpler chain rule. In addition to this conformable fractional derivative definition, the conformable integral definition, Rolle theorem, and mean value theorem for conformable fractional differentiable functions were given. Abdeljawad[10] improved this new theory. For instance, definitions of left and right conformable fractional derivatives and fractional integrals of higher order (ie, of order $\alpha > 1$), the fractional power series expansion and the fractional transform Laplace definition, fractional integration by parts formulas, chain rule, and Gronwall inequality are also provided by him. Moreover, the conformable partial derivative of the order $\alpha \in$ of the real value of several variables and conformable gradient vector are defined[11,12]; and a conformable version of Clairaut's theorem for partial derivatives of conformable fractional orders is proved.[11,12] In short time, many studies[13-21] about theory and application of the fractional differential equations are based on this new fractional derivative definition.

On the other hand, Euler's theorem on homogeneous functions is used to solve many problems in engineering, science, and finance. Hiwarekar[22] discussed the extension and applications of Euler's theorem for finding the values of higher-order expressions for two variables. In a later work, Shah and Sharma[23] extended the results from the function of two variables to n variables and obtain results for higher-order derivatives.

This paper is organized as follows. In Section 2, the main concepts of the conformable fractional calculus are presented. In Section 3, some classical results on homogeneous functions are recalled, then two new results on homogeneous functions involving their conformable partial derivatives are introduced, specifically, the homogeneity of the conformable partial derivatives of a homogeneous function and the conformable version of Euler's theorem. In Section 4, the conformable version of Euler's theorem is introduced and proved. In addition, this result is extended to higher-order derivatives. The conclusions are given in Section 5.

2 | BASIC DEFINITIONS AND TOOLS

Definition 1. Give a function $f : [0, \infty) \to \mathbb{R}$. Then, the conformable fractional derivative of f of order α is defined[9] by

$$(T_{\alpha} f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \tag{1}$$

for all $t > 0$, $0 < \alpha \leq 1$.

If f is α-differentiable in some $(0, a)$, $a > 0$, and $\lim_{t\to 0^+}(T_{\alpha} f)(t)$ exist, then it is defined as

$$(T_{\alpha} f)(0) = \lim_{t\to 0^+}(T_{\alpha} f)(t). \tag{2}$$

As a consequence of the aforementioned definition, the following useful theorem is obtained.[9]

Theorem 1. *If a function $f : [0, \infty) \to \mathbb{R}$ is α-differentiable at $t_0 > 0$, $0 < \alpha \leq 1$, then f is continous at t_0.*

It is easily shown than T_{α} satisfies the following properties.[9]

Theorem 2. *Let $0 < \alpha \leq 1$ and f, g be a α-differentiable at a point $t > 0$. Then, we have the following:*

(i) $T_{\alpha}(af + bg) = a(T_{\alpha} f) + b(T_{\alpha} g)$, *for all* $a, b \in \mathbb{R}$;
(ii) $T_{\alpha}(t^p) = pt^{p-\alpha}$, *for all* $p \in \mathbb{R}$;
(iii) $T_{\alpha}(\lambda) = 0$, *for any constant* $\lambda \in \mathbb{R}$;
(iv) $T_{\alpha}(f \cdot g) = f \cdot (T_{\alpha} g) + g \cdot (T_{\alpha} f)$;
(v) $T_{\alpha}\left(\frac{f}{g}\right) = \frac{g\cdot(T_{\alpha} f) - f\cdot(T_{\alpha} f)}{g^2}$;
(vi) *if, in addition, f is differentiable, then* $(T_{\alpha} f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$.

The conformable fractional derivative of certain functions for the aforementioned definition is given as follows:

(i) $T_\alpha(1) = 0$;
(ii) $T_\alpha(\sin(at)) = at^{1-\alpha}\cos(at), a \in \mathbb{R}$;
(iii) $T_\alpha(\cos(at)) = -at^{1-\alpha}\sin(at), a \in \mathbb{R}$;
(iv) $T_\alpha(e^{at}) = at^{1-\alpha}e^{at}, a \in \mathbb{R}$.

Furthermore, many functions behave as in the usual derivative. The following are some formulas:

(i) $T_\alpha\left(\frac{1}{\alpha}t^\alpha\right) = 1$;
(ii) $T_\alpha\left(e^{\frac{1}{\alpha}t^\alpha}\right) = e^{\frac{1}{\alpha}t^\alpha}$;
(iii) $T_\alpha\left(\sin\left(\frac{1}{\alpha}t^\alpha\right)\right) = \cos\left(\frac{1}{\alpha}t^\alpha\right)$;
(iv) $T_\alpha\left(\cos\left(\frac{1}{\alpha}t^\alpha\right)\right) = -\sin\left(\frac{1}{\alpha}t^\alpha\right)$.

