Structure Constants for New Infinite-Dimensional Lie Algebras of $U(N_+,N_-)$
Tensor Operators and Applications

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Abstract

The structure constants for Moyal brackets of an infinite basis of functions on the algebraic manifolds $M$ of pseudo-unitary groups $U(N_+,N_-)$ are provided. They generalize the Virasoro and $W_\infty$ algebras to higher dimensions. The connection with volume-preserving diffeomorphisms on $M$, higher generalized-spin and tensor operator algebras of $U(N_+,N_-)$ is discussed. These centrally-extended, infinite-dimensional Lie-algebras provide also the arena for non-linear integrable field theories in higher dimensions, residual gauge symmetries of higher-extended objects in the light-cone gauge and $C^*$-algebras for tractable non-commutative versions of symmetric curved spaces.
The general study of infinite-dimensional algebras and groups, their quantum deformations (in particular, central extensions) and representation theory has not progressed very far, except for some important achievements in one- and two-dimensional systems, and there can be no doubt that a breakthrough in the subject would provide new insights into the two central problems of modern physics: unification of all interactions and exact solvability in QFT and statistics.

The aforementioned achievements refer mainly to Virasoro and Kac-Moody symmetries (see e.g. [1, 2]), which have played a fundamental role in the analysis and formulation of conformally-invariant (quantum and statistical) field theories in one and two dimensions, and systems in higher dimensions which in some essential respects are one- or two-dimensional (e.g. String Theory). Generalizations of the Virasoro symmetry, as the algebra \( \text{diff}(S^1) \) of reparametrisations of the circle, lead to the infinite-dimensional Lie algebras of area-preserving diffeomorphisms \( \text{sdiff}(\Sigma) \) of two-dimensional surfaces \( \Sigma \). These algebras naturally appear as a residual gauge symmetry in the theory of relativistic membranes [3], which exhibits an intriguing connection with the quantum mechanics of space constant (e.g. vacuum configurations) \( SU(N) \) Yang-Mills potentials in the limit \( N \to \infty \) [4]; the argument that the internal symmetry space of the \( U(\infty) \) pure Yang-Mills theory must be a functional space, actually the space of configurations of a string, was pointed out in Ref. [5]. Moreover, the \( \mathcal{W}_\infty \) and \( \mathcal{W}_{1+\infty} \) algebras of area-preserving diffeomorphisms of the cylinder [6] generalize the underlying Virasoro gauged symmetry of the light-cone two-dimensional induced gravity discovered by Polyakov [7] by including all positive conformal-spin currents [8], and induced actions for these \( \mathcal{W} \)-gravity theories have been proposed [9, 10]. Also, the \( \mathcal{W}_{1+\infty} \) (dynamical) symmetry has been identified by [8] as the set of canonical transformations that leave invariant the Hamiltonian of a two-dimensional electron gas in a perpendicular magnetic field, and appears to be relevant in the classification of all the universality classes of \textit{incompressible quantum fluids} and the identification of the quantum numbers of the excitations in the Quantum Hall Effect. Higher-spin symmetry algebras where introduced in [11] and could provide a guiding principle towards the still unknown “M-theory”.

It is remarkable that area-preserving diffeomorphisms, higher-spin and \( \mathcal{W} \) algebras can be seen as distinct members of a one-parameter family \( \mathcal{L}_\mu(su(2)) \) — or the non-compact version \( \mathcal{L}_\mu(su(1,1)) \) — of non-isomorphic [13] infinite-dimensional Lie-algebras of \( SU(2) \) — and \( SU(1,1) \) — tensor operators, more precisely, the factor algebra \( \mathcal{L}_\mu(su(2)) = \mathcal{U}(su(2))/I_\mu \) of the universal enveloping algebra \( \mathcal{U}(su(2)) \) by the ideal \( I_\mu = (\hat{C} - \hbar^2 \mu)\mathcal{U}(su(2)) \) generated by the Casimir operator \( \hat{C} \) of \( su(2) \) (\( \mu \) denotes an arbitrary complex number). The structure constants for \( \mathcal{L}_\mu(su(2)) \) and \( \mathcal{L}_\mu(su(1,1)) \) are well known for the Racah-Wigner basis of tensor operators [14], and they can be written in terms of Clebsch-Gordan and (generalized) 6\( j \)-symbols [3, 8].