Remark 1. One should notice that a function could be α-differentiable at a point but not differentiable. For example, take $f(t) = 3\sqrt[3]{t}$. Then, $(T_{\frac{1}{3}}f)(0) = \lim_{t\to 0^+}(T_{\frac{1}{3}}f)(t) = 1$, where $(T_{\frac{1}{3}}f)(t) = 1$, for t>0, but $\frac{df}{dt}(0)$ does not exist.

Definition 2. The (left) conformable derivartive starting from a of a function $f : [0,\infty) \to \mathbb{R}$ de f of order $0 < \alpha \leq 1$ is defined[10] by

$$\left(T_\alpha^a f\right)(t) = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon(t-a)^{1-\alpha}) - f(t)}{\varepsilon}. \tag{3}$$

When $a = 0$, it is written as $(T_\alpha f)(t)$. If f is α-differentiable in some (a, b), then define

$$\left(T_\alpha^a f\right)(a) = \lim_{t\to a^+}(T_\alpha f)(t). \tag{4}$$

Note that, if f is differentiable, then $(T_\alpha^a f)(t) = (t-a)^{1-\alpha}\frac{df}{dt}(t)$. All properties in Theorem 2 are valid also for Definition 2 when t is replaced by $(t-a)$. Conformable fractional derivative of certain functions for Definition 2 is given as follows:

(i) $T_\alpha^a((t-a)^p) = p(t-a)^{p-\alpha}, \forall p \in \mathbb{R}$;
(ii) $T_\alpha^a\left(e^{\lambda\frac{(t-a)^\alpha}{\alpha}}\right) = \lambda e^{\lambda\frac{(t-a)^\alpha}{\alpha}}$;
(iii) $T_\alpha^a\left(\sin\left(\omega\frac{(t-a)^\alpha}{\alpha} + C\right)\right) = \omega\cos\left(\omega\frac{(t-a)^\alpha}{\alpha} + C\right), \forall\omega, C \in \mathbb{R}$;
(iv) $T_\alpha^a\left(\cos\left(\omega\frac{(t-a)^\alpha}{\alpha} + C\right)\right) = -\omega\sin\left(\omega\frac{(t-a)^\alpha}{\alpha} + C\right), \forall\omega, C \in \mathbb{R}$;
(v) $T_\alpha^a\left(\frac{(t-a)^\alpha)}{\alpha}\right) = 1$.

Theorem 3 (Chain rule[10]). *Assume* $f, g : (a,\infty) \to \mathbb{R}$ *be (left)* α*-differentiable functions, where* $0 < \alpha \leq 1$. *Let* $h(t) = f(g(t))$. *Then,* $h(t)$ *is* α*-differentiable for all* $t \neq a$ *and* $g(t) \neq 0$ *and*

$$\left(T_\alpha^a h\right)(t) = \left(T_\alpha^a f\right)(g(t)) \cdot \left(T_\alpha^a g\right)(t) \cdot (g(t))^{\alpha-1}. \tag{5}$$

If $t = a$, *then*

$$\left(T_\alpha^a h\right)(a) = \lim_{t\to a^+}\left(T_\alpha^a f\right)(g(t)) \cdot \left(T_\alpha^a g\right)(t) \cdot (g(t))^{\alpha-1}. \tag{6}$$

Remark 2. In the work of Abdeljawad,[10] the left conformable fractional derivative at a for some smooth funcions is discussed. Let $0 < \alpha \leq 1$ and $n \in \mathbb{Z}^+$, then the left sequential conformable fractional derivative of order n is defined by the expression

$$^{(n)}T_\alpha^a f(t) = \underbrace{T_\alpha^a T_\alpha^a \ldots T_\alpha^a(t)}_{n\text{-times}}. \tag{7}$$

Proceeding inductively, it is easy to show that, if f is n-continuously differentiable and $0 < \alpha \leq \frac{1}{n}$, then the nth-order sequential conformable fractional derivative is continuous and vanishes at end point a.

Finally, the conformable partial derivative of a real valued function with several variables is defined[11,12] as follows.

Definition 3. Let f be a real valued function with n variables and $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$ be a point whose ith component is positive. Then, the limit

$$\lim_{\varepsilon \to 0} \frac{f\left(a_1, \dots, a_i + \varepsilon a_i^{1-\alpha}, \dots, a_n\right) - f(a_1, \dots, a_n)}{\varepsilon}, \tag{8}$$

if it exists, is denoted $\frac{\partial^\alpha}{\partial x_i^\alpha} f(\mathbf{a})$, and called the ith conformable partial derivative of f of the order $\alpha \in (0, 1]$ at point $\mathbf{a}$.

Remark 3. If a real valued function f with n variables has all conformable partial derivatives of the order α at $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$, each $a_i > 0$, then the conformable α-gradient of f of the order $\alpha \in (0, 1]$ at $\mathbf{a}$ is

$$\nabla f(\mathbf{a}) = \left(\frac{\partial^\alpha}{\partial x_1^\alpha} f(\mathbf{a}), \dots, \frac{\partial^\alpha}{\partial x_n^\alpha} f(\mathbf{a}) \right). \tag{9}$$

In the work of Atangana et al,[11] Clairaut's theorem for partial derivatives of conformabe fractional orders is probed.

Theorem 4. *Let be α, β positive constants, such that $0 < \alpha, \beta \leq 1$. Assume that $f(x, y)$ is a function for which $\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{\partial^\beta f(x,y)}{\partial y^\beta}\right)$ and $\frac{\partial^\beta}{\partial y^\beta}\left(\frac{\partial^\alpha f(x,y)}{\partial x^\alpha}\right)$ exist and are continuous over a domain $D \subset \mathbb{R}^2$, then*

$$\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{\partial^\beta f(x, y)}{\partial y^\beta}\right) = \frac{\partial^\beta}{\partial y^\beta}\left(\frac{\partial^\alpha f(x, y)}{\partial x^\alpha}\right). \tag{10}$$

3 | CONFORMABLE PARTIAL DERIVATIVES OF HOMOGENEOUS FUNCTIONS

In this section, some classic results on homogeneous functions are recalled.[24]

Definition 4. Let f be a real valued function with n variables defined on a set $D \subset \mathbb{R}^n$ for which $(tx_1, \dots, tx_n) \in D$ whenever $t > 0$ and $(x_1, \dots, x_n) \in D$. Then, f is homogeneous function of degree r if

$$f(tx_1, \dots, tx_n) = t^r f(x_1, \dots, x_n) \text{ for all } (x_1, \dots, x_n) \in D \text{ and } t > 0. \tag{11}$$

Theorem 5. *Let f, g be a real valued functions with n variables defined on a set $D \subset \mathbb{R}^n$ for which $(tx_1, \dots, tx_n) \in D$ whenever $t > 0$ and $(x_1, \dots, x_n) \in D$. Suposse that f and g are homogeneous functions of degree r. Then, $f + g$ and λf are homogeneous functions of degree r.*

Proof.

$$\begin{aligned}
(f + g)(tx_1, \dots, tx_n) &= f(tx_1, \dots, tx_n) + g(tx_1, \dots, tx_n) = t^r f(x_1, \dots, x_n) + t^r g(x_1, \dots, x_n) \\
&= t^r(f(x_1, \dots, x_n) + g(x_1, \dots, x_n)) = t^r(f + g)(x_1, \dots, x_n). \\
(\lambda f)(tx_1, \dots, tx_n) &= \lambda f(tx_1, \dots, tx_n) = \lambda t^r f(x_1, \dots, x_n) = t^r(\lambda f(x_1, \dots, x_n)) = t^r(\lambda f)(x_1, \dots, x_n).
\end{aligned}$$

□

Theorem 6. *Let f, g be a real valued functions with n variables defined on a set $D \subset \mathbb{R}^n$ for which $(tx_1, \dots, tx_n) \in D$ whenever $t > 0$ and $(x_1, \dots, x_n) \in D$. Suppose that f and g are homogeneous functions of degree r and s, respectively. Then, $f \cdot g$ and f/g (for all $(x_1, \dots, x_n) \in D$ such that $g(x_1, \dots, x_n) \neq 0$) are homogeneous functions of degree $r + s$ and $r - s$, respectively.*

Proof.

$$(f \cdot g)(tx_1, \dots, tx_n) = f(tx_1, \dots, tx_n) \cdot g(tx_1, \dots, tx_n) = t^r f(x_1, \dots, x_n) \cdot t^s g(x_1, \dots, x_n)$$
$$= t^{r+s}(f(x_1, \dots, x_n) \cdot g(x_1, \dots, x_n)) = t^{r+s}(f \cdot g)(x_1, \dots, x_n).$$
$$\left(\frac{f}{g}\right)(tx_1, \dots, tx_n) = \frac{f(tx_1, \dots, tx_n)}{g(tx_1 \dots, tx_n} = \frac{t^r f(x_1, \dots, x_n)}{t^s g(x_1, \dots, x_n)} = t^{r-s}\frac{f(x_1, \dots, x_n)}{g(x_1, \dots, x_n)} = t^{r-s}\left(\frac{f}{g}\right) f(x_1, \dots, x_n).$$