Another interesting feature of \( \mathcal{L}_\mu(su(2)) \) is that, when \( \mu \) coincides with the eigenvalue of \( C \) in an irrep \( D_j \) of \( SU(2) \), that is \( \mu = j(j+1) \), there exists an ideal \( \chi \) in \( \mathcal{L}_\mu(su(2)) \) such that the quotient \( \mathcal{L}_\mu(su(2))/\chi \simeq sl(2j+1, C) \) or \( su(2j+1) \), by taking a compact real form of the complex Lie algebra [13]. That is, for \( \mu = j(j+1) \) the infinite-dimensional algebra \( \mathcal{L}_\mu(su(2)) \) collapses to a finite-dimensional one. This fact was used in [3] to approximate \( \lim_{\hbar \to 0} \mathcal{L}_\mu(su(2)) \simeq \text{sdiff}(S^2) \) by \( su(N)|_{N \to \infty} \) ("large number of colors").

The generalization of these constructions to general unitary groups proves to be quite unwieldy, and a canonical classification of \( U(N) \)-tensor operators has, so far, been proven to exist only for \( U(2) \) and \( U(3) \) (see [14] and references therein). Tensor labeling is provided in these cases by the Gel’fand-Weyl pattern for vectors in the carrier space of the irreps of \( U(N) \).
In this letter, a quite appropriate basis of operators for $\mathcal{L}_\mu(u(N_+, N_-))$, $\mu = (\mu_1, \ldots, \mu_N)$, $N \equiv N_+ + N_-$, is provided and the structure constants, for the particular case of the boson realization of quantum associative operatorial algebras on algebraic manifolds $M_{N_+ N_-} = U(N_+, N_-)/U(1)^N$, are calculated. The particular set of operators in $\mathcal{U}(u(N_+, N_-))$ is the following:

$$L^I_{|m|} = \prod_\alpha (\hat{G}_{\alpha\alpha})^{I_\alpha - (\sum_{\beta > \alpha} |m_{\alpha\beta}| + \sum_{\beta < \alpha} |m_{\beta\alpha}|)/2} \prod_{\alpha < \beta} (\hat{G}_{\alpha\beta})^{|m_{\alpha\beta}|}$$

$$\hat{L}^I_{|m|} = \prod_\alpha (\hat{G}_{\alpha\alpha})^{I_\alpha - (\sum_{\beta > \alpha} |m_{\alpha\beta}| + \sum_{\beta < \alpha} |m_{\beta\alpha}|)/2} \prod_{\alpha < \beta} (\hat{G}_{\beta\alpha})^{|m_{\alpha\beta}|}$$

(1)

where $\hat{G}_{\alpha\beta}$, $\alpha, \beta = 1, \ldots, N$, are the $U(N_+, N_-)$ Lie-algebra (step) generators with commutation relations:

$$[\hat{G}_{\alpha_1\beta_1}, \hat{G}_{\alpha_2\beta_2}] = \hbar(\eta_{\alpha_1\beta_2} \hat{G}_{\alpha_2\beta_1} - \eta_{\alpha_2\beta_1} \hat{G}_{\alpha_1\beta_2}),$$

(2)

and $\eta = \text{diag}(1, N_+, 1, -1, N_-, -1)$ is used to raise and lower indices; the upper (generalized spin) index $I \equiv (I_1, \ldots, I_N)$ of $\hat{L}$ in (1) represents a $N$-dimensional vector which, for the present, is taken to lie on an half-integral lattice; the lower index $m$ symbolizes a integral upper-triangular $N \times N$ matrix, and $|m|$ means absolute value of all its entries. Thus, the operators $L^I_m$ are labeled by $N + N(N - 1)/2 = N(N + 1)/2$ indices, in the same way as wave functions $\psi_m^I$ in the carrier space of irreps of $U(N)$. An implicit quotient by the ideal $I_\mu = \prod_{j=1}^N (\hat{C}_j - h\mu_j)\mathcal{U}(u(N_+, N_-))$ generated by the Casimir operators

$$\hat{C}_1 = \hat{G}_a^a = h\mu_1, \quad \hat{C}_2 = \hat{G}_a^a \hat{G}_a^a = h^2\mu_2, \ldots$$