□

Theorem 7. *Let f be a real valued function with n variables defined on a set $D_1 \subset \mathbb{R}^n$ for which $(tx_1, \dots, tx_n) \in D_1$ whenever $t > 0$ and $(x_1, \dots, x_n) \in D_1$. Let g be a real valued function on a set $D_2 \subset \mathbb{R}$ for which $ty \in D_2$, whenever $t > 0$ and $y \in D_2$. Assume that $f(D_1) \subset D_2$ and that f and g are homogeneous functions of degree r and s, respectively, then $g \circ f$ is a homogeneous function of degree rs.*

Proof.

$$(g \circ f)(tx_1, \dots, tx_n) = g(f(tx_1, \dots, tx_n)) = g\left(t^r f(x_1, \dots, tx_n)\right) = (t^r)^s g(f(x_1, \dots, x_n)) = t^{rs}(g \circ f)(x_1, \dots, x_n).$$

□

Now, two new results on homogeneous functions involving conformable partial derivatives are presented.

Remark 4. Let $\alpha \in (0, \frac{1}{p}]$, $p \in \mathbb{Z}^+$ and f be a real valued function with n variables defined on an open set $D \subset \mathbb{R}^n$, such that, for all $(x_1, \dots, x_n) \in D$, each $x_i > 0$. Function f is said to be in $C^p_\alpha(D, \mathbb{R})$ if all its conformable fractional partial derivatives of order $\leq p$ exist and are continuous on D.

Theorem 8. *Let $\alpha \in (0, \frac{1}{p}]$, $p \in \mathbb{Z}^+$ and f be a real valued function with n variables defined on an open set $D \subset \mathbb{R}^n$ for which $(tx_1, \dots, tx_n) \in D$ whenever $t > 0$ and $(x_1, \dots, x_n) \in D$, each $x_i > 0$, that satisfies the following.*

(i) f is homogeneous function of degree r.
(ii) $f \in C^p_\alpha(D, \mathbb{R})$.

Then, each of its conformable partial derivatives of order q, with $q \leq p$, is homogeneous functions of degree $r - q\alpha$.

Proof. Next, the Principle of Mathematical Induction on q is used. For $q = 1$, computing[10] the conformable partial derivative of Equation 11 with respect to x_i, for $i = 1, 2, \dots, n$, then

$$\frac{\partial^\alpha f(tx_1, \dots, tx_n)}{\partial(tx_i)^\alpha} \cdot (tx_i)^{\alpha-1} \cdot \frac{\partial^\alpha(tx_i)}{\partial x_i^\alpha} = t^r \cdot \frac{\partial^\alpha f(x_1, \dots, x_n)}{\partial(x_i)^\alpha},$$

using $\frac{\partial^\alpha(tx_i)}{\partial x_i^\alpha} = tx_i^{1-\alpha}$ and dividing both sides by t^α to get

$$\frac{\partial^\alpha f(tx_1, \dots, tx_n)}{\partial(tx_i)^\alpha} = t^{r-\alpha} \cdot \frac{\partial^\alpha f(x_1, \dots, x_n)}{\partial(x_i)^\alpha}.$$

Therefore, the result is true for $q = 1$.

Assume that this result is true for $q = k$, that is,

$$\left(\frac{{}^{(k)}\partial^\alpha f(tx_1, \dots, tx_n)}{\partial(tx_{i_1})^\alpha \cdot \partial(tx_{i_2})^\alpha \dots \partial(tx_{i_k})^\alpha}\right) = t^{r-k\alpha} \cdot \left(\frac{{}^{(k)}\partial^\alpha f(x_1, \dots, x_n)}{\partial(x_{i_1})^\alpha \cdot \partial(x_{i_2})^\alpha \dots \partial(x_{i_k})^\alpha}\right).$$

Let $n = k + 1$, computing the conformable partial derivative of the aforementioned equation with respect to x_i, for $i = 1, 2, \dots, n$,[10] then

$$\left(\frac{{}^{(k+1)}\partial^\alpha f(tx_1, \dots, tx_n)}{\partial(tx_{i_1})^\alpha \cdot \partial(tx_{i_2})^\alpha \dots \partial(tx_{i_{k+1}})^\alpha}\right)(tx_i)^{\alpha-1}\frac{\partial^\alpha(tx_i)}{\partial x_i^\alpha} = t^{r-k\alpha} \cdot \frac{{}^{(k+1)}\partial^\alpha f(x_1, \dots, x_n)}{\partial(x_{i_1})^\alpha \cdot \partial(x_{i_2})^\alpha \dots \partial(x_{i_{k+1}})^\alpha},$$

using $\frac{\partial^\alpha(tx_i)}{\partial x_i^\alpha} = tx_i^{1-\alpha}$ and dividing both sides by t^α to get

$$\left(\frac{{}^{(k+1)}\partial^\alpha f(tx_1,\ \dots,tx_n)}{\partial(tx_{i_1})^\alpha\cdot\partial(tx_{i_2})^\alpha\dots\partial(tx_{i_{k+1}})^\alpha}\right) = t^{r-(k+1)\alpha}\cdot\frac{{}^{(k+1)}\partial^\alpha f(x_1,\ \dots,x_n)}{\partial(x_{i_1})^\alpha\cdot\partial(x_{i_2})^\alpha\dots\partial(x_{i_{k+1}})^\alpha}.$$

Thus, this result is true for $q = k + 1$.