(3)

is understood. The manifest expression of the structure constants $f$ for the commutators

$$\left\{ L^I_m, L^J_n \right\} = \mu(L^I_m \hat{L}^J_n - \hat{L}^J_n L^I_m = f_{mnK}^{IL} \hat{L}^K_l$$

(4)

of a pair of operators (1) of $\mathcal{L}_\mu(u(N_+, N_-))$ entails an unpleasant and difficult computation, because of inherent ordering problems. However, the essence of the full quantum algebra $\mathcal{L}_\mu(u(N_+, N_-))$ can be still captured in a classical construction by extending the Poisson-Lie bracket

$$\left\{ L^I_m, L^J_n \right\}_{PL} = (\eta_{\alpha_1\beta_1} G_{\alpha_2\beta_1} - \eta_{\alpha_2\beta_1} G_{\alpha_1\beta_2}) \frac{\partial L^I_m}{\partial G_{\alpha_1\beta_1}} \frac{\partial L^J_n}{\partial G_{\alpha_2\beta_2}}$$

(5)

of a pair of functions $L^I_m, L^J_n$ on the commuting coordinates $G_{\alpha\beta}$ to its deformed version, in the sense of Ref. [17]. To perform calculations with (5) is still rather complicated because of non-canonical brackets for the generating elements $G_{\alpha\beta}$. Nevertheless, there is a standard boson operator realization $G_{\alpha\beta} \equiv a_\alpha a_\beta$ of the generators of $u(N_+, N_-)$ for which things simplify greatly. Indeed, we shall understand that the quotient by the ideal generated by polynomials $G_{\alpha_1\beta_1} G_{\alpha_2\beta_2} - G_{\alpha_1\beta_2} G_{\alpha_2\beta_1}$ is taken, so that the Poisson-Lie bracket (5) coincides with the standard Poisson bracket

$$\left\{ L^I_m, L^J_n \right\}_P = \eta_{\alpha\beta} \left( \frac{\partial L^I_m}{\partial a_\alpha} \frac{\partial L^J_n}{\partial a_\beta} - \frac{\partial L^I_m}{\partial a_\beta} \frac{\partial L^J_n}{\partial a_\alpha} \right)$$

(6)

for the Heisenberg-Weyl algebra. There is basically only one possible deformation of the bracket (6) —corresponding to a normal ordering— that fulfills the Jacobi identities [17], which is the
Moyal bracket \([15]\):

\[
\{ L^I_m, L^J_n \} = L^I_m * L^J_n - L^J_n * L^I_m = \sum_{r=0}^{\infty} 2 \frac{(\hbar/2)^{2r+1}}{(2r+1)!} P^{2r+1}(L^I_m, L^J_n),
\]

where \( L * L' \equiv \exp(\hbar P/2)(L, L') \) is an invariant associative \(*\)-product and

\[
P^r(L, L') \equiv \Upsilon_{1,1} \cdots \Upsilon_{r, r} \frac{\partial^r L}{\partial x_{11} \cdots \partial x_{1r}} \frac{\partial^r L'}{\partial x_{j1} \cdots \partial x_{jr}},
\]

with \( x \equiv (a, \bar{a}) \) and \( \Upsilon_{2N \times 2N} \equiv \begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix} \). We set \( P^0(L, L') \equiv LL' \); see also that \( P^1(L, L') = \{L, L'\}_P \). Note the resemblance between the Moyal bracket (7) for \([\cdot, \cdot]_\text{Moyal} \) with \( \ast \) being the classical Moyal bracket (7), for operators \( \hat{L}_m^I \) and the standard commutator (4) for operators \( \hat{L}_m^I \). It is worthwhile mentioning that Moyal brackets where identified as the primary quantum deformation \( \mathcal{W}_\infty \) of the classical algebra \( w_\infty \) of area-preserving diffeomorphisms of the cylinder (see Ref. [19]).

With this information at hand, the manifest expression of the structure constants \( f \) for the Moyal bracket (7) is the following:

\[
\left\{ L^I_m, L^J_n \right\}_M = \sum_{\varphi \in \Pi_2^{(2r+1)}} (-1)^{t_{\varphi} + 1} \prod_{s=0}^{2r} f^I_{m, n} (\varphi, I_{\varphi(s)}, m) f^J_{n, m} (\varphi, J_{\varphi(s)}, -n),
\]

\[
f^I_{m, n} (\alpha_0, \ldots, \alpha_{2r}) = \sum_{s=0}^{\infty} 2 \frac{(\hbar/2)^{2r+1}}{(2r+1)!} \eta_{\alpha_0} \cdots \eta_{\alpha_2} f^I_{m, n} (\alpha_0, \ldots, \alpha_{2r}) \sum_{h=0}^{\infty} \delta_{\alpha_0, \alpha_2} / 2,
\]

\[
f^J_{n, m} (\alpha_0, \ldots, \alpha_{2r}) = \sum_{s=0}^{\infty} 2 \frac{(\hbar/2)^{2r+1}}{(2r+1)!} \eta_{\alpha_0} \cdots \eta_{\alpha_2} f^J_{n, m} (\alpha_0, \ldots, \alpha_{2r}) \sum_{h=0}^{\infty} \delta_{\alpha_0, \alpha_2} / 2,
\]

\[
\theta(s - \ell_{\varphi}) = \begin{cases} 0, & s \leq \ell_{\varphi} \\ 1, & s > \ell_{\varphi} \end{cases}, \quad \delta_{\alpha_j} = (\delta_{1, \alpha_j}, \ldots, \delta_{N, \alpha_j}),
\]

where \( \Pi_2^{(2r+1)} \) denotes the set of all possible partitions \( \varphi \) of a string \( (\alpha_0, \ldots, \alpha_{2r}) \) of length \( 2r + 1 \) into two substrings

\[
\ell_{\varphi} \quad \text{and} \quad 2r + 1 - \ell_{\varphi},
\]

of length \( \ell_{\varphi} \) and \( 2r + 1 - \ell_{\varphi} \), respectively. The number of elements \( \varphi \) in \( \Pi_2^{(2r+1)} \) is clearly

\[
\dim(\Pi_2^{(2r+1)}) = \sum_{\ell=0}^{2r+1} \frac{(2r+1)!}{(2r+1-\ell)!\ell!} = 2^{2r+1}.
\]

For \( r = 0 \), there are just 2 partitions: \( (\alpha), (\alpha) \), and the leading (classical, \( \hbar \to 0 \)) structure constants are, for example:

\[
f^I_{m, n} (\alpha) = J_\alpha \left( \sum_{\beta > \alpha} m_{\alpha \beta} - \sum_{\beta < \alpha} m_{\beta \alpha} \right),
\]

\[
f^J_{n, m} (\alpha) = I_\alpha \left( \sum_{\beta > \alpha} n_{\alpha \beta} - \sum_{\beta < \alpha} n_{\beta \alpha} \right).
\]

They reproduce in this limit the Virasoro commutation relations for the particular generators \( V_k^{(\alpha \beta)} \equiv \delta_{\alpha \beta} \), where \( k \in \mathcal{Z} \) and \( e_{\alpha \beta} \) denotes an upper-triangular matrix with zero entries.
except for 1 at \((\alpha/\beta)\)-position, if \(\alpha < \beta\), or 1 at the \((\beta/\alpha)\)-position if \(\alpha > \beta\). Indeed, there are \(N(N-1)/2\) non-commuting Virasoro sectors in \([9]\), corresponding to each positive root in \(SU(N_+,N_-)\), with classical commutation relations:
\[
\{V^{(\alpha/\beta)}_k, V^{(\alpha/\beta)}_l\} = \eta^{\alpha} \text{sign}(\beta - \alpha) (k - l)V^{(\alpha/\beta)}_{k+l}.
\]

For large \(r\), we can benefit from the use of algebraic-computing programs like [24] to deal with the high number of partitions.