Therefore, by the Principle of Mathematical Induction, this result is true for any positive integer $q \leq p$. This completes the proof of the theorem. □

4 | CONFORMABLE EULER'S THEOREM ON HOMOGENEOUS FUNCIONS AND ITS EXTENSION

In this section, the version conformable of Euler's theorem on homogeneous functions is proposed.

Theorem 9. *Let $\alpha \in (0,1]$ and f be a real valued function with n variables defined on an open set $D \subset \mathbb{R}^n$ for which $(tx_1,\ \dots,tx_n) \in D$ whenever $t > 0$ and $(x_1,\ \dots,x_n) \in D$, each $x_i > 0$. Assuming that $f \in C_\alpha(D,\mathbb{R})$, then f is homogeneous function of degree r if and only if*

$$\sum_{i=1}^{n} x_i^\alpha\cdot\frac{\partial^\alpha f(x_1,\ \dots,x_n)}{\partial x_i^\alpha} = rf(x_1\dots,x_n),\quad ,\forall(x_1,\ \dots,x_n)\in D. \tag{12}$$

Proof. Let us probe that, if f is homogeneous function of degree r, then 12 holds. Let $(x_1,\ \dots,x_n) \in D$.

Consider the function

$$g(t) = f(tx_1,\ \dots,tx_n),\forall t > 0.$$

Since g is α-differentiable on $(0,\infty)$, then, applying the conformable chain rule,[10] it is given that

$$(T_\alpha g)(t) = \sum_{i=1}^{n}\frac{\partial^\alpha f(tx_1,\ \dots,tx_n)}{\partial(tx_i)^\alpha}\cdot(tx_i)^{\alpha-1}\cdot T_\alpha(tx_i) = \sum_{i=1}^{n}\frac{\partial^\alpha f(tx_1,\ \dots,tx_n)}{\partial(tx_i)^\alpha}\cdot(tx_i)^{\alpha-1}\cdot x_i\cdot t^{1-\alpha} = \sum_{i=1}^{n}x_i^\alpha\cdot\frac{\partial^\alpha f(tx_1,\ \dots,tx_n)}{\partial(tx_i)^\alpha}.$$

Since f is homogeneous function of degree r, then

$$g(t) = f(tx_1,\ \dots,tx_n) = t^r f(x_1,\ \dots,x_n) \Rightarrow (T_\alpha g)(t) = rt^{r-\alpha}f(x_1,\ \dots,x_n).$$

Taking $t = 1$, our results follows.

Now, let us probe that, if 13 holds, then f is an homogeneous function of degree r. Let $(x_1,\ \dots,x_n) \in D$ and consider function $h(t)$ defined as

$$h(t) = \frac{f(tx_1,\ \dots,tx_n)}{t^r} - f(x_1,\ \dots,x_n),\forall t > 0.$$

Since h is α-differentiable on $(0,\infty)$, then applying the conformable chain rule[10] produces

$$\begin{aligned}(T_\alpha h)(t) &= \frac{\left(\sum_{i=1}^{n} x_i^\alpha\cdot\frac{\partial^\alpha f(tx_1,\ \dots,tx_n)}{\partial(tx_i)^\alpha}\right)\cdot t^r - rt^{r-\alpha}f(tx_1,\ \dots,tx_n)}{t^{2r}}\\ &= \frac{t^{r-\alpha}\left(\sum_{i=1}^{n}(tx_i)^\alpha\cdot\frac{\partial^\alpha f(tx_1,\ \dots,tx_n)}{\partial(tx_i)^\alpha} - rf(tx_1,\ \dots,tx_n)\right)}{t^{2r}} = 0,\end{aligned}$$

since, by hypothesis $\sum_{i=1}^{n} x_i^\alpha\cdot\frac{\partial^\alpha f(x_1,\dots,x_n)}{\partial(x_i)^\alpha} = rf(x_1,\ \dots,x_n)$ and, therefore, at point $(tx_1,\ \dots,tx_n)$ produce $\sum_{i=1}^{n}(tx_i)^\alpha\cdot\frac{\partial^\alpha f(tx_1,\dots,tx_n)}{\partial(tx_i)^\alpha} = rf(tx_1,\ \dots,tx_n)$.