Our boson operator realization \(G_{\alpha/\beta} = a_\alpha \bar{a}_\beta\) of the \(u(N_+,N_-)\) generators corresponds to the particular case of \(\bar{\mu}_0 = (N,0,\ldots,0)\) for the Casimir eigenvalues, so that the commutation relations \([4]\) are related to the particular algebra \(\mathcal{L}_{\bar{\mu}_0}(u(N_+,N_-))\) (see below for more general cases). A different (minimal) realization of \(\mathcal{L}_{\mu}(su(1,1))\) in terms of a single boson \((a,\bar{a})\), which corresponds to \(\mu_c = \bar{s}(\bar{s} - 1) = -3/16\) for the critical value \(\bar{s} = 3/4\) of the symplin degree of freedom \(s\), was given in [23]; this case is also related to the symplection algebra of [14]. Note the close resemblance between the algebra \([4]\) —and the leading structure constants \([11]\) —and the quantum deformation \(W_\infty \simeq L_0(su(1,1))\) of the algebra of area-preserving diffeomorphisms of the cylinder \([8, 12]\), although we recognize that the case discussed in this letter is far richer.

If the analyticity of the symbols \(L^{I_m}_m\) of \([1]\) is taken into account, then one should worry about a restriction of the range of the indices \(I_\alpha, m_\alpha\beta\). The subalgebra \(\mathcal{L}^{A}_{\bar{\mu}_0}(u(N_+,N_-)) \equiv \{L^{I_m}_m | \Lambda_\alpha = I_\alpha - (\sum_{\beta > \alpha} |m_{\alpha/\beta}| + \sum_{\beta < \alpha} |m_{\beta/\alpha}|/2 \in N\}\) of polynomial functions on \(G_{\alpha/\beta}\), the structure constants \(f^{I_{n,m}}_{I_{m,n}}(0\alpha,\ldots,0\beta)\) of which are zero for \(r > (\sum_\alpha (I_\alpha + J_\alpha) - 1)/2\), can be extended beyond the “wedge” \(\Lambda \geq 0\) by analytic continuation, that is, by revoking this restriction to \(\Lambda \in \mathbb{Z}/2\). The aforementioned “extension beyond the wedge” (see \([11, 12]\) for similar concepts) makes possible the existence of conjugated pairs \((L^{I_m}_m, L^{I'_m}_m)\), with \(\sum_\alpha I_\alpha + I'_\alpha = 2r + 1\) and \(I_\alpha + I'_\alpha \equiv r_\alpha \in \mathbb{N}\), that give rise to central terms under commutation:
\[
\Xi(L^{I_m}_m, L^{I'_n}_n) = \frac{h^{2r+1}2^{2r}(-1)^{\sum_{\alpha=1}^{N} r_\alpha}}{\prod_{n=1}^{N} (2r + 1 - r_\alpha)!} f^{I_{m-n}}_{I_{n-m}}(1^{(r_1)},\ldots,N^{(r_N)}) \delta_{m+n,0} \hat{1},
\]
where \((1^{(r_1)},\ldots,N^{(r_N)})\) is a string of length \(\sum_{\alpha=1}^{N} r_\alpha = 2r + 1\) and \(\alpha^{(r_\alpha)}\) denotes a substring made of \(r_\alpha\)-times \(\alpha\), for each \(\alpha\). The generator \(\hat{1} \equiv I^0_0\) is central (commutes with everything) and the Lie algebra two-cocycle \([13]\) defines a non-trivial central extension of \(\mathcal{L}_{\bar{\mu}_0}(u(N_+,N_-))\) by \(U(1)\).

A thorough study of the Lie-algebra cohomology of \(\mathcal{L}_{\bar{\mu}_0}(u(N_+,N_-))\) and its irreps still remains to be accomplished; it requires a separate attention and shall be left for future works. Two-cocycles like \([13]\) provide the essential ingredient to construct invariant geometric action functionals on coadjoint orbits of \(\mathcal{L}_{\bar{\mu}_0}(u(N_+,N_-))\) —see e.g. \([11]\) for the derivation of the WZNW action of \(D = 2\) matter fields coupled to chiral \(W_\infty\) gravity background from \(W_\infty \simeq L_0(su(1,1))\).