If $(T_\alpha h)(t) = 0$ on $(0, \infty)$, then h is a constant and, as $h(1) = 0$, then $h(t) = 0$ on $(0, \infty)$. Therefore,

$$\frac{f(tx_1, \dots, tx_n)}{t^r} - f(x_1, \dots, x_n) = 0 \Rightarrow f(tx_1, \dots, tx_n) = t^r f(x_1, \dots, x_n).$$

Then, f is homogeneous function of degree r. □

Example 1. Using Theorem 9, it is easy to prove the following statement. Let $\alpha \in (0, 1]$ and f be a real valued function with n variables defined on an open set $D \subset \mathbb{R}^n$ for which $(tx_1, \dots, tx_n) \in D$ whenever $t > 0$ and $(x_1, \dots, x_n) \in D$, each $x_i > 0$, that satisfies the following:

(i) f is homogeneous function of degree r;
(ii) $f \in C_\alpha(D, \mathbb{R})$;
(iii) $rf > 0$.

Then,

$$f(x_1, \dots, x_n) = A(x_1, \dots, x_n) \cdot \prod_{i=1}^{n} \left(x_i^\alpha\right)^{\delta_i(x_1, \dots, x_n)}, \forall (x_1, \dots, x_n) \in D, \tag{13}$$

where

$$A(x_1, \dots, x_n) = \prod_{i=1}^{n} \left(\frac{\frac{\partial^\alpha f(x_1, \dots, x_n)}{\partial x_i^\alpha}}{r\delta_i(x_1, \dots, x_n)} \right)^{\delta_i(x_1, \dots, x_n)}, \tag{14}$$

with

$$\delta_i(x_1, \dots, x_n) = \frac{x_i^\alpha \frac{\partial^\alpha f(x_1, \dots, x_n)}{\partial x_i^\alpha}}{\sum_{j=1}^{n} x_j^\alpha \frac{\partial^\alpha f(x_1, \dots, x_n)}{\partial x_j^\alpha}} = \frac{x_i^\alpha \frac{\partial^\alpha f(x_1, \dots, x_n)}{\partial x_i^\alpha}}{r f(x_1, \dots, x_n)}, \forall i = 1, 2, \dots, n, \tag{15}$$

and

$$\sum_{i=1}^{n} \delta_i(x_1, \dots, x_n) = 1.$$

Solution. From Equation 15, we obtain

$$\frac{f(x_1, \dots, x_n)}{x_i^\alpha} = \frac{\frac{\partial^\alpha f(x_1, \dots, x_n)}{\partial x_i^\alpha}}{r\delta_i(x_1, \dots, x_n)}.$$

Now, it is straithforward to use those expression in 14 to get

$$A(x_1, \dots, x_n) = \prod_{i=1}^{n} \left(\frac{f(x_1, \dots, x_n)}{x_i^\alpha} \right)^{\delta_i(x_1, \dots, x_n)} = f(x_1, \dots, x_n) \prod_{i=1}^{n} \left(\frac{1}{x_i^\alpha} \right)^{\delta_i(x_1, \dots, x_n)}$$

and the result follows.

Finally, conformable Euler's theorem on homogeneous functions for higher-order derivative is extended.

Remark 5. Let $\alpha \in (0, \frac{1}{p}], p \in \mathbb{Z}^+$ and f be a real valued function with n variables defined on an open set $D \subset \mathbb{R}^n$. If f is a $C_\alpha^p(D, \mathbb{R})$ function on D, then we use following notation for simplicity:

(i)

$$\left(x_1^\alpha \frac{\partial^\alpha}{\partial x_1^\alpha} + x_2^\alpha \frac{\partial^\alpha}{\partial x_2^\alpha} + \dots + x_n^\alpha \frac{\partial^\alpha}{\partial x_n^\alpha} \right)^{(1)} f = \left(\sum_{i=1}^{n} x_i^\alpha \frac{\partial^\alpha}{\partial x_i^\alpha} \right) f = \sum_{i=1}^{n} x_i^\alpha \frac{\partial^\alpha f}{\partial x_i^\alpha};$$

(ii)

$$\left(x_1^\alpha \frac{\partial^\alpha}{\partial x_1^\alpha} + x_2^\alpha \frac{\partial^\alpha}{\partial x_2^\alpha} + \dots + x_n^\alpha \frac{\partial^\alpha}{\partial x_n^\alpha} \right)^{(2)} f = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} x_i^\alpha x_j^\alpha \frac{\partial^\alpha}{\partial x_i^\alpha} \frac{\partial^\alpha}{\partial x_j^\alpha} \right) f = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i^\alpha x_j^\alpha \frac{{}^{(2)}\partial^\alpha f}{\partial x_i^\alpha x_j^\alpha};$$