In order to deduce the structure constants for general \(\mathcal{L}_{\bar{\mu}_0}(u(N_+,N_-))\) from \(\mathcal{L}_{\bar{\mu}_0}(u(N_+,N_-))\), a procedure similar to that of Ref. \([12]\), for the particular case of \(U(sl(2,\mathbb{R}))\), can be applied. Special attention must be paid to the limit \(\lim_{\bar{\mu}_0 \to \bar{\mu}} \mathcal{L}_{\bar{\mu}_0}(u(N_+,N_-)) \simeq P_C(M_{N_+,N_-})\), which coincides with the Poisson algebra of complex (wave) functions \(\psi_{|m\rangle}^I, \psi_{-|m\rangle}^I \equiv \bar{\psi}_{|m\rangle}^I\) on algebraic manifolds (coadjoint orbits \([23]\)) \(M_{N_+,N_-} \simeq U(N_+,N_-)/U(1)^N\). \(|\rangle\) It is well known that there

\(^{1}\mathbb{N}, \mathbb{Z}, \mathbb{R}\) and \(\mathbb{C}\) denote the set of natural, integer, real and complex numbers, respectively

\(^{2}\)For \(N_- \neq 0\), other cases could be also contemplated (e.g. continuous series of \(SU(1,1)\))
exists a natural symplectic structure \((M_{N+ N}, \Omega)\), which defines the Poisson algebra

\[
\{ \psi^I_m, \psi^J_n \} = \Omega^{\alpha_1 \beta_1 \alpha_2 \beta_2} \frac{\partial \psi^I_m}{\partial g^{\alpha_1 \beta_1}} \frac{\partial \psi^J_n}{\partial g^{\alpha_2 \beta_2}}
\]

(14)

and an invariant symmetric bilinear form \(\langle \psi^I_m | \psi^J_n \rangle = \int v(g) \bar{\psi}_m^I (g) \psi_n^J (g)\) given by the natural invariant measure \(v(g) \sim \Omega^{N(N-1)/2}\) on \(U(N_+, N_-)\), where \(g^{\alpha \beta} = \bar{g}^{\beta \alpha} \in \mathbb{C}, \alpha \neq \beta\), is a (local) system of complex coordinates on \(M_{N+ N}\). The structure constants for (14) can be obtained through \(f^{IJK}_{mn} = \langle \psi^K_I | \{ \psi^I_m, \psi^J_n \} \rangle\). Also, an associative \(*\)-product can be defined through the convolution of two functions \((\psi^I_m \ast \psi^J_n)(g') \equiv \int v(g) \psi^I_m (g) \psi^J_n (g^{-1} \cdot g')\), which gives the algebra \(\mathcal{P}_C(M_{N+ N})\) a non-commutative character — \(g \cdot g'\) denotes the group composition law of \(U(N_+, N_-)\). A manifest expression for all these structures is still in progress [23].

Taking advantage of all these geometrical tools, action functionals for \(\mathcal{L}_\infty (u(N_+, N_-))\) Yang-Mills gauge theories in \(D\) dimensions could be built as:

\[
S = \int d^D x (F_{\nu \gamma} (x, g) F^{\nu \gamma} (x, g)),
\]

\[
F_{\nu \gamma} = \partial_\nu A_\gamma - \partial_\gamma A_\nu + \{ A_\nu, A_\gamma \},
\]

\[
A_\nu (x, g) = A^m_\nu (x) \psi^I_m (g), \ \nu, \gamma = 1, \ldots, D,
\]

the ‘vacuum configurations’ (spacetime-constant potentials \(X_\nu (g) \equiv A_\nu (0, g)\)) of which, define the action for higher-extended objects: \(N(N - 1)\)-branes’, in the usual nomenclature. Here, \(\mathcal{L}_\infty (u(N_+, N_-))\) plays the role of gauge symplectic (volume-preserving) diffeomorphisms \(L_\psi \equiv \{ \psi, \cdot \}\) on the \(N(N - 1)\)-brane \(M_{N+ N}\). A particularly interesting case might be \(SU(2, 2) = U(2, 2)/U(1)\): the conformal group in \(3 + 1\) (or the AdS group in \(4 + 1\)) dimensions, in an attempt to construct ‘conformal gravities’ in realistic dimensions. The infinite-dimensional algebra \(\mathcal{L}_\mu(u(2, 2))\) might be seen as the generalization of the Virasoro (two-dimensional) conformal symmetry to \(3 + 1\) dimensions.

Finally, let me comment on the potential relevance of the \(C^*\)-algebras \(\mathcal{L}_\mu(\mathcal{G})\) on tractable non-commutative versions [24] of symmetric curved spaces \(M = G/H\), where the notion of a pure state \(\psi^I_m\) replaces that of a point. The possibility of describing phase-space physics in terms of the quantum analog of the algebra of functions (the covariant symbols \(L^I_m\)), and the absence of localization expressed by the Heisenberg uncertainty relation, was noticed a long time ago by Dirac [24]. Just as the standard differential geometry of \(M\) can be described by using the algebra \(C^\infty (M)\) of smooth complex functions \(\psi\) on \(M\) (that is, \(\lim_{\hbar \to 0} \mathcal{L}_\mu(\mathcal{G})\), when considered as an associative, commutative algebra), a non-commutative geometry for \(M\) can be described by using the algebra \(\mathcal{L}_\mu(\mathcal{G})\), seen as an associative algebra with a non-commutative \(*\)-product like (18). The appealing feature of a non-commutative space \(M\) is that a \(G\)-invariant ‘lattice structure’ can be constructed in a natural way, a desirable property as regards finite models of quantum gravity (see e.g. [26] and Refs. therein). Indeed, as already mentioned, \(\mathcal{L}_\mu(\mathcal{G})\) collapses to Mat\(_d\)(\(\mathcal{C}\)) (the full matrix algebra of \(d \times d\) complex matrices) whenever \(\mu_\alpha\) coincides with the eigenvalue of \(\hat{C}_\alpha\) in a \(d\)-dimensional irrep \(D_\mu\) of \(G\). This fact provides a finite \((d\)-points\) ‘fuzzy’ or ‘cellular’ description of the non-commutative space \(M\), the classical (commutative) case being recovered in the limit \(\mu \to \infty\). The notion of space itself could be the collection of all of them, enclosed in a single irrep of \(\mathcal{L}_\mu(\mathcal{G})\) for general \(\mu\), with different multiplicities, as it actually happens with the reduction of an irrep of the centrally-extended
Virasoro group under its $SL(2,\mathbb{R})$ subgroup \[27\]; The multiplicity should increase with $\vec{\mu}$ (‘the density of points’), so that classical-like spaces are more abundant. It is also a very important feature of $\mathcal{L}_\mu(u(N_+, N_-))$ that the quantization deformation scheme (7) does not affect the maximal finite-dimensional subalgebra $su(N_+, N_-)$ (‘good observables’ or preferred coordinates \[17\]) of non-commuting ‘position operators’

$$
y_{\alpha\beta} = \frac{i}{\lambda}(\hat{G}_{\alpha\beta} + \hat{G}_{\beta\alpha}), \quad y_{\beta\alpha} = \frac{i}{\lambda}(\hat{G}_{\alpha\beta} - \hat{G}_{\beta\alpha}), \quad \alpha < \beta,
$$

$$
y_{\alpha} = \frac{\lambda}{\hbar}(\eta_{\alpha\alpha} \hat{G}_{\alpha\alpha} - \eta_{\alpha+1,\alpha+1} \hat{G}_{\alpha+1,\alpha+1}),
$$

(16)
on the algebraic manifold $M_{N_+,N_-}$, where $\lambda$ denotes a parameter that gives $y$ dimensions of length (e.g., the square root of the Planck area $\hbar G$). The ‘volume’ $v_j$ of the $N - 1$ submanifolds $M_j$ of the flag manifold $M_{N_+,N_-} = M_N \supset \ldots \supset M_2$ (see e.g. \[28\] for a definition of flag manifolds) is proportional to the eigenvalue $\mu_j$ of the $su(N_+, N_-)$ Casimir operator $\hat{C}_j$ in those coordinates: $v_j = \lambda^j \mu_j$. Large volumes (flat-like spaces) correspond to a high density of points (large $\mu$). In the classical limit $\lambda \to 0, \mu \to \infty$, the coordinates $y$ commute.

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**References**