(iii)

$$\left(x_1^\alpha \frac{\partial^\alpha}{\partial x_1^\alpha} + x_2^\alpha \frac{\partial^\alpha}{\partial x_2^\alpha} + \ldots + x_n^\alpha \frac{\partial^\alpha}{\partial x_n^\alpha}\right)^{(p)} f = \left(\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_p=1}^{n} x_{i_1}^\alpha x_{i_2}^\alpha \ldots x_{i_p}^\alpha \frac{\partial^\alpha}{\partial x_{i_1}^\alpha}\frac{\partial^\alpha}{\partial x_{i_2}^\alpha}\cdots\frac{\partial^\alpha}{\partial x_{i_p}^\alpha}\right) f$$
$$= \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_p=1}^{n} x_{i_1}^\alpha x_{i_2}^\alpha \ldots x_{i_p}^\alpha \frac{{}^{(p)}\partial^\alpha f}{\partial x_{i_1}^\alpha x_{i_2}^\alpha \ldots x_{i_p}^\alpha}.$$

These will help to prove extension of conformable Euler's theorem on homogeneous functions.

Theorem 10 (Extension of conformable Euler's theorem on homogeneous functions)**.** *Let $\alpha \in (0, \frac{1}{p}]$, $p \in \mathbb{Z}^+$ and f be a real valued function with n variables defined on an open set $D \subset \mathbb{R}^n$ for which $(tx_1, \ldots, tx_n) \in D$ whenever $t > 0$ and $(x_1, \ldots, x_n) \in D$, each $x_i > 0$, that satisfies the following:*

(i) f is homogeneous function of degree r;
(ii) $f \in C_\alpha^p(D, \mathbb{R})$.

Then,

$$\left(x_1^\alpha \frac{\partial^\alpha}{\partial x_1^\alpha} + x_2^\alpha \frac{\partial^\alpha}{\partial x_2^\alpha} + \cdots + x_n^\alpha \frac{\partial^\alpha}{\partial x_n^\alpha}\right)^{(p)} f = r(r-\alpha)(r-2\alpha)\ldots(r-(p-1)\alpha)\cdot f. \tag{16}$$

Proof. This result is proved by the Principle of Mathematical Induction on p.

Let $p = 1$. By Theorem 9,

$$\left(\sum_{i=1}^{n} x_i^\alpha \frac{\partial^\alpha}{\partial x_i^\alpha}\right) f = \sum_{i=1}^{n} x_i^\alpha \cdot \frac{\partial^\alpha f}{\partial x_i^\alpha} = r\cdot f. \tag{17}$$

Therefore, these results are true for $p = 1$.

Now, take $p = 2$. Compute the conformable partial derivative[10] of Equation 17 with respect to x_i, then multiply it by x_i^α, for $i = 1, 2, \ldots, n$, then we get

$$\sum_{j=1}^{n} x_i^\alpha \cdot x_j^\alpha \frac{{}^{(2)}\partial^\alpha f}{\partial x_i^\alpha x_j^\alpha} + x_i^\alpha \frac{\partial^\alpha f}{\partial x_i^\alpha} = r\cdot x_i^\alpha \cdot \frac{\partial^\alpha f}{\partial x_i^\alpha}.$$

Adding all equations for $i = 1, \ldots, n$, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n} x_i^\alpha \cdot x_j^\alpha \frac{{}^{(2)}\partial^\alpha f}{\partial x_i^\alpha x_j^\alpha} + \sum_{i=1}^{n} x_i^\alpha \frac{\partial^\alpha f}{\partial x_i^\alpha} = r\sum_{i=1}^{n} x_i^\alpha \cdot \frac{\partial^\alpha f}{\partial x_i^\alpha},$$

and solving

$$\sum_{i=1}^{n}\sum_{j=1}^{n} x_i^\alpha \cdot x_j^\alpha \frac{{}^{(2)}\partial^\alpha f}{\partial x_i^\alpha x_j^\alpha} = r(r-\alpha)\cdot f$$

or

$$\left(\sum_{i=1}^{n}\sum_{j=1}^{n} x_i^\alpha x_j^\alpha \frac{\partial^\alpha}{\partial x_i^\alpha}\frac{\partial^\alpha}{\partial x_j^\alpha}\right)^{(2)} = r(r-\alpha)\cdot f.$$

Therefore, this result is true for $p = 2$.

Assume now that the result is true for $p = k$, that is,

$$\left(x_1^\alpha \frac{\partial^\alpha}{\partial x_1^\alpha} + x_2^\alpha \frac{\partial^\alpha}{\partial x_2^\alpha} + \dots + x_n^\alpha \frac{\partial^\alpha}{\partial x_n^\alpha}\right)^{(k)} f = \left(\sum_{i_1=1}^{n}\sum_{i_2=1}^{n} \dots \sum_{i_k=1}^{n} x_{i_1}^\alpha x_{i_2}^\alpha \dots x_{i_k}^\alpha \frac{\partial^\alpha}{\partial x_{i_1}^\alpha}\frac{\partial^\alpha}{\partial x_{i_2}^\alpha} \dots \frac{\partial^\alpha}{\partial x_{i_k}^\alpha}\right) f$$
$$= \sum_{i_1=1}^{n}\sum_{i_2=1}^{n} \dots \sum_{i_k=1}^{n} x_{i_1}^\alpha x_{i_2}^\alpha \dots x_{i_k}^\alpha \frac{{}^{(k)}\partial^\alpha f}{\partial x_{i_1}^\alpha x_{i_2}^\alpha \dots x_{i_k}^\alpha}$$
$$= r(r-\alpha)(r-2\alpha)\dots(r-(k-1)\alpha)\cdot f.$$

Let $p = k+1$. Compute the conformable partial derivative[10] of the aforementioned equation with respect to x_i, then multiply it by x_i^α, for $i = 1, 2, \dots, n$, and adding all equations, we find

$$\sum_{i_1=1}^{n}\sum_{i_2=1}^{n} \cdots \sum_{i_{k+1}=1}^{n} x_{i_1}^\alpha x_{i_2}^\alpha \cdots x_{i_{k+1}}^\alpha \frac{{}^{(k+1)}\partial^\alpha f}{\partial x_{i_1}^\alpha x_{i_2}^\alpha \dots x_{i_{k+1}}^\alpha} + r\cdot \sum_{i_1=1}^{n}\sum_{i_2=1}^{n} \cdots \sum_{i_k=1}^{n} x_{i_1}^\alpha x_{i_2}^\alpha \cdots x_{i_k}^\alpha \frac{{}^{(k)}\partial^\alpha f}{\partial x_{i_1}^\alpha x_{i_2}^\alpha \dots x_{i_{k+1}}^\alpha}$$
$$= r(r-\alpha)(r-2\alpha)\dots(r-(k-1)\alpha)\cdot \sum_{i=1}^{n} x_i^\alpha \cdot \frac{\partial^\alpha f}{\partial x_i^\alpha}.$$

Solve

$$\sum_{i_1=1}^{n}\sum_{i_2=1}^{n} \cdots \sum_{i_{k+1}=1}^{n} x_{i_1}^\alpha x_{i_2}^\alpha \cdots x_{i_{k+1}}^\alpha \frac{{}^{(k+1)}\partial^\alpha f}{\partial x_{i_1}^\alpha x_{i_2}^\alpha \dots x_{i_{k+1}}^\alpha} = r(r-\alpha)(r-2\alpha)\dots(r-k\alpha)\cdot f.$$

Thus, the result is true for $p = k+1$. Therefore, by the Principle of Mathematical Induction, this result is true for any positive integer p.

This completes the proof of the theorem. □

Example 2. Let $\alpha \in (0, \frac{1}{3}]$. Since $f(x_1, x_2, x_3) = x_1^2 x_2^3 + x_3^5$ is homogeneous function of degree 5 and class C_α^p on $\mathbb{R}^3$, for all $p \in \mathbb{Z}^+$, then

$$\sum_{i=1}^{3} x_i^\alpha \cdot \frac{\partial^\alpha f}{\partial x_i^\alpha} = 5f$$
$$\sum_{i=1}^{3}\sum_{j=1}^{3} x_i^\alpha \cdot x_j^\alpha \cdot \frac{{}^{(2)}\partial^\alpha f}{\partial x_i^\alpha \partial x_j^\alpha} = 5(5-\alpha)f$$
$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} x_i^\alpha \cdot x_j^\alpha \cdot x_k^\alpha \cdot \frac{{}^{(3)}\partial^\alpha f}{\partial x_i^\alpha \partial x_j^\alpha \partial x_k^\alpha} = 5(5-\alpha)(5-2\alpha)f.$$

5 | CONCLUSIONS

Properties of homogeneous functions that involve their conformable partial derivatives are proposed and proven in this paper, specifically, the homogeneity of the conformable partial derivatives of a homogeneous function and the conformable version of Euler's theorem. In addition, this last result is extended to higher-order derivatives. The findings of this study indicate that the results obtained in fractional case conform with the results obtained in ordinary case. Finally, this work is applicable to thermodynamics and various areas of finance, where function is dependent on more than three variables.

CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

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How to cite this article: Martínez F, Martínez-Vidal I, Paredes S. Conformable Euler's theorem on homogeneous functions. *Comp and Math Methods*. 2019;1:e1048. https://doi.org/10.1002/cmm4.1048